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On Simultaneously Remotal in $L^1(\mu, X)$

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Abstract

Let X be a Banach space, G be a closed and bounded subset of X, and (Ω, Σ, μ) be a $\sigma - finite$ measure space. In this paper we shall show that if G is seperable, then $L^1(\mu, G)$ is simultaneously remotal in $L^1(\mu, X)$ if and only if G is simultaneously remotal in X.

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1 Introduction

Let G be a closed and bounded subset in X, the farthest distance map ρ is defined as

$$\rho(x, G) = \sup\{\|k - x\| : k \in G\}, \quad x \in X\}.$$

For a finite subset H of X, we set

$$\rho(H, G) = \sup\left\{\sum_{h \in H} \|g - h\| : g \in G\right\}.$$

Let n be any integer, the $n-$ farthest point map for $x = (x_1, x_2, ..., x_n) \in \underset{n}{\oplus_1} X$ (a product of $m-$copies of X, equipped with ℓ_1- norm) defined as

$$F_{n,G}(x) = \left\{g \in G : \sum_i \|g - x_i\| = \rho(\{x_i : 1 \leq i \leq n\}, G)\right\},$$

note that this set may be empty. Let $R_n(G, X) = \left\{x \in \underset{n}{\oplus_1} X : F_{n,G}(x) \neq \phi\right\}$. We call a closed and bounded set G simultaneously remotal if $R_n(G, X) =$

$\underset{n}{\oplus_1} X$, for any integer n. If there exists at least $y \in G$ such that

$$\sum_{i=1}^{n} \|x_i - y\| = \rho\left(\{x_i : 1 \leq i \leq n\}, G\right)$$

, then the element y is called a farthest simultaneous point of $x_1, x_2, ..., x_n$ in G. Of course, for $n = 1$ the preceding concepts are just remotal set and farthest point. The study of remotal sets goes back to sixties, see [2,8]. Most of the results are concerned the topological properties of remotal sets in general Banach space see [4, 9, 12]. Recently it has been studed in vector valued function spaces, see [10,11,15,18].

Let X be a Banach space, G a closed and bounded subset of X, and (Ω, Σ, μ) be a measure space.

Definition 1 *Let (M, d) be a metric space. A Borel measurable function from Ω to M is called strongly measurable if it is the pointwise limit of a sequence of simple Borel measurable functions from Ω to M.*

$L^1(\mu, X)$ is denoted to the Banach space consisting of (equivalent classes of) strongly measurable functions $f : \Omega \to X$ such that $\int_\Omega \|f(t)\| \, d\mu$ is finite, with the usual norm

$$\|f\|_1 = \int_\Omega \|f(t)\| \, d\mu.$$

If X is the Banach space of real numbers, we simply write $L^1(\mu)$. For $A \in \Sigma$ and a strongly measurable function $f : \Omega \to X$, we write χ_A for the charateristic function of A and $\chi_A f$ denote the function $\chi_A(s) f(s)$. In particular, for $x \in X$, $\chi_A x(s) = \chi_A(s) x$.

In [17], It is shown that for a closed and bounded subset G of X and (Ω, Σ, μ) a finite measure space, if G is seperable remotal set in X, then $L^1(\mu, G)$ is remotal in $L^1(\mu, X)$. In [10], It is shown that if the span of G is finite dimesinal, then $L^1(\mu, G)$ is remotal in $L^1(\mu, X)$. Also it is shown that if E is l-summand of X, then $L^1(\mu, E)$ is remotal in $L^1(\mu, X)$. For more results on remotal sets in $L^1(\mu, X)$ see [12]. The first paper on simultaneously remotal set appeared in 2010, see [18], the concept of simultaneously remotal (densely remotal) were formulated and studed in $L^\infty(\mu, E)$. In this paper we study simultaneous remotal set in $L^1(\mu, X)$.

