Int. Journal of Math. Analysis, Vol. 6, 2012, no. 45, 2199 - 2216

# Generalized Potential Function in Linear Mixed Integral Equation

**M. A. Abdou* [a] and F. M. Al-Saedy [b]**

[a] Department of Mathematics, Faculty of Education
University of Alexandria, Alexandria, Egypt
abdella_777@ yahoo.com

[b] Department of Mathematics, Faculty of Sciences
Umm Al-Qura University , Makkah, Saudi Arabia
f_saedy21@ hotmail.com

**Abstract**

In this work, the Fredholm-Volterra integral equation of the second kind with a generalized potential kernel is considered. The existence of a unique solution of the integral equation, under certain conditions, is discussed and proved in the space $L_2(\Omega)\times C[0,T]$, $\Omega=\left\{(x,y,z)\in\Omega:\sqrt{x^2+y^2}\le a\,,\, z=0\right\}$ and $T<\infty$. By using a numerical method ,the integral equation can be represented as a system of Fredholm integral equations of the second kind and the solution can be obtained by using the degenerate kernel method . Also, the kernel of position is represented in the Weber-Sonien integral formula and nonhomogeneous wave equation. Many special and new cases are derived from the kernel.

**Mathematics Subject Classification:** 45B05, 45E10

**Keywords:** Fredholm-Volterra integral equation (F-VIE); Generalized potential; Weber-Sonien integral; Legendre polynomials; Wave equation

## 1. Introduction

Many problems of mathematical physics , contact problems in the theory of elasticity and mixed problems of mechanics of continuous media reduce to an integral equation with a kernel that have either of the following forms,

$$K^{\nu,\gamma}_{n,m}(x,y)=\frac{x^{\gamma}}{y^{\varepsilon+\gamma-1}}W^{\nu}_{n,m}(x,y) \qquad , \quad 0<\varepsilon<<1$$

$$W^{\nu}_{n,m}(x,y)=\int_0^{\infty} t^{2\nu-1}J_n(xt)\,J_m(yt)\,dt \qquad , \qquad (1.1)$$

where $J_n(x)$ is a Bessel function of the first kind and order $n$ .

Kavalenko in [1] , developed the Fredholm integral equation of the first kind for the mechanics mixed problems of continuous media and obtained an approximate solution with an elliptic kernel. In [5] , Abdou discussed the solution of the integral equation of the second kind in three dimensions with a potential kernel. The resolvent kernel of the potential function is obtained and discussed in the work of Abdou [7]. Mkhitarian, in [11] , used potential theory method to obtain the spectral relationships for a Fredholm integral equation of the first kind with Carleman kernel. Also, many spectral relationships were obtained in Abdou [8], when the kernel of Fredholm integral equation takes the Carleman function. Abdou and Hassan [10] , solved the Fredholm integral equation of the first kind with logarithmic kernel. Moreover Abdou in [9], obtained and discussed many spectral relationships for the Fredholm equation with logarithmic kernel. Also the same author, in [6], obtained numerically the solution of F-VIE of the second kind with logarithmic kernel, using Legendre polynomials. Abdou , in his work [4] , obtained the spectral relationships of Fredholm-Volterra integral equation of the first kind with a generalized potential kernel.

In this work , we establish the F-VIE of the second kind with a generalized potential kernel with respect to position, for the Fredholm integral term, and a continuous kernel in time, for the Volterra integral term. Using a numerical method, the F-VIE will be reduced to a finite system of Fredholm integral equations. The kernel of the system, after using the separation of variable method, with some special polynomials, can be reduced to a general form of Weber-Sonien integral formula. Also, using a degenerate method, the solution of Fredholm system can be discussed. Moreover the structure resolvent of the generalized potential kernel is obtained. Numerical results are considered to obtain the potential function numerically.

## 2. Formulation of the problem

The Fredholm-Volterra integral equation,

$$(\lambda_1+\lambda_2)\phi(x,y;t)-(\theta_1+\theta_2)\iint_{\Omega}k(x-\zeta,y-\eta)\phi(\zeta,\eta;t)\,d\zeta\,d\eta-(\theta_1+\theta_2)\int_0^t F(t,\tau)\phi(x,y;\tau)\,d\tau$$
$$=\pi\,[\,\gamma(t)+\beta(t)x-f_1(x,y)-f_2(x,y)\,] \qquad (2.1)$$

is investigated from the semi-symmetric contact problem of Hertz type of two rigid surfaces $(G_i,\nu_i \;\; ,i=1,2)$ having two elastic materials occupying the domain $\Omega$. Here $f_i(x,y)\in L_2(\Omega)$, $i=1,2$ , describing the two surfaces . If the upper surface is impressed into the lower surface by a variable force $p(t)$, whose eccentricity of application $e(t)$ and a moment $M(t)$ that case rigid displacements $\gamma(t)$ and $\beta(t)x$ respectively, through the time interval $0 \le t < T < \infty$ and $\gamma(t), \beta(t) \in C[0,T]$ . In the absence of body forces, and when the frictional forces in the domain of contact between the two surfaces are so small in which it can be neglected. The unknown function $\phi(x,y;t)$ represents the unknown normal stresses between the two surfaces through the time $t$ , $t \in [0,T]$, $t < \infty$. The positive continuous function $F(t,\tau) \in C([0,T] \times [0,T])$ represents the resistance force of the lower material against the total pressure and moment. Here $\lambda_i$, $i = 1,2$ are the coefficients bed of the compressible materials that depend on its geometry and its physical properties $\theta_i = \dfrac{1-\mu_i^2}{\pi E_i}$, $i = 1.2$ where $\mu_i, \nu_i$ are the poisson's coefficients and $E_i$ are the coefficients of Young .

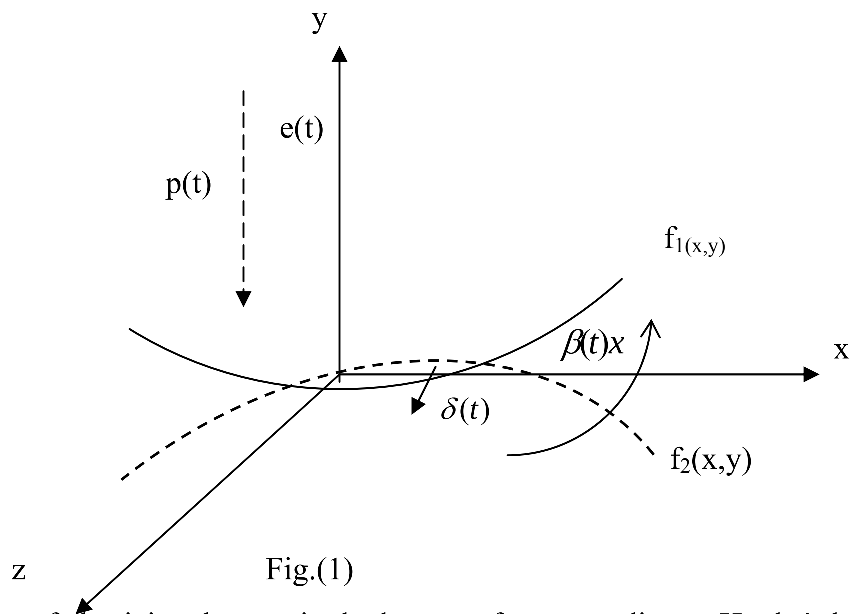

Fig.(1)

