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Results Connected to the Riemann Hypothesis

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Abstract

In his famous presentation at the International Congress of Mathematicians held in Paris in 1900, David Hilbert included the Riemann Hypothesis (all the non-trivial zeros of the zeta function have real part $\frac{1}{2}$) as number 8 in his list of 23 challenging problems for mathematicians for the 20th century. Over 100 years later, it is one of the few on that list that have not been solved. Nowadays, many mathematicians consider it the most important unsolved problem in mathematics. We give results connected to the Riemann Hypothesis.

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1 Introduction

Let A be any compact subset of the plane that avoids the integers. Enclose A in an open disk centered at $(0,0)$ of some radius R. For each point z of A, we have $|n-z| \geq n - |z| > n - R$ and for each integer $t > R$ we then have for integers n with $|n| \geq t$:

$$\sum_{-\infty}^{\infty} \frac{1}{|n-z|^2} \leq 2 \sum_{n=t}^{\infty} \frac{1}{(n-R)^2}$$

which is the remainder of an absolutely convergent series and converges to 0 as t tends to infinity and since this does not depend on z, the series $\sum\limits_{-\infty}^{\infty} \frac{1}{(n-z)^2}$ converges uniformly A.
This leads us to the following

Theorem 1.1.

$$\sum_{-\infty}^{\infty} \frac{1}{(n-z)^2} = \frac{\pi^2}{\sin^2 \pi z}$$

Proof. Since the series

$$\sum_{-\infty}^{\infty} \frac{1}{(n-z)^2}$$

converges absolutely on its domain of definition, we can replace n by $n+1$ below to get

$$\sum_{-\infty}^{\infty} \frac{1}{(n-(z+1))^2} = \sum_{-\infty}^{\infty} \frac{1}{((n+1-(z+1))^2} = \sum_{-\infty}^{\infty} \frac{1}{(n-z)^2}.$$

Now let $b > 0$ and let z=x+iy with $|y| \geq b$. Let k be the integral part of x. Then

$$\begin{aligned}
&|\sum_{-\infty}^{\infty} \frac{1}{(n-(x+iy))^2}| = |\sum_{-\infty}^{\infty} \frac{1}{(n-(x-p+iy))^2}| \\
&= \lim_{d\to\infty,\ t\to\infty} |\sum_{t}^{d} \frac{1}{(n-(x-p+iy))^2}| \\
&\leq \lim_{d\to\infty, t\to\infty} \sum_{t}^{d} \frac{1}{|(n-(x-p+iy))|^2} \\
&= \sum_{-\infty}^{\infty} \frac{1}{|n-x+p-iy|^2} \\
&= \sum_{1}^{\infty} \frac{1}{|n-x+p-iy|^2} + \sum_{0}^{\infty} \frac{1}{|-n-x+p-iy|^2} \\
&\leq \sum_{1}^{\infty} \frac{1}{(n-1)^2+b^2} + \sum_{0}^{\infty} \frac{1}{n^2+b^2} = 2\sum_{0}^{\infty} \frac{1}{n^2+b^2}.
\end{aligned}$$

Next the function

$$g(z) = \sum_{-\infty}^{\infty} \frac{1}{(n-z)^2} - \frac{\pi^2}{\sin^2 \pi z}$$

is holomorphic except on the integers and satisfies

$$\sum_{-\infty}^{\infty} \frac{1}{(n-(z+1))^2} - \frac{\pi^2}{\sin^2 \pi(z+1)} = \sum_{-\infty}^{\infty} \frac{1}{(n-z)^2} - \frac{\pi^2}{\sin^2 \pi z}$$

and

$$|\sum_{-\infty}^{\infty} \frac{1}{(n-z)^2} - \frac{\pi^2}{\sin^2 \pi z}| \leq \sum_{0}^{\infty} \frac{1}{b^2+n^2} + \frac{2\pi^2}{(e^{\pi b} - e^{-\pi b})^2}$$

which tends to 0 when b tends to infinity. So the function

$$g(z) = \sum_{-\infty}^{\infty} \frac{1}{(n-z)^2} - \frac{\pi^2}{\sin^2 \pi z}$$

is bounded on $[0,1]+i\mathbb{R}$ and thus on $\mathbb{C}$. Thus the function g is constant; since $g(ib)$ tends to 0 as b tends to infinity, g must be the zero function. Thus