2 Preliminary Results

Throughout this section X is a Banach space and G is a closed and bounded subset of X. Let $f_1, f_2, ..., f_m$ be any finite number of elements in $L^1(\mu, X)$, and set

$$\phi(s) = \rho(\{f_i(s) : 1 \leq i \leq m\}, G).$$

Theorem 2.1 Let (Ω, Σ, μ) be a finite measure space, $f_1, f_2, ..., f_m$ be any finite number of elements in $L^1(\mu, X)$, and $\phi(s)$ as defined above. Then, $\phi \in L^1(\mu)$ and

$$\rho\left(\{f_i : 1 \leq i \leq m\},\ L^1(\mu, G)\right) = \int_\Omega |\phi(s)|\, d\mu\ .$$

Proof. Since $f_1, f_2, ..., f_m \in L^1(\mu, X)$, there exist sequences of simple functions

$$(f_{(i,n)})_{n=1}^{\infty}, i = 1, 2, ..., m,$$

such that

$$\lim \left\| f_{(i,n)}(s) - f_i(s) \right\| = 0,$$

for $i = 1, 2, ..., m$, and for almost all s. We may write

$$f_{(i,n)} = \sum_{j=1}^{k(n)} \chi_{A(n,j)}(\cdot)\, x_{(i,n,j)}\ , \quad i = 1, 2, ..., m,$$

$\sum_{j=1}^{k(n)} \chi_{A(n,j)}(\cdot) = 1$, and that $\mu(A(n,j)) > 0$. Then

$$\rho\left(\left\{f_{(i,n)}(s) : 1 \leq i \leq m\right\},\ G\right) = \sum_{j=1}^{k(n)} \chi_{A(n,j)} \rho\left(\left\{x_{(i,n,j)} : 1 \leq i \leq m\right\},\ G\right)$$

and by the continuity of ρ

$$\lim \rho\left(\left\{f_{(i,n)}(s) : 1 \leq i \leq m\right\},\ G\right) = \rho(\{f_i(s) : 1 \leq i \leq m\},\ G),$$

for almost all s. Thus ϕ is measurable and $\phi \in L^1(\mu)$.
Now, for any $h \in L^1(\mu, G)$,

$$\begin{aligned} \int_\Omega \rho(\{f_i(s) : 1 \leq i \leq m\},\ G)\, d\mu &\geq \int_\Omega \sum_{i=1}^{m} \|f_i(s) - h(s)\|\, d\mu \\ &= \sum_{i=1}^{m} \|f_i - h\|_1 \end{aligned}$$

Hence

$$\int_{\Omega} \rho\left(\{f_i(s) : 1 \leq i \leq m\},\ G\right) d\mu \geq \rho\left(\{f_i : 1 \leq i \leq m\},\ L^1(\mu, G)\right).$$

To prove the reverse inequality, let $\epsilon > 0$ be given and $w_i,\ i = 1, 2, ..., m$, be simple functions in $L^1(\mu, X)$ such that

$$\|f_i - w_i\|_1 < \frac{\epsilon}{3m}.$$

We may write $w_i = \sum_{k=1}^{\ell} \chi_{A_k}(\cdot)\ x_{(i,k)}$, $\sum\limits_{k=1}^{\ell} \chi_{A_k}(\cdot) = 1$, and that $\mu(A_k) > 0$. Since $w_i \in L^1(\mu, X)$ for all i, we have $\left\|x_{(i,k)}\right\| \mu(A_k) < \infty$ for all k and i. If $\mu(A_k) < \infty$, select $h_k \in G$ so that

$$\sum_{i=1}^{m} \left\|x_{(i,k)} - h_k\right\| < \rho\left(\left\{x_{(i,k)} : 1 \leq i \leq m\right\},\ G\right) - \frac{\epsilon}{3\mu(A_k)},$$

for all k.