If the modulus of elasticity changes in the layer surface according to Hooke's law : $\sigma_i = k_0 \varepsilon_i^{\nu}$ , $i = 1,2,3$ $(0 < \nu < 1)$ where $\sigma_i$ and $\varepsilon_i$ are the stress and strain rate intensities , respectively , while $k_0$ depends on the physical properties of the elastic layer , the kernel, in this case, will take the form :

$$k(x-\zeta, y-\eta) = [\ (x-\zeta)^2 + (y-\eta)^2\ ]^{-\nu} \ , \ 0 < \nu < 1 \qquad (2.2)$$

## 3. Existence and uniqueness solution of the integral equation

Consider the linear integral equation of (2.1) in the form

$$\phi(x,y;t)= f(x,y;t)+\lambda\iint_{\Omega}k(x-\zeta,y-\eta)\phi(\zeta,\eta;t)\,d\zeta\,d\eta+\lambda\int_0^t F(t,\tau)\phi(x,y;\tau)\,d\tau \qquad (3.1)$$

under the conditions

$$\iint_{\Omega}\phi(x,y;t)\,dx\,dy = N(t),$$
$$\iint_{\Omega}x\,y\,\phi(x,y;t)\,dx\,dy = M(t). \qquad (3.2)$$

where, $f(x,y;t)=\dfrac{\pi}{\lambda_1+\lambda_2}[\gamma(t)+\beta(t)-f_1(x,y)-f_2(x,y)]$ , $\lambda=\dfrac{\theta_1+\theta_2}{\lambda_1+\lambda_2};\lambda$ is a constant, may be complex, and has many physical meaning. The given functions $f(x,y;t), k(x-\zeta,y-\eta)$ and $F(t,\tau)$ are called respectively the free term , the kernel of Fredholm integral term and the kernel of Volterra.While ,$\Omega$ is the domain of integration with respect to position definedas $\Omega=\left\{(x,y,z)\in\Omega:\sqrt{x^2+y^2}\le a\,,z=0\right\}$ , and $t$ is the time , such that $t\in[0,T], T<\infty$ . The unknown function $\phi(x,y;t)$ will be obtained in the space $L_2(\Omega)\times C[0,T]$.The existence and uniqueness solution of Eq.(2.1) and consequently its equivalent Eq.(3.1), under certain conditions, will be discussed and proved in the space $L_2(\Omega)\times C[0,T]$ , using Banach fixed point theorem . For this , we write Eq. (3.1) in the integral operator form ,

$$\overline{W}\phi = W\phi + f \quad , W\phi = K\phi + F\phi \qquad (3.3)$$

where ,

$$K\phi=\lambda\iint_{\Omega}k(x-\zeta,y-\eta)\phi(\zeta,\eta;t)\,d\zeta\,d\eta \quad , \qquad (3.4)$$

$$F\phi=\lambda\int_0^t F(t,\tau)\phi(x,y;\tau)\,d\tau \qquad (3.5)$$

We assume the following conditions :

i. The kernel of Fredholm integral term satisfies the discontinuity condition,

$$\left[\iint_{\Omega}\iint_{\Omega}k^2(x-\zeta,y-\eta)\,d\zeta\,d\eta\,dx\,dy\right]^{\frac{1}{2}}=c\,, c \text{ is constant .}$$

ii. The kernel of Volterra integral term $F(t,\tau)\in C([0,T]), 0\le\tau\le t<\infty$, and satisfies $|F(t,\tau)|\le M \quad \forall\, t,\tau\in[0,T], M$ is constant.

iii. The given function $f(x,y;t)$ with its partial derivatives with respect to $x, y$ and $t$ are continuous in $L_2(\Omega)\times C[0,T]$ and its norm can be defined as,

$$\|f(x,y;t)\| = \max_{0\le t\le T}\int_0^t\left[\int_{\Omega}f^2(x,y;\tau)\,dx\,dy\right]^{\frac{1}{2}}d\tau = H\,, H \text{ is a constant .}$$

iv. The unknown function $\phi(x,y;t)$, in the space $L_2(\Omega)\times C[0,T]$, behaves as the given function $f(x,y;t)$ .

**Theorem( 3.1)**

If the conditions (i) – (iv) are satisfied , then Eq. (3.1 ) has a unique solution $\phi(x,y;t)$ in the Banach space $L_2(\Omega)\times C[0,T]$, under the condition,

$$|\lambda| < \frac{1}{(c+MT)} \ . \tag{3.6}$$

**Proof :**

To prove the existence of a unique solution of Eq. (3.1 ) using Banach fixed point theorem , we must prove the normality and continuity of the integral operator (3.3):

(a) For the normality of the integral operator $W\phi$ , we write ,

$$\|W\phi\| \le |\lambda| \left\| \iint_{\Omega} k(x-\zeta, y-\eta)\phi(\zeta,\eta;t)\,d\zeta\, d\eta \right\| + |\lambda| \left\| \int_0^t F(t,\tau)\phi(x,y;\tau)\,d\tau \right\|$$

By using the conditions (i) and (ii) , we have

$$\|W\phi\| \le \beta\|\phi\|, \qquad \beta = |\lambda|\,(c+M\,T) \tag{3.7}$$

Hence, $W$ is a norm operator, which leads directly, after using condition(iii), to the normality of the operator $\overline{W}$ .

(b) For the continuity of the integral operator $\overline{W}$ we assume the two potential functions $\phi_1(x,y;t), \phi_2(x,y;t)$ satisfies the formula (3.3) then ,

$$\begin{aligned} \left\|\overline{W}(\phi_1-\phi_2)\right\| &\le |\lambda| \left\| \iint_{\Omega} k(x-\zeta, y-\eta)\big(\phi_1(\zeta,\eta;t)-\phi_2(\zeta,\eta;t)\big)\,d\zeta\, d\eta \right\| \\ &+ |\lambda| \left\| \int_0^t F(t,\tau)\big(\phi_1(x,y;\tau)-\phi_2(x,y;\tau)\big)\,d\tau \right\|. \end{aligned} \tag{3.8}$$

Using the conditions (i) and (ii) , we get

$$\left\|\overline{W}(\phi_1-\phi_2)\right\| \le \beta\|\phi_1-\phi_2\| \qquad ,\beta \text{ is constant .} \tag{3.9}$$

So , $\overline{W}$ is a continuous operator. By using the condition (3.6) we have $\beta<1$ , i.e. $\overline{W}$ is a contraction operator , then it has a unique solution .