$$\sum_{-\infty}^{\infty} \frac{1}{(n-z)^2} = \frac{\pi^2}{\sin^2 \pi z}$$

Remark 1.2. *To motivate an important corollary, observe that the function* $\frac{z}{e^z-1}$ *has its nearest singularities at* $z=-2\pi i$ *and* $z=2\pi i$ *and so is holomorphic in the disk* $|z|<2\pi$. *Therefore we can represent it there as* $\frac{z}{e^z-1} = \sum\limits_{0}^{\infty} B_n \frac{z^n}{n!}$ *where the coefficients* $B_k's$ *are nothing but the Bernoulli numbers the first of which are* $B_0=1, B_1=-\frac{1}{2}, B_2=\frac{1}{6}, B_3=0, B_4=-\frac{1}{30}, B_5=0, B_6=\frac{1}{42}, B_7=0, B_8=-\frac{1}{30}, B_9=0$ *and* $B_{10}=\frac{5}{66}$.

Corollary 1.3. $\sum\limits_{k=1}^{\infty} \frac{1}{k^{2n}} = \frac{(-1)^{n-1}2^{2n-1}B_{2n}\pi^{2n}}{(2n)!}$.

Proof. The function

$$\frac{\pi^2}{\sin^2 \pi z} - \sum_{-\infty}^{\infty} \frac{1}{(n-z)^2}$$

is the derivative of

$$\frac{1}{z} + \sum_{1}^{\infty} (\frac{1}{z+n} + \frac{1}{z-n}) - \pi \cot \pi z$$

and this derivative is zero. This implies that $\pi \cot \pi z - \frac{1}{z} - \sum_{1}^{\infty} (\frac{1}{z+n} + \frac{1}{z-n})$ is constant and, moreover, it turns out that this constant is 0. Therefore,

$$\pi z \cot \pi z = 1 + \sum_{1}^{\infty} \frac{2z^2}{z^2 - k^2}.$$

Let a_n be the coefficient of z^n in the power series expansion of the function $\pi z \cot \pi z$.

Since

$$a_n = \frac{1}{2\pi i} \int\limits_{|z|=r} \frac{dz}{z^{n+1}} + \frac{1}{2\pi i} \sum_{k=1}^{\infty} \int\limits_{|z|=r} \frac{2z^{1-n} dz}{z^2 - k^2}$$

for every $n = 0, 1, 2, \ldots$, $a_0 = 1$ and $a_n = 0$ for odd n. Now for even n we have

$$a_n = 2 \sum_{k=1}^{\infty} \frac{1}{k^n}.$$

Comparing with the power series expansion

$$\pi z \cot \pi z = 1 + \sum_{1}^{\infty} \frac{(-4)^n B_{2n} \pi^{2n} z^{2n}}{(2n)!},$$

we get

$$\sum_{k=1}^{\infty} \frac{1}{k^{2n}} = \frac{(-1)^{n-1} B_{2n} 2^{2n-1} \pi^{2n}}{(2n)!}.$$

Remark 1.4. *The above formula for* $\sum_{k=1}^{\infty} \frac{1}{k^{2n}}$ *was due to Euler. However, despite trying hard to obtain a closed formula for* $\sum_{k=1}^{\infty} \frac{1}{k^3}$*, he was not successful as was reflected in a relevant translation of his Latin* 1785 *Monograph "Opiscula analytica" [1] "At this point I will examine in rather more detail a unique case, which does not seem alien to follow such a relation, namely the sum of the series of the reciprocals of cubes* $1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \cdots$ *etc., which so far in no way I could reduce to the circle or to logarithms." Euler tried, unsuccessfully, to look for a linear relation with integer coefficients among* $\pi^3, \pi^2 \log 2, (\log 2)^3$*, due to the following connections*

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} - \cdots$$

and

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

More precisely, the series for $\pi^2 \log 2$ *or that for* $(\log 2)^3$ *contains the series* $\sum_{k=1}^{\infty} (-1)^k \frac{1}{k^3}$. *For over* 225 *years now, all the cases* $\sum_{k=1}^{\infty} \frac{1}{k^{2n+1}}, n = 1, 2, ...$ *remain open.*