Let $g = \sum_{k=1}^{\ell} \chi_{A_k}(\cdot)\, h_k$. It is clear that $g \in L^1(\mu, G)$. Then

$$\begin{aligned}
\rho\left(\{f_i : 1 \leq i \leq m\},\ L^1(\mu, G)\right) &\geq \sum_{i=1}^{m} \|f_i - g\|_1 \\
&= \sum_{i=1}^{m} \|f_i - w_i + w_i - g\|_1 \\
&\geq -\sum_{i=1}^{m} \|f_i - w_i\|_1 + \sum_{i=1}^{m} \|w_i - g\|_1 \\
&> \sum_{i=1}^{m} -\left(\frac{\epsilon}{3m}\right) + \sum_{i=1}^{m} \|w_i - g\|_1 \\
&= -\frac{\epsilon}{3} + \sum_{i=1}^{m} \sum_{k=1}^{\ell} \mu(A_k) \left\|x_{(i,k)} - h_k\right\| \\
&= -\frac{\epsilon}{3} + \sum_{k=1}^{\ell} \sum_{i=1}^{m} \mu(A_k) \left\|x_{(i,k)} - h_k\right\| \\
&= -\frac{\epsilon}{3} + \sum_{k=1}^{\ell} \mu(A_k) \sum_{i=1}^{m} \left\|x_{(i,k)} - h_k\right\| \\
&\geq \frac{\epsilon}{3} + \sum_{k=1}^{\ell} \mu(A_k)\, \rho\left(\left\{x_{(i,k)} : 1 \leq i \leq m\right\},\ G\right) - \frac{\epsilon}{3}
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{2\epsilon}{3} + \int\limits_{\Omega} \sum_{k=1}^{\ell} \chi_{A_k}(s)\, \rho\left(\left\{x_{(i,k)} : 1 \le i \le m\right\},\ G\right) d\mu \\
&= -\frac{2\epsilon}{3} + \int\limits_{\Omega} \rho\left(\left\{w_i(s) : 1 \le i \le m\right\},\ G\right) d\mu \\
&\ge -\frac{2\epsilon}{3} + \left(\int\limits_{\Omega} \left[\rho\left(\left\{f_i(s) : 1 \le i \le m\right\},\ G\right) + \left(\sum_{i=1}^{m} \left\| f_i(s) - w_i(s) \right\| \right) \right] d\mu \right) \\
&\ge \frac{2\epsilon}{3} + \left[\begin{array}{r} \int\limits_{\Omega} \rho\left(\left\{f_i(s) : 1 \le i \le m\right\},\ G\right) d\mu \\ + \int\limits_{\Omega} \sum_{i=1}^{m} \left\| f_i(s) - w_i(s) \right\| d\mu \end{array} \right] \\
&\ge \frac{2\epsilon}{3} + \int\limits_{\Omega} \rho\left(\left\{f_i(s) : 1 \le i \le m\right\},\ G\right) d\mu + \sum_{i=1}^{m} \left\| f_i - w_i \right\|_1 \\
&\ge \frac{2\epsilon}{3} + \int\limits_{\Omega} \rho\left(\left\{f_i(s) : 1 \le i \le m\right\},\ G\right) d\mu - \frac{\epsilon}{3}
\end{aligned}
$$

$$
\ge -\epsilon + \int\limits_{\Omega} \rho\left(\left\{f_i(s) : 1 \le i \le m\right\},\ G\right) d\mu
$$

This ends the proof.

Corollary 2.2 Let (Ω, Σ, μ) be a finite measure space, $f_1, f_2, ..., f_m$ be any finite number of elements in $L^1(\mu, X)$. Let $g : \Omega \to G$ be a measurable function such that $g(s)$ is a simultaneous farthest point of $f_1(s), f_2(s), ..., f_n(s)$ for almost all s. Then g is a simultaneous farthest point of $f_1, f_2, ..., f_n$ in $L^1(\mu, G)$ (and therefore $g \in L^1(\mu, G)$).