## 4. The system of Fredholm integral equations

For representing ( 3.1 ) as a finite system of Fredholm integral equations , we divide the interval $[\ 0,T\ ]$ , $0 \le t \le T < \infty$ as $0 = t_0 < t_1 < ... < t_N = T$ , where $t = t_k \in [\ 0,T\ ]$ , $k = 0,1,2,...,N$ . Then using the quadratic formula, the Volterra integral term of (3.1 ) yields, see[6,4],

$$\int_0^{t_k} F(t_k,\tau)\phi(x,y,\tau)d\tau = \sum_{j=0}^{k} u_j F(t_k,t_j)\phi(x,y,t_j) + O(\hbar_k^{p+1}) \tag{4.1}$$

where ,

$$h_k \to 0 \quad , \quad p>0 \quad , \quad h_j = t_{j+1}-t_j \quad , \quad j=0,1,...,k \quad , \quad \hbar_k = \max_{0\le k\le N} h_k$$

$$u_j = \begin{cases} \frac{h}{2} & , \quad j = 0, k \\ h & , \quad j \neq 0, k \end{cases} \tag{4.2}$$

The values of $u_j$ and $p \approx k$ depend on the number of derivatives of $F(t,\tau)$ with respect to $t$ and $\tau$ .
Using the following notations

$$\phi(x,y,t_j) = \phi_j(x,y) \quad , \; F(t_k,t_j) = F_{k,j} \quad , \; f(x,y;t_i) = f_i(x,y) \quad , \tag{4.3}$$

Eq.(3.1), yields

$$\mu\phi_k(x,y) - \lambda \iint_\Omega \frac{\phi_k(\zeta,\eta) d\zeta d\eta}{[(x-\zeta)^2 + (y-\eta)^2]^\nu} - \lambda \sum_{j=0}^{k} u_j F_{k,j} \phi_j(x,y) = f_k(x,y) \quad . \tag{4.4}$$

Also , the condition ( 3.2 ) becomes ,

$$\iint_\Omega \phi_k(x,y) dx\, dy = N_k \quad , \quad (N(t_k) = N_k) \quad . \tag{4.5}$$

To separate the variables , we assume in Eqs.(4.4) and (4.5)

$$\phi_k(x,y) = \phi_{km}(r) \begin{cases} \cos m\theta \\ \sin m\theta \end{cases} , \quad f_k(x,y) = f_{km}(r) \begin{cases} \cos m\theta \\ \sin m\theta \end{cases} , \tag{4.6}$$

to have,

$$\mu\phi_{km}(r) - \lambda \int_0^a W_m^\nu(r,\rho)\, \phi_{km}(\rho)\, \rho\, d\rho - \lambda \sum_{j=0}^{k} u_j F_{k,j} \phi_{jm}(r) = f_{km}(r) \quad , \tag{4.7}$$

and

$$\int_0^a \rho\, \phi_{km}(\rho)\, d\rho = \begin{cases} \frac{N_k}{2\pi} & , \quad m = 0 \\ 0 & , \quad m \geq 1 \end{cases} \tag{4.8}$$

where

$$W_m^\nu(r,\rho) = \int_{-\pi}^{\pi} \frac{\cos m\omega}{\left[ r^2 + \rho^2 - 2r\rho\cos\omega \right]^\nu} d\omega \ . \tag{4.9}$$

To write the integral (4.9 ) in the Bessel function form, we use the following relations , see [2] ,

$$\int_0^{2\pi} \frac{\cos m\psi}{\left[1-2z\cos\psi+z^2\right]^{\alpha}}\,d\psi = \frac{2\pi(\alpha)_m}{m!}\,z^m\;{}_2F_1\left(\alpha, m+\alpha, m+1, z^2\right),$$
$$(\alpha)_m = \frac{\Gamma(\alpha+m)}{\Gamma(\alpha)}, \tag{4.10}$$

and

$${}_2F_1\left(\alpha, \alpha+\tfrac{1}{2}-\beta, \beta+\tfrac{1}{2}, z^2\right) = (1+z)^{-2\alpha}\;{}_2F_1\left(\alpha, \beta, 2\beta, \frac{4z}{(1+z)^2}\right), \quad \left(|z|<1 \;,\; \operatorname{Re}\alpha > 0\right) \tag{4.11}$$

where $\Gamma(x)$ is the gamma function, $(\alpha)_m$ is called pochammer symbol and ${}_2F_1(a,b,c;z)$ is the Gauss hypergeometric function, to get

$$W_m^{\nu}(r,\rho) = 2\pi\frac{\Gamma(m+\nu)}{m!\,\Gamma(\nu)(r+\rho)^{2(m+\nu)}}(r\rho)^m\;{}_2F_1\left(m+\nu, m+\tfrac{1}{2}, 2m+1, \frac{4r\rho}{(r+\rho)^2}\right), \tag{4.12}$$

Secondly, using the relation , see [3]

$$\int_0^{\infty} J_{\alpha}(ax)\,J_{\alpha}(bx)\,x^{-\beta}\,dx = \frac{2^{-\beta}\,a^{\alpha}\,b^{\alpha}\,\Gamma\left(\alpha+\frac{1-\beta}{2}\right)}{(a+b)^{2\alpha+\beta+1}\,\Gamma(1+\alpha)\,\Gamma\left(\frac{1+\beta}{2}\right)} \times {}_2F_1\left(\alpha+\tfrac{1-\beta}{2}, \alpha+\tfrac{1}{2}, 2\alpha+1, \frac{4ab}{(a+b)^2}\right), \tag{4.13}$$

Eq.(4.12) becomes

$$W_m^{\nu}(r,\rho) = c\int_0^{\infty} t_1^{\,2l}\,J_m(r\,t_1)\,J_m(\rho\,t_1)\,dt_1 \;,\left( c = \frac{\pi\,\Gamma(\frac{1}{2}-l)}{2^{2l-1}\,\Gamma(\frac{1}{2}+l)} \;,\; 0<\nu<1 \;,\; 0<l<\frac{1}{2} \right), \tag{4.14}$$

where $J_m(x)$ is the Bessel function of the first type and order $m$ ; $m \ge 0$ .
Using the following notations ,

$$\psi_{k,m}(r) = \sqrt{r}\,\phi_{k,m}(r) \;,\; g_{k,m}(r) = \sqrt{r}\,f_{k,m}(r) \;,\; r = au \;,\; \rho = av \;,\; t_1 = \frac{t}{a} \;,\; Q_k = \frac{N_k}{2\pi a} \;, \tag{4.15}$$

the formula (4.7) and the condition (4.8), respectively, become

$$\mu_k\psi_{k,m}(u) - \lambda\int_0^1 K_{m,m}^{\nu}(u,v)\,\psi_{k,m}(v)\,dv = G_{k,m}(u) \;,\left(\mu_k = \mu - \lambda u_k F_{k,k}\right) \tag{4.16}$$

and

$$\int_0^1 \sqrt{v}\,\psi_{km}(v)\,dv = \begin{cases} Q_k \;, & m=0 \\ 0 \;, & m \ge 1 \end{cases} \tag{4.17}$$

where

$$K_m^\nu(u,v)=c^*\sqrt{uv}\int_0^\infty t^{2l}\,J_m(u\,t)\,J_m(v\,t)\,dt \quad , \quad c^*=a^{1-2l}\,c\ ,\ K_{m,m}^\nu(u,v)=K_m^\nu(u,v) \qquad (4.18)$$

and

$$G_{k,m}(u)=g_{k,m}(u)+\lambda\sum_{j=0}^{k-1}u_j F_{k,j}\psi_{j,m}(u) \qquad (4.19)$$

The formula (4.16) has a unique solution under the condition,

$$|\lambda|\le|\mu_k|/\{\int_0^1\int_0^1[K_m^\nu(u,v)]^2\,du\,dv\ \}^{\frac{1}{2}} \quad ,(\,0\le k\le N\,)\,. \qquad (4.20)$$

Also, it represents a finite linear system of Fredholm integral equations of the second kind when $\mu_k\neq 0$ for all values of $k$ , $0\le k\le N$, and of the first kind when $\mu_k=0$ for $0\le k\le N$.