2 Riemann zeta function with some properties

The **Riemann zeta function** ζ is defined by $\zeta(z) = \sum_{k=1}^{\infty} \frac{1}{k^z}$. Set $z = x + iy$. Now $|\frac{1}{k^z}| = \frac{1}{|e^{z \log k}|} = \frac{1}{|e^{x \log k}|} = \frac{1}{k^x}$. So the series $\sum_{k=1}^{\infty} \frac{1}{k^z}$ converges absolutely in the half-plane $x > 1$. By Weierstrass test, this series converges uniformly on every compact subset of this half-plane. With some work, this function can be continued analytically to all complex $z \neq 1$ (see for instance [6].) (For $z = 1$, the outcome turns out to be the harmonic series which diverges to infinity. As a result, the zeta function is a meromorphic function of the complex variable z, which is holomorphic everywhere except for a simple pole at $z = 1$ with residue 1).

Theorem 2.1. *For* $Re(z) > 1$ *we have* $\zeta^{'}(z) = -\sum_{n=1}^{\infty} \frac{\log n}{n^z}$;*that is, for* $Re(z) > 1, \zeta^{'}(z)$ *can be obtained by differentiating the series of* $\zeta(z)$ *term-by-term and taking its negative.*

Proof. Let K be a compact subset of $\{z : Re(z) > 1\}$. Then K is a subset of some half-plane $H = \{z : Re(z) \geq c > 1\}$. Consider the sequence of functions $\{f_k(z)\}_1^{\infty}$ defined by $f_k(z) = \sum_{n=1}^{k} \frac{1}{n^z}$. It is easy to see that this sequence $\{f_k(z)\}_1^{\infty}$ converges uniformly to $\zeta(z)$ on H. Then the sequence $\{f_k^{'}(z)\}_1^{\infty} = \{-\sum_{n=1}^{k} \frac{\log n}{n^z}\}$ converges to $-\sum_{n=1}^{\infty} \frac{\log n}{n^z}$. Consequently, for $Re(z) > 1, \zeta^{'}(z) = -\sum_{n=1}^{\infty} \frac{\log n}{n^z}$;

Definition 2.2. *The von Mangoldt function* Λ *is defined by*

$$\Lambda(n) = \begin{cases} \log p & \textit{for} \quad n = p, p^2, \cdots ; \\ 0 & \textit{otherwise}, \end{cases}$$

It turns out that, for $Re(z) > 1$, the logarithmic derivative of the zeta function $\log' \zeta(z)$ is

$$\frac{\zeta'(z)}{\zeta(z)} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^z}$$

which plays a crucial role in the proof of the Prime Number Theorem. We'll elaborate more on this function in a later section.
A particularly striking formula was proven by von Mangoldt:
For $x > 1$,

$$\sum_{1<n<x} \Lambda(n) + \frac{1}{2}\Lambda(x) = x - \sum_{\text{primes } p, \zeta(p)=0, Re(p)>0} \frac{x^p}{p} - \log(2\pi) - \frac{1}{2}\log(1-\frac{1}{x^2}).$$

Definition 2.3. *The Möbius function μ is defined by*

$$\mu(n) = \begin{cases} 0 & \textit{if} \quad n \textit{ is divisible by a square of a prime}; \\ 1 & \textit{if} \quad n \textit{ is a product of an even number of distinct prime numbers}; \\ -1 & \textit{if} \quad n \textit{ is a product of an odd number of distinct prime numbers} \end{cases}$$

It turns out that, for $Re(z) > 1$, the reciprocal of the zeta function $\frac{1}{\zeta(z)}$ is

$$\frac{1}{\zeta(z)} = -\sum_{n=1}^{\infty} \frac{\mu(n)}{n^z}$$

which converges and represents a holomorphic function on $\{z : Re(z) > 1\}$. We'll elaborate more on this function in a later section.

Definition 2.4. *The Gamma function Γ is defined as*

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{1}^{\infty} (1+\frac{z}{n})^{-1} e^{\frac{z}{n}},$$

where γ is Euler constant $\gamma = \lim_{n\to\infty}(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n)$.