Proof. Assume that $g(s)$ is a simultaneous farthest point of $f_1(s), f_2(s), ..., f_m(s)$ for almost all s.Since G is bounded and (Ω, Σ, μ) is finite, then $g \in L^1(\mu, G)$. By Theorem 2.1,

$$
\begin{aligned}
\rho\left(\left\{f_i : 1 \le i \le m\right\},\ L^1(\mu, G)\right) &= \int\limits_{\Omega} \rho\left(\left\{f_i(s) : 1 \le i \le m\right\},\ G\right) d\mu \\
&= \int\limits_{\Omega} \sum_{i=1}^{m} \left\| f_i(s) - g(s) \right\| d\mu \\
&= \sum_{i=1}^{m} \left\| f_i - g \right\|_1 .
\end{aligned}
$$

Therefore g is a simultaneous farthest point of $f_1, f_2, .., f_m$ in $L^1(\mu, G)$. ■

The condition in Corollary 2.2 is sufficient ; $g(s)$ is is a simultaneous farthest point of $f_1(s), f_2(s), ..., f_m(s)$ for almost all s in G, implies g is a simultaneous farthest point of $f_1, f_2, ..., f_m$ in $L^1(\mu, G)$. In fact we have the following theorem :

Theorem 2.3. Let (Ω, Σ, μ) be a finite measure space. Then $L^1(\mu, G)$ is simultaneously remotal in $L^1(\mu, X)$ if and only if for any finite number of elements $f_1, f_2, ..., f_m$ in $L^1(\mu, X)$, there exists $g \in L^1(\mu, G)$ such that $g(s)$ is a simultaneous farthest point of $f_1(s), f_2(s), ..., f_n(s)$ for almost all s.

Proof. Sufficiency of the condition is an immediate consequence of Corollary 2.2. We will show the necessity. Assume that $L^1(\mu, G)$ is simultaneously remotal in $L^1(\mu, X)$ and let $f_1, f_2, ..., f_m$ be any finte number of elements in $L^1(\mu, X)$. Then there exists $g \in L^1(\mu, G)$ such that

$$\begin{aligned}\sum_{i=1}^{m} \|f_i - g\|_1 &= \rho\left(\{f_i : 1 \le i \le m\},\ L^1(\mu, G)\right) \\ &= \int_\Omega \rho\left(\{f_i(s) : 1 \le i \le m\},\ G\right) d\mu,\end{aligned}$$

hence

$$\int_\Omega \left(-\sum_{i=1}^{m} \|f_i(s) - g(s)\| + d\left(\{f_i(s) : 1 \le i \le m\},\ G\right) \right) d\mu = 0.$$

Thus

$$\sum_{i=1}^{m} \|f_i(s) - g(s)\| = d\left(\{f_i(s) : 1 \le i \le m\},\ G\right),$$

for allmost all s.

3 Main Result.

Let (Ω, Σ, μ) be a measure space and X be a Banach space. We say that $f : \Omega \to X$ is measurable in the classical sense if $f^{-1}(O)$ is measurable for every open set $O \subset X$.

The following lemmas will be used to prove our main result.

Lemma 3.1 [3] . Let (Ω, Σ, μ) be a complete measure space and X be a Banach space. If f is a strongly measurable functon from Ω to X, then f is measurable in the classical sense.

Lemma 3.2 [3] . Let (Ω, Σ, μ) be a complete measure space and X be a Banach space. If $f : \Omega \to X$ is measurable in the classical sense and has essetially separable range, then f is srongly measurable.

Let Φ be a set-valued mapping, taking each piont of a measurable space Ω into a subset of a metric space X. We say that Φ is weakly measurable if $\Phi^{-1}(O)$ is measurable in Ω whenever O is open in X. Here we have put, for any $A \subset X$,

$$\Phi^{-1}(A) = \{s \in \Omega : \Phi(s) \cap A \neq \phi\}.$$

The following theorem is due to Kuratowski [15], it is known as Measurable Selection Theorem.