## 5. Generalized kernel and its structure resolvent

The kernel (4.18) of the linear system takes a generalized from of Weber-Sonien integral formula and it is not difficult to prove that:

(i) For the first derivatives, we have

$$(\frac{\partial}{\partial r}+\frac{\partial}{\partial\rho})K_{m,m}^\nu(u,v)=[a_m^-(r)+a_m^-(\rho)]K_{m,m}^\nu(r,\rho)+K_{m-1,m}^\nu(r,\rho)+K_{m,m-1}^\nu(r,\rho) \qquad (5.1)$$

or

$$(\frac{\partial}{\partial r}+\frac{\partial}{\partial\rho})K_{m,m}^\nu(u,v)=[a_m^+(r)+a_m^+(\rho)]K_{m,m}^\nu(r,\rho)-(K_{m+1,m}^\nu(r,\rho)+K_{m,m+1}^\nu(r,\rho)) \qquad (5.2)$$

where $a^\pm(x)=(\frac{1}{2x}\pm\frac{m}{x})$ ,

The formulas (5.1) and (5.2) are called Cauchy problem of the first order.

In the logarithmic kernel or Carleman function $a^\pm(x)=0$, where, in this case, $m=\pm\frac{1}{2}$.

(ii) Also, for the second derivatives, we get

$$(\frac{\partial^2}{\partial u^2}-\frac{\partial^2}{\partial v^2})\,K_m^\nu(u,v)=[\ h_m(u)-h_m(v)\ ]K_m^\nu(u,v) \qquad (5.3)$$

where

$$h_m(x)\ =\ (\ m^2-\frac{1}{4}\ )\ x^{-\frac{1}{2}} \quad , \quad (m\neq\pm\frac{1}{2}) \quad , \quad K_{m,m}^\nu(u,v)=K_m^\nu(u,v)$$

The formula (5.3) represents a nonhomogeneous wave equation for $m \neq \pm\frac{1}{2}$, and a homogeneous wave equation when $m = \pm\frac{1}{2}$ . ( i.e. for logarithmic kernel and Carleman function )

The resolvent kernel of Eq.(4.16) takes the form

$$R^{\nu}_{m,m}(u,v;\alpha) = K^{\nu}_{m,m}(u,v) + \alpha\int_0^1 K^{\nu}_{m,m}(u,s)\, R^{\nu}_{m,m}(s,v,\alpha)\, ds \ ,$$

$$((I-\alpha K)\,(I+R) = I\ ,\ \alpha = \frac{\lambda}{\mu_k}\ ,\ u\ ,\ v \in (0,1);\ R^{\nu}_{m,m}(u,v,\alpha) = R_m(u,v,\alpha)\,) \tag{5.4}$$

and

$$R^{\nu}_{m,m}(u,v,\alpha) = K^{\nu}_{m,m}(u,v) + \alpha\int_0^1 R^{\nu}_{m,m}(u,s;\alpha)\, K^{\nu}_{m,m}(s,v)\, ds$$

$$((I+R)\,(I-\alpha K)) = I \ , \tag{5.5}$$

where $R^{\nu}_m(u,v,\alpha)$ is the resolvent kernel of the Fredholm integral equation with a generalized kernel $K^{\nu}_m(u,v)$ of Eq.(4.18), and $I$ is the unit vector ( identity matrix ).

The resolvent kernel $R^{\nu}_m(u,v;\alpha)$ and the kernel $K^{\nu}_m(u,v)$ are bounded, continuous and differentiable functions belong to the class $C^l(0,1)$ , the class of continuous functions with its derivatives $l$ times.

To obtain the system of differentiable equations of the resolvent, let $\Omega = (0,1)\times(0,1)$, and consider the two differentiable operators

$$L\,f = \sum_{i=0}^{l} A_i(u)\partial^i_u f(u) \quad , \quad L\,f = \sum_{i=0}^{l} (-1)^i \partial^i_u (f(u)A_i(u)) \tag{5.6}$$

where $A_i(u)$ and $f(u)$ are functions contain in $C^l(0,1)$ .

The proof of the following theorem can be obtained with the aid of Abdou[3].

**Theorem (5.1)**

If $K^{\nu}_m(u,v) \in C^l(\Omega) \cap C^{l-1}(\overline{\Omega})$,

$$L\,K^{\nu}_m(u,v) \overset{def}{=} L_u\, K^{\nu}_m(u,v) - L_v\, K^{\nu}_m(u,v) = N(u)M(v)\ ,\ (u,v \in \Omega) \tag{5.7}$$

Then, the differentiable operator of the resolvent takes the form,

$$L\,R^{\nu}_m(u,v;\alpha) = \alpha\{\ \sum_{i=1}^{l}\ \sum_{j=1}^{i}\ (-1)^{i+j+2}\partial^{i-j}_u[R^{\nu}_m(u,s)\,A_i(s)]\partial^{i-1}_s R^{\nu}_m(s,v)\ \}^1_{s=0} + N(u)M(v)\,, \tag{5.8}$$

where $N(u)$ , $M(v)$ are continuous function, defined from integral equations,

$$(I-\alpha\, K^{\nu}_m)\,N(u) = p(u) \quad , \quad (I-\alpha\, K^{\nu}_m)\,M(v) = q(v) \tag{5.9}$$

As special case of the theorem, we let in (5.8) $N(u) = M(v) = 0$ and $A_i(s) = 1$, to have

$$L\,R_m^\nu(u,v;\alpha)=\alpha[\sum_{i=1}^{l}\sum_{j=1}^{i}(-1)^{i+j+2}\,\partial_s^{i-j}\,R_m^\nu(u,s)\partial_s^{i-1}R_m^\nu(s,v)]_{s=0}^{1} \tag{5.10}$$

For the first derivatives $l=1$, we obtain

$$(\frac{\partial}{\partial u}+\frac{\partial}{\partial v})R_m^\nu(u,v;\alpha)=\alpha[R_m^\nu(u,1;\alpha)\,R_m^\nu(1,v;\alpha)-R_m^\nu(u,0;\alpha)\,R_m^\nu(0,v;\alpha)]. \tag{5.11}$$

The first derivative of the resolvent kernel represents Cauchy problem in the sence of partial differential equation.