We now mention the recurrence formula for the Gamma Function

$$\Gamma(z+1) = z\Gamma(z)$$

(for a proof see [3], p. 50) and the duplication formula

$$\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2})$$

(for a proof see [3], p. 51). Our further study requires the following definitions

Definition 2.5. *Let $f(z)$ be an entire function. The maximum modulus function, denoted by $M(r)$, is defined by $M(r) = max\{|f(z)| : |z| = r\}$.*

Definition 2.6. *Let $f(z)$ be a non-constant entire function. The order ρ of $f(z)$ is defined by*

$$\rho = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r}.$$

The order of any constant function is 0, by convention.

Definition 2.7. *An entire function $f(z)$ of positive order ρ is said to be of type τ if*

$$\tau = \limsup_{r\to\infty} \frac{\log M(r)}{r^{\rho}}.$$

Definition 2.8. *If $0 \leq \tau < \infty$, then $f(z)$ is said to be of finite type.*
If $\tau = 0$, then $f(z)$ is said to be of minimal type.
If $0 < \tau < \infty$, then $f(z)$ is said to be of normal type.
If $\tau = \infty$, then $f(z)$ is said to be of infinite type.

Definition 2.9. *Let $z_1,\ z_2, \cdots$ be a sequence of non-zero complex numbers. The greatest lower bound of positive numbers α for which $\sum_0^\infty \frac{1}{|z_n|^\alpha}$ is convergent is called the exponent of convergence of the sequence $\{z_n\}$ and is denoted by ρ_1. The smallest positive integer α for which the series is convergent is denoted by $p+1$ and p is called the genus of $\{z_n\}$.*

We can now state and prove the following

Theorem 2.10.

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}.$$

Proof. The function $\sin(\pi z)$ is an entire function which vanishes at $0, \pm 1, \pm 2, ...$ and since $\sum_{-\infty}^{\infty} \frac{1}{v^2} = 2\sum_1^\infty \frac{1}{v^2}$ $(v \neq 0)$ converges while $\sum_{-\infty}^{\infty} \frac{1}{v}$ diverges (since the partial sum $S_{(-k,2k)}$ diverges) , the genus of the sequence of the zeros is 1. Hence

$$\sin(\pi z) = z e^{\varphi(z)} \prod_{-\infty}^{\infty} (1 - \frac{z}{v}) e^{\frac{z}{v}}.$$

Since the product is unconditionally convergent, then by rearranging the factors we get

$$\sin(\pi z) = z e^{\varphi(z)} \prod_{1}^{\infty} (1 - \frac{z^2}{v^2}).$$

To determine $\varphi(z)$ take the logarithmic derivative. Then

$$\pi\cot(\pi z)=\frac{1}{z}+\varphi^{'}(z)+\sum_{v=1}^{\infty}\frac{2z}{z^2-v^2}.$$

We have seen that

$$\pi\cot(\pi z)=\frac{1}{z}+\sum_{v=1}^{\infty}\frac{2z}{z^2-v^2}.$$

It follows that $\varphi^{'}(z)=0$. Thus $\varphi(z)=K$, a constant. Now

$$\sin(\pi z)=ze^K\prod_{1}^{\infty}(1-\frac{z^2}{v^2}).$$

Since $\lim_{z\longrightarrow 0}\frac{\sin(\pi z)}{\pi z}=1, e^K=\pi$. Consequently,

$$\sin(\pi z)=\pi z\prod_{1}^{\infty}(1-\frac{z^2}{v^2}).$$

The proof of the following theorem is quite involved and can be found for instance in [6], pp. $23-28$

Theorem 2.11. *Riemann Functional Equation.*

$$\zeta(z)\Gamma(\frac{1}{2}z)\pi^{-\frac{1}{2}z}=\pi^{-\frac{1}{2}(1-z)}\Gamma(\frac{1}{2}(1-z))\zeta(1-z).$$

We now state and prove an equivalent form of Riemann Functional Equation which is needed to prove a few properties about the zeta function:

Theorem 2.12. *Alternative form of Riemann Functional Equation.*

$$\zeta(1-z)=2(2\pi)^{-z}\cos\frac{\pi z}{2}\Gamma(z)\zeta(z).$$

Proof. By theorem 2.10 we have $\Gamma(\frac{z}{2})\Gamma(1-\frac{z}{2})=\frac{\pi}{\sin\frac{\pi z}{2}}$. Now let $s=1-z$ in the latter formula to get $\Gamma(\frac{1}{2}-\frac{s}{2})\Gamma(\frac{1}{2}+\frac{s}{2})=\frac{\pi}{\cos\frac{\pi s}{2}}$. Let $z=\frac{s}{2}$ in the duplication formula to get $\Gamma(\frac{1}{2}+\frac{s}{2})=\frac{2^{1-s}\sqrt{\pi}\Gamma(s)}{\Gamma(\frac{s}{2})}$. Now

$$2(2\pi)^{-z}\cos\frac{\pi z}{2}\Gamma(z)\zeta(z)=2^{1-z}\pi^{1-z}\frac{\Gamma(z)\zeta(z)}{\Gamma(\frac{1}{2}-\frac{z}{2})\Gamma(\frac{1}{2}+\frac{z}{2})}=\zeta(1-z).$$

We now use $\zeta(2n)$ with the alternative form of Riemann functional equation as follows

$$\zeta(1-2n)=2(2\pi)^{-2n}\cos n\pi\Gamma(2n)\zeta(2n)=-\frac{B_{2n}}{2n}.$$

Let $k = 2n$. Then $\zeta(1-k) = -\frac{B_k}{k}$. If we now let $m = k-1$, then we get $\zeta(-m) = -\frac{B_{m+1}}{m+1}$. In particular, for $m = 0$, we get $\zeta(0) = \frac{1}{2}$. Now let us prove, in general, that $B_{2k+1} = 0$ for $k = 1, 2, 3, ...$ First, $\frac{t}{e^t-1} + \frac{t}{2} = \frac{t}{2}\coth\frac{t}{2}$ is an even function. Next, $\frac{t}{2}\coth\frac{t}{2} = \sum_0^\infty B_n\frac{t^n}{n!} + \frac{t}{2} = \sum_{n=0,n\neq 1}^\infty B_n\frac{t^n}{n!}$. Now replacing t with $-t$ we get $\frac{t}{2}\coth\frac{t}{2} = \sum_{n=0,n\neq 1}^\infty(-1)^n B_n\frac{t^n}{n!}$. Therefore, $(-1)^n B_n = B_n$ for $n = 0, 2, 3, 4, ...$ which for odd $n = 2k+1$ implies the result. In particular, $\zeta(-2n) = 0$ for natural numbers n, (these are called the trivial zeros of the zeta function).

3 Riemann Hypothesis

In 1859 Bernhard Riemann [5] conjectured that the ALL non-trivial zeros of his zeta function have real part equal to 1/2 (the trivial zeros being at the negative even integers and recently computer calculations have shown that the first 10 trillion non-trivial zeros lie on the critical line $1/2 + it$, where t is a real number). This has been known as the Riemann Hypothesis. In the year 2000, Fields Medalist Enrico Bombieri stated that "In the opinion of many mathematicians the Riemann Hypothesis is probably the most important open problem in pure mathematics today." Not only that but it has a million-dollar tag attached to whether or not it is true. Indeed, it is one of seven problems each worth a million dollars (one of them, the Poincare Conjecture, was solved by Grigori Perelman in 2002. Eight years later, on March 18, 2010, the Clay Mathematics Institute-named after the Boston businessman Landon Clay-awarded Perelman the one million dollar Prize in recognition of his proof but he rejected it quoting that Richard Hamilton who had a major contribution for the proof, is equally deserving. Earlier, in August 2006, Perelman was awarded for his proof the Fields Medal, the highest award in mathematics, but he declined it as well.)

In what follows, we mention an essential result for an analytic proof of the Prime Number Theorem. On the basis of counting primes, one may be led to suspect that $\pi(x)$ increases somehow like $\frac{x}{\log x}$. As a matter of fact, in 1791 at the age of 14, Gauss conjectured that $\lim_{x\to\infty}\frac{\pi(x)\log x}{x} = 1$. In 1850, trying to settle the Gauss conjecture, Tchebycheff showed that there exist positive constants c and C such that

$$c\frac{x}{\log x} < \pi(x) < C\frac{x}{\log x}$$

for $x \geq 2$ with $c = .92$ and $C = 1.11$
It was not until 1896 that the Gauss conjecture was settled by Hadamard and, simultaneously by, de la Vallée Poussin and from then on it has been known as the Prime Number Theorem. Both Hadamard and de la Vallée Poussin employed complex-variable methods. They proved that $\zeta(1+it) \neq 0$ from which they deduced the Prime Number Theorem.