Theorem 3.3 [5] . Let Φ be a weakly measurable set-valued map which carries each piont of a measurable space Ω to a closed nonvoid subset of a complete separable metric space X. Then Φ has a measurable selection; i.e., there exists a function $f : \Omega \to X$ such that $f(s) \in \Phi(s)$ for each $s \epsilon \Omega$ and $f^{-1}(O)$ is measurable in Ω whenever O is open in X.

Theorem 3.4. Let X be a Banach space and G be a closed bounded separable subset of X, and (Ω, Σ, μ) be a finite complete measure space. Then the following are equivalent :

1. G is simultaneously remotal in X.
2. $L^1(\mu, G)$ is simultaneously remotal in $L^1(\mu, X)$.

Proof. (2) $\Rightarrow$ (1) : Let $x_1, x_2, ..., x_m$ be any finite number of elements in X. Define $f_{x_i} : \Omega \to X$, $i = 1, 2, ..., m$, by

$$f_{x_i}(s) = \chi_\Omega(s)\, x_i,$$

for all $s \in \Omega$. Then $f_{x_i} \in L^1(\mu, X)$ for all i. By the assumption, there exists $f_0 \in L^1(\mu, G)$ such that

$$\sum_{i=1}^{m} \|f_{x_i} - f_0\|_1 = \rho\left(\{f_{x_i} : 1 \leq i \leq m\}, L^1(\mu, G)\right).$$

By Theorem 2.3, $f_0(s)$ is a best simultaneous approximation of $f_{x_1}(s), f_{x_2}(s), ..., f_{x_m}(s)$ for almost all s. Then

$$\sum_{i=1}^{m} \|f_{x_i}(s) - f_0(s)\| \geq \sum_{i=1}^{m} \|f_{x_i}(s) - \chi_\Omega(s) g\|,$$

for almost all s and for any $g \in G$, hence $f_0(s_0)$ is a simultaneous farthest point of $x_1, x_2, ..., x_m$ in G for some $s_0 \in \Omega$.

(1) $\Rightarrow$ (2) : Let $f_1, f_2, ..., f_m$ be any finite number of elements in $L^1(\mu, X)$. For each $s \in \Omega$ define

$$\Phi(s) = \left\{ g \in G : \sum_{i=1}^{m} \|f_i(s) - g\| = \rho(\{f_i(s) : 1 \leq i \leq m\}, \ G) \right\}.$$

For each $s \in \Omega$, $\Phi(s)$ is closed, bounded, and nonvoid subset of G. We shall show that Φ is weakly measurable. Let O be an open set in X, the set

$$\Phi^{-1}(O) = \{s \in \Omega : \Phi(s) \cap O \neq \phi\}$$

can be also be described as

$$\Phi^{-1}(O) = \{s \in \Omega : \sup_{g \in G} \sum_{i=1}^{m} \|f_i(s) - g\| = \sup_{g \in O} \sum_{i=1}^{m} \|f_i(s) - g\|\}.$$

Since (Ω, Σ, μ) is complete, f_i is measurable in the classical sence for $i = 1, 2, ..., m$ by Lemma 3.1. Since subtraction in G, sum, and the norm in X are continuous, then the map

$$s \to \sup_{g \in A} \sum_{i=1}^{m} \|f_i(s) - g\|$$

is measurable for any set A. It follows that $\Phi^{-1}(O)$ is measurable. By Theorem 3.3, Φ has a measurable selection; i.e., there exists a function $f : \Omega \to G$ such that $f(s) \in \Phi(s)$ for each $s \epsilon \Omega$ and f is measurable in the classical sense. By Lemma 3.2, f is strongly measurable. Hence f is a simultaneous farthest point for $f_1, f_2, ..., f_m$ in $L^1(\mu, G)$ by Corollary 2.2.

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