Also, for $l=2$, we have

$$(\frac{\partial^2}{\partial u^2}-\frac{\partial^2}{\partial v^2})\,R_m^\nu(u,v;\alpha)=F(u,v;\alpha)\quad,(\alpha=\frac{\lambda}{\mu_k},\,k=0,1,\ldots,N) \tag{5.12}$$

where

$$F(u,v;\alpha)=\alpha[R_m^\nu(u,s;\alpha)R_m^\nu(s,v;\alpha)-R_m^\nu(s,v;\alpha)\partial_s R_m^\nu(u,s;\alpha)+R_m^\nu(u,s;\alpha)\partial_s R_m^\nu(s,v;\alpha)]_{s=0}^{1}$$

Eq.(5.12) represents a nonhomogenuous wave equation where the given function $F(u,v;\alpha)$ is known.

Many special and new cases can be derived from Eq.(4.18),

i. For elliptic kernel,

$$K_0^0(x,y)=\frac{2\sqrt{xy}}{\pi(x+y)}K\left(\frac{\sqrt{2xy}}{x+y}\right)=\int_0^\infty J_0(xt)\,J_0(yt)\,dt\ \ ,(m=0\,,\,\nu=\frac{1}{2}\ ) \tag{5.13}$$

ii. For potential function, $(\nu=\frac{1}{2})$, see Fig.(2),

$$K_m^{\frac{1}{2}}(x,y)=\sqrt{xy}\int_0^\infty J_m(xt)\,J_m(yt)\,dt \tag{5.14}$$

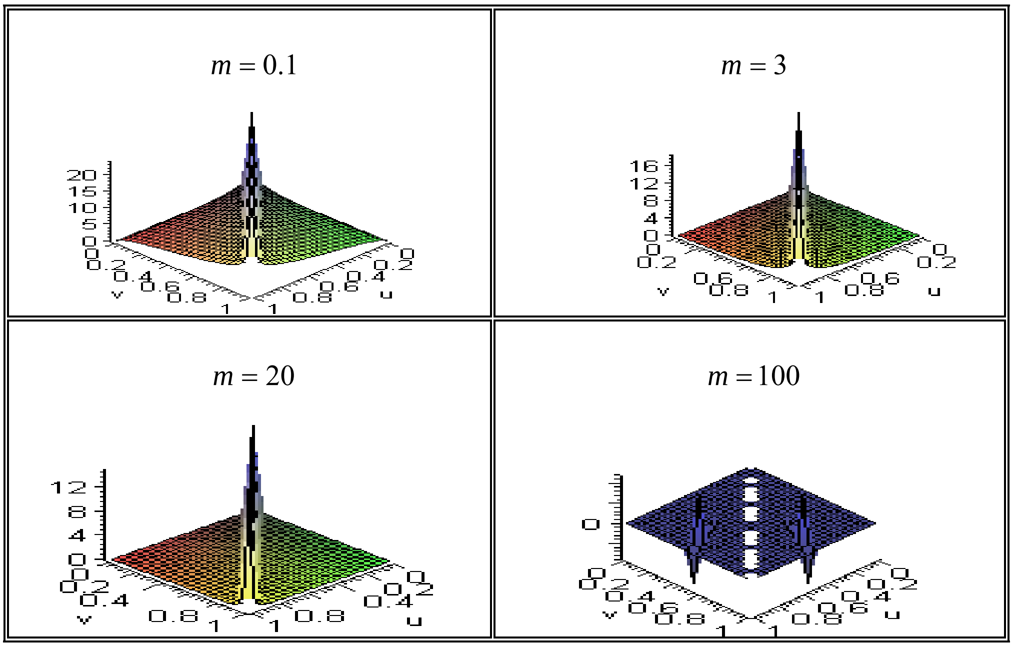


Fig.(2)

iii. For Carleman function, see Fig.(3),

$$K^{\nu}_{\pm 1/2}(x,y) = |\, x - y \,|^{-\nu} = \sqrt{xy}\int_0^{\infty} t^{2\nu-1} J_{\pm 1/2}(xt)\, J_{\pm 1/2}(yt)\, dt \tag{5.15}$$

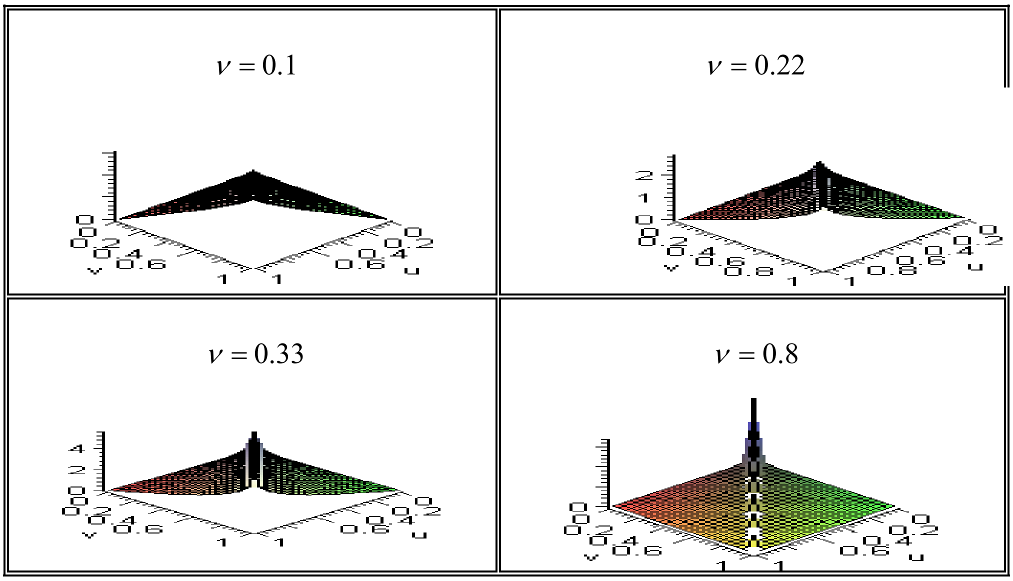


Fig.(3)

iv. For logarithmic kernel

$$K^{0}_{\pm 1/2}(x,y) = -\ln\left| x-y \right| = \sqrt{xy}\int_{0}^{\infty} J_{\pm 1/2}(xt)\, J_{\pm 1/2}(yt)\, dt \;\;,\;\; (m = \pm\frac{1}{2}, \nu = \frac{1}{2}) \qquad (5.16)$$

v. A generalized potential function for the higher-order ( $m \geq 1$ ) harmonic with different values of $\nu$ , ( $0 < \nu < 1$ ) can be found in Fig.(4)

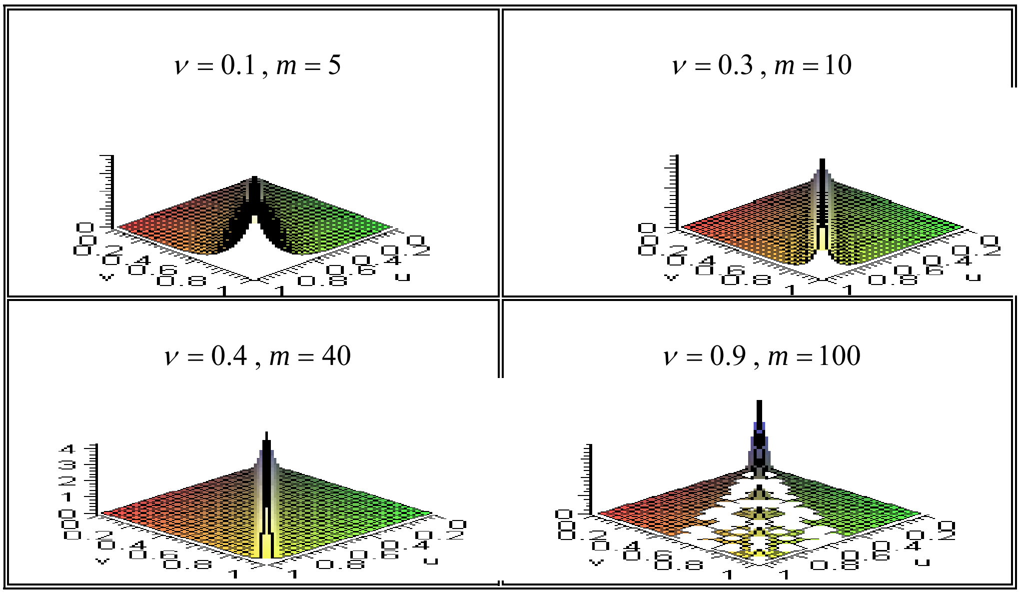


Fig.(4)

vi. A generalized potential function for a negative fractional of higher-order harmonic with different value of $\nu$ $(0 < \nu < 1)$ is represented in Fig. (5)

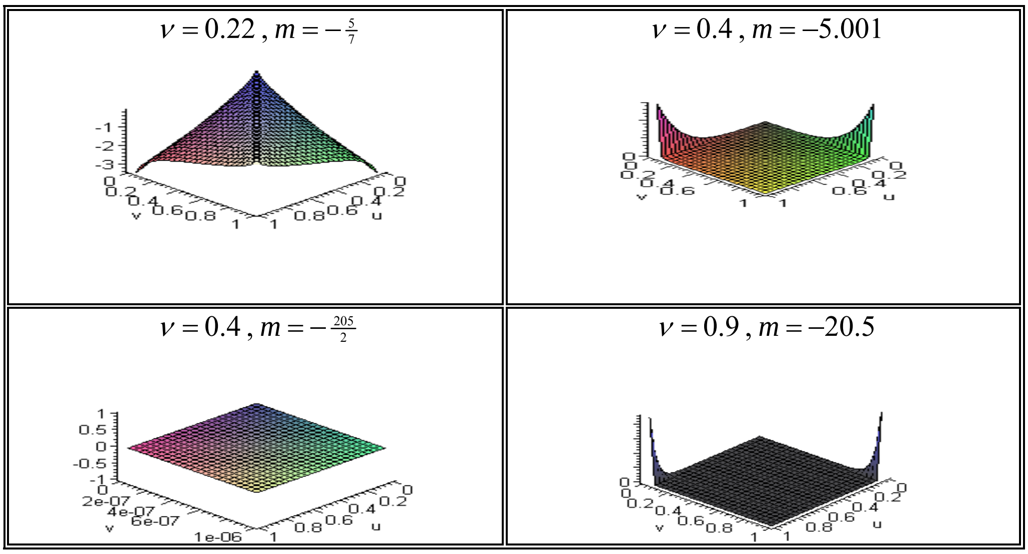


Fig.(5)

## 6. Solution of the linear system

The solution of the linear system of Fredholm integral equations of the second kind (4.16) depends on the kernel (4.18) and the surface $f_m(r)$ . When the initial and the tangent points of the surface are in contact with the origin $0$ , we can expand $f_m(u)$ in Macklorien expansion :

$$f_m(u) \cong \frac{f_m''(u)}{2!}u^2 + \frac{f_m'''(u)}{3!}u^3 + \ldots + \frac{f_m^n(0)}{n!}u^n + \ldots \tag{6.1}$$

The last equation can be approximated to any degree of displacement of the surface. For example , if the displacement is very small and $\frac{f_m''(0)}{(2m)!} = A_2 \neq 0$ , we obtain $f_m(u) = A_2\, u^2$.
In general , we write

$$f_m(u) = A_{2m}\, u^{2m} \quad , \quad A_{2m} = \frac{f^{2m}(0)}{(2m)!} \quad , \quad (m \geq 0) \quad , \tag{6.2}$$

where $m$ is the order harmonic of the contact problem .
Hence, the function $g_m(u)$ takes the form

$$g_m(u) = (\sqrt{u}\Delta - \beta A_{2m}\, u^{2m+\frac{1}{2}}) \ , \ \Delta_0 = \beta\delta_0 \ , \beta = \frac{\pi}{\lambda_1 + \lambda_2} \quad . \tag{6.3}$$

To obtain the solution of Eq.(4.16), using degenerate kernel method we write the kernel in the form ,

$$K_m^\nu(u,v) = c^* 2^{-2\omega}(uv)^{m+\frac{1}{2}} \sum_{j=0}^{\infty} \frac{\Gamma^2(j+m+1-\omega^-)\, p_j^m(u) p_j^m(v)}{\Gamma^2(j+m+1)(2j+m+1-\omega^-)^{-1}} \ , \ \omega^\pm = (\frac{1 \pm \nu}{2}) \ , \tag{6.4}$$

$$p_n^m(u) = p_n^{(m-\omega^\pm)}(1-2u^2) \ , \tag{6.5}$$

where $p_n^m(u)$ is the Legendre polynomial and $p_n^{(m-\omega^\pm)}(u)$ is the Jacobi polynomial .

The simplicity of finding a solution of the Fredholm system of Eq.(4.16) with the kernel $K_m^\nu(u,v)$ of Eq.(4.18), under the condition (4.17), with a degenerate method naturally leads one to think of replacing the given kernel $K_m^\nu(u,v)$ approximately by a degenerate kernel $L_{m,n}^\nu(u,v)$ , that is

$$L_{m,n}^\nu(u,v) = \sum_{i=1}^{n} B_{m,i}^\nu(u) C_{m,i}^\nu(v) \quad . \tag{6.6}$$

Here, the set of functions $\{B_{m,i}^\nu(u)\}$ and $\{C_{m,i}^\nu(v)\}$ are assumed to be linearly independent , such that

$$\left\{ \int_0^1 \int_0^1 \left| K_m^\nu(u,v) - L_{m,n}^\nu(u,v) \right|^2 du\, dv \right\}^{\frac{1}{2}} \to 0 \ \text{ as } \ n \to \infty \quad . \tag{6.7}$$

Hence, the formula (4.16) in view of the approximate kernel (6.6) can be adapted in the form,

$$\mu_k \, H^{(n)}_{k,m}(u) - \lambda \sum_{i=0}^{n} D^{\nu}_{k,m,i} \, B^{\nu}_{k,m,i}(u) = G_{k,m}(u) \quad , \tag{6.8}$$

where

$$D^{\nu}_{k,m,i} = \int_0^1 C^{\nu}_{m,i}(v) H^{(n)}_{k,m}(v)\, dv \quad , \tag{6.9}$$

and

$$\left| \psi_{k,m}(u) - H^{(n)}_{k,m}(u) \right| \to 0 \ \text{as} \ \ n \to \infty \quad .$$

Here, $D^{(\nu)}_{k,m,i}$ are constants to be determined from the following formula,

$$D^{\nu}_{k,m,i} = \int_0^1 \frac{1}{\mu_k} C^{\nu}_{m,i}(v) \{ \, G_{k,m}(v) + \lambda \sum_{j=0}^{n} D^{\nu}_{k,m,j} B^{\nu}_{m,j}(v) \ \} dv \quad , \ (\mu_k \neq 0) \ . \tag{6.10}$$

Adapting the formula (6.10) to take the form

$$\mu_k \, D^{\nu}_{k,m,i} = Z^{\nu}_{k,m,i} + \lambda \sum_{j=0}^{n} F^{\nu}_{m,i,j} \, D_{k,m,j} \quad , \tag{6.11}$$

where

$$Z^{\nu}_{k,m,i} = \int_0^1 C^{\nu}_{m,i}(v) G_{k,m}(v)\, dv \quad ,$$

$$F^{\nu}_{m,i,j} = \int_0^1 C^{\nu}_{m,i}(v) B^{\nu}_{m,j}(v)\, dv \quad . \tag{6.12}$$

Hence, Eq.(6.11) represents a finite linear algebraic system for the approximate kernel of Eq.(6.4), which can be solved numerically under the condition,

$$|\lambda| \left| \sum_{j=0}^{n} F^{\nu}_{m,i,j} \right| < |\mu_k| \ , \ i = 0,1,2,\ldots,n \quad ; \quad k = 1,2,\ldots,N \ .$$

Moreover, at $n \to \infty$, the solution of Eq.(4.16) with the kernel (6.4), under the condition (4.17), is equivalent to the solution of the following linear algebraic equations,

$$\mu_k X^{\nu}_{k,m,j} \ = \ h^{\nu}_{k,m,j} + \lambda \, c^* \sum_{i=0}^{\infty} A^{\nu}_{m,j} \ B^{\nu}_{m,i,j} \ X^{\nu}_{k,m,i} \tag{6.13}$$

where

$$h^{\nu}_{k,m,j} \ = \ (\, 2i + m + 1 - \omega^- \,)^{\frac{3}{4}} \int_0^1 G_{K,m}(u) \, u^{m+\frac{1}{2}} \ p^m_j(u)\, du \quad ,$$

$$B^{\nu}_{m,i,j} \ = \ (2\,j + m + 1 - \omega^-)^{\frac{3}{4}} \ (2i + m + 1 - \omega^-)^{\frac{3}{4}} \int_0^1 u^{2m+1} \ p^m_j(u) \ p^m_i(u)\, du$$

and

$$A^{\nu}_{m,j} \ = \ 2^{-2\omega} \, \frac{\Gamma^2(\,j + m + 1 - \omega^-)\,(2\,j + m + 1 - \omega^-)^{\frac{1}{4}}}{\Gamma^2(\,j + m + 1)} \tag{6.14}$$

The infinite linear system of Eq.(6.13) has a unique solution under the convergence condition,

$$|\lambda| c^* \sum_{j=0}^{\infty} \left| A^{\nu}_{m,i} \ B^{\nu}_{m,i,j} \right| < |\mu_k| \ , \ ( \ m \geq 0 \quad , \quad i \geq 0 \quad , \quad 0 \leq k \leq N \ ) \tag{6.15}$$

Hence, the general solution of Eq.(4.16), under the condition (4.17), takes the form,

$$\mu_k \psi_{m,}(u) = u^{2m+(\frac{1}{2})} + \lambda\, c^* \sum_{j=0}^{\infty} 2^{-2\omega} \frac{\Gamma^2(j+m+1-\omega^-)\,u^{m+\frac{1}{2}}\, p_j^m(u)\, X_{k,m,j}^m}{\Gamma^2(j+m+1).\,(2j+m+1-\omega^-)^{-\frac{1}{4}}} \qquad (6.16)$$

## 7. Numerical Results

**Case ( 1 ) :**

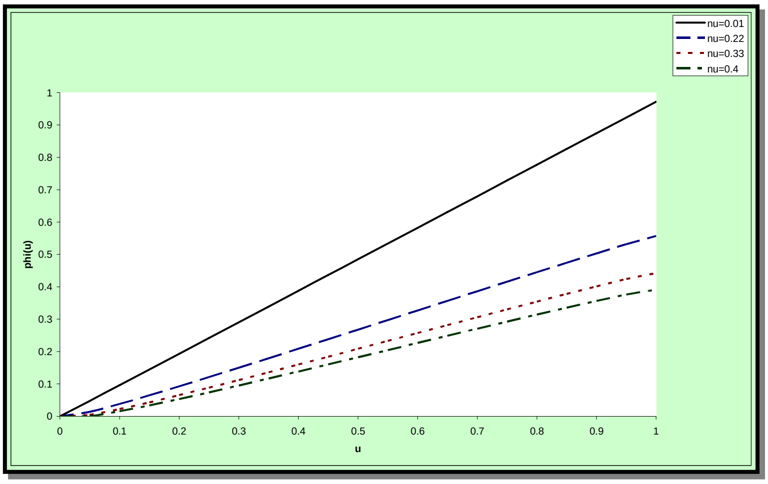


Fig.(6)

The Fig.(6) contains the values of $\phi_m(u)$ when $k=1$, i.e. $F(t,\tau)=t$ , $\mu=\lambda=1$, $m=0.25$, for some different values of $\nu$.

**Case ( 2 ) :**

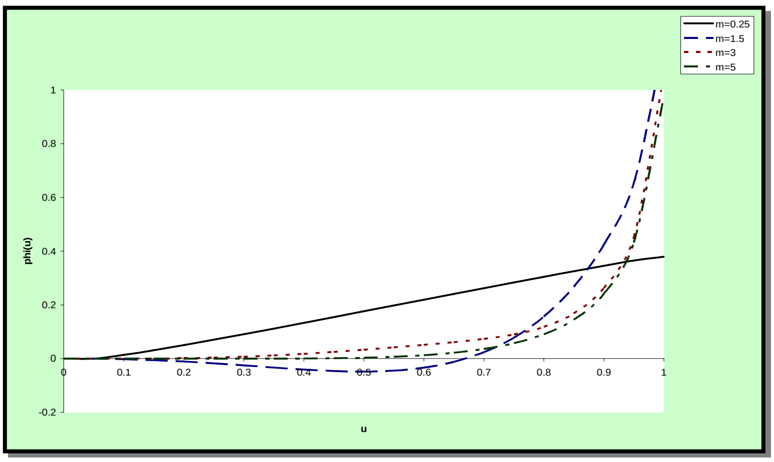


Fig.(7)

The Fig.(7) contains the values of $\phi_m(u)$ when $k=1$, i.e. $F(t,\tau)=t$ , $\mu=\lambda=1$, $\nu=0.42$, for some different values of $m$.

**Case ( 3 ) :**

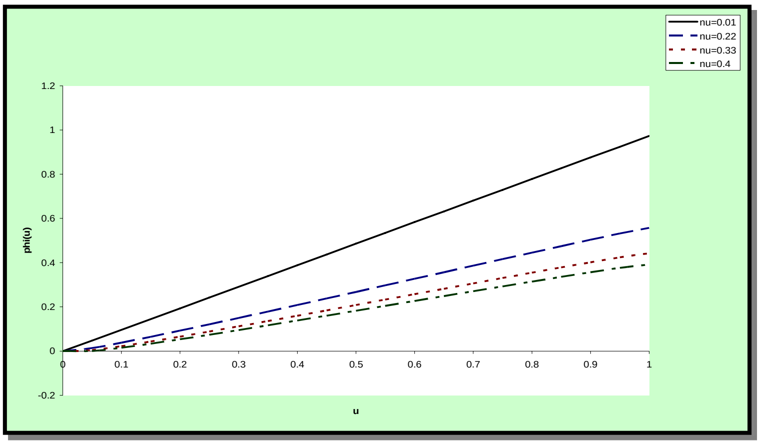


Fig.(8)

The Fig.(8) contains the values of $\phi_m(u)$ when $k=2$, i.e. $F(t,\tau)=t^2$ , $\mu=\lambda=1$, $m=0.25$, for some different values of $\nu$.

**Case ( 4 ) :**

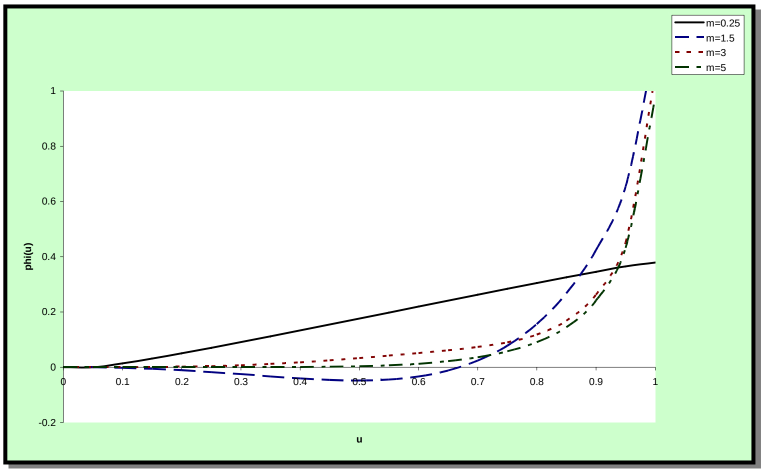


Fig.(9)

The Fig.(9) contains the values of $\phi_m(u)$ when $k=2$, i.e. $F(t,\tau)=t^2$ , $\mu=\lambda=1$, $\nu=0.42$, for some different values of $m$.

## 8. Conclusions

From the above results and discussion , the following may be concluded :

i. The Fredholm–Volterra integral equation of the second kind, by using numerical method, reduced to a system of Fredholm integral equations of the second kind .

ii. The kernel of position is represented in the form of Weber – Sonien integral formula,

$$K_m^{\nu}(u,v)=c^*\sqrt{uv}\int_0^{\infty}t^{2l}\,J_m(u\,t)\,J_m(v\,t)\,dt \quad ,(\; c^*=a^{1-2l}\,\frac{\pi\,\Gamma(\frac{1}{2}-l)}{2^{2l-1}\,\Gamma(\frac{1}{2}+l)}\;,\; 0<\nu<1\;,$$

$$0<l<\frac{1}{2}\;)\,,$$

and it represents a Cauchy problem for the first derivatives and a nonhomogeneous wave equation for the second derivatives .

iii. When the kernel takes a form of logarithmic kernel, Carleman function, elliptic kernel, potential kernel are considered as special cases of this work. Moreover, we can obtain many new and different case for different harmonic order $m$ , such that, $m \neq -1,-2,-3,...$ .

iv. The generalized potential kernel can be written in the form ,

$$K_m^{\nu}(u,v)=c^*\,2^{-2\omega}(uv)^{m+\frac{1}{2}}\sum_{j=0}^{\infty}\frac{\Gamma^2(j+m+1-\omega^-)\,p_j^m(u)\,p_j^m(v)}{\Gamma^2(j+m+1)\,.(2j+m+1-\omega^-)^{-1}}\;,\;\omega^{\pm}=(\frac{1\pm\nu}{2})\,.$$

v. We deduce that , when the values of $\nu$ increase the corresponding values of the function $\phi_m(u)$ is decrease , see Fig.(6).

vi. For specific values of $\nu$ and different values of $m$ the values of the function $\phi_m(u)$ deduce that the pressure force is grater than the resistance force at some values of $u$ , $0 \le u \le 1$ , see Fig.(7).

vii. The structure resolvent of the generalized potential kernel represents in the sence of partial differential equation Cauchy problem for the first derivatives and nonhomogeneous wave equation for the second derivatives.

## References


[1] E . V . Kovalenko, Some approximate methods of solving integral equation of mixed problem, J.Appl. Math. Mech. 53 (1989), 85-92 .

[2] G . Bateman, A . Ergelyi, Higher Transcendental Function, Vol. II , Reprinted. McGraw-Hill, New York, 1989.

[3] I . S . Gradshteyn and I . M . Ryzhik, Table of Integrals, Series and Product ; Academic Press, New York, 1980 .

[4] M.A.Abdou, Fredholm–Volterra integral equation and generalized potential kernel, J. Appl. Math. Comput. 131 (2002), 81–94.

[5] M . A . Abdou , Fredholm integral equation of the second kind with potential kernel, J . Comp . Appl . Math . 72 (1996) ,161-167 .

[6]M.A.Abdou, Fredholm-Volterra integral equation with singular kernel,Appl.Math.Comput.137(2003),231-143.

[7] M.A.Abdou, Fredholm integral equation with potential kernel and its structure resolvent, Appl. Math.Comput.107(2000)169-180 .

[8] M.A.Abdou, Integral equation of mixed type and integrals of orthogonal polynomials, J.Comp.Appl.Math.138(2002),273-285.

[9] M.A.Abdou, Spectral relationships of integral operators in contact problem of impressing stamps,Appl.Math.Comput.118(2001),95-111.

[10] M.A.Abdou, S.A. Hassan, Fredholm integral equation with singular kernel, Korean J.Comput. Appl. Math. Vol. 7No. 11(2000),223-236.

[11] S. M . Mkhitarian, Spectral relationships for the integral operators generated by a kernel in the form of a Weber Sonin integral and their applications to contact problems, Appl. Meth. Mech. 48 (1984) , 67-74.

**Received: March, 2012**