Remark. Hadamard and de la Vallée Poussin also showed the converse to be true and for a while it appeared that the Prime Number Theorem was impossible to prove without using $\zeta(1+it) \neq 0$. However, in 1949, Erdös and Selberg proved the Prime Number Theorem by "elementary" methods meaning without using functions of a complex variable. Below we state Hadamard and de la Vallée Poussin key result:

Theorem 3.1. $\zeta(1+it) \neq 0$. *That is, no zeros of the zeta function could lie on the line* $1+it$.

It is worth mentioning here that the Riemann Hypothesis is indirectly related to prime numbers (which of course are of great interest to number theorists) via Analytic Number Theory. In his famous 1859 paper, Riemann unveiled a close relationship between prime numbers and zeros of his zeta function; more specifically, through "Euler" Product $\zeta(z) = \Pi_{p \text{ prime}}(\frac{1}{1-\frac{1}{p^z}})$ (earlier, Euler had considered this function only as a real function). Since none of the factors have zeros, there are no zeros of the zeta function with real part greater than 1. Combined with our earlier results, this shows that all non-trivial zeros must lie inside the critical strip $0 \leq Re(z) \leq 1$.
At one point, G. H. Hardy made headlines within the mathematical community when he claimed to have proved the Riemann Hypothesis. In fact, he was able to prove that there were infinitely many zeros on the critical line, but was unable to prove that there did not exist other zeros that were NOT on the line (or even infinitely many off the line). Hardy and Ramanujan collaboration included the Riemann Hypothesis which still defied this joint attempt.
We conclude this section with the following interesting result by Speiser [8] which also ties this section with the previous one:

Theorem 3.2. *Riemann Hypothesis is true if and only if* ζ' *has no zeros in the strip* $\{z : 0 < Re(z) < \frac{1}{2}.\}$ *Thus the zeta function* ζ *has only simple zeros* *on the critical line if and only if its derivative ζ' has no zeros on the critical line.*

4 The completed zeta function

Definition 4.1. *The completed zeta function is defined by*

$$\xi(z) = \frac{1}{2}z(z-1)\pi^{-\frac{z}{2}}\zeta(z)\Gamma(\frac{z}{2})$$

Theorem 4.2. $\xi(z) = \xi(1-z)$. *Thus the function $\xi(z)$ is symmetric about the critical line $Re(z) = \frac{1}{2}$;*

Proof. Rewrite Riemann's Functional Equation as

$$\pi^{-\frac{1}{2}z}\Gamma(\frac{1}{2}z)\zeta(z) = \pi^{-\frac{1}{2}(1-z)}\Gamma(\frac{1}{2}(1-z))\zeta(1-z)$$

from which the relation $\xi(z) = \xi(1-z)$ follows easily.

Theorem 4.3. *The function $\xi(z)$ is entire.*

Proof. Using the analytic continuation of $\Gamma(z)$ to the whole complex plane and the fact that $\zeta(z)$ is holomorphic in the whole complex plane except for a simple pole at $z = 1$ which is removable because of the factor $z-1$, it follows that $\xi(z)$ is entire.

Theorem 4.4. $\xi(z)$ *is of order* 1.

Proof. With $[t]$ denoting the greatest integer function, we have

$$z\int_1^{N+1}[t]t^{-1-z}dt = z\sum_{n=1}^{N}\int_n^{n+1}[t]t^{-1-z}dt = z\sum_{n=1}^{N}n\int_n^{n+1}t^{-1-z}dt$$

$$= z\sum_{n=1}^{N}n(n^{-z}-(n+1)^{-z}) = 1-2^{-z}+2(2^{-z}-3^{-z})+...N[N^{-z}-(N+1)^{-z}]$$

$$= \sum_{n=1}^{N}n^{-z} - N(N+1)^{-z}.$$

When $x > 1, \lim_{N\to\infty}|N(N+1)^{-z}| = \lim_{N\to\infty}N(N+1)^{-x} = 0$. Hence for $x > 1$ we have

$$\zeta(z) = z\int_1^{\infty}[t]t^{-1-z}dt.$$

Therefore

$$\zeta(z) = \frac{z}{z-1} + z\int_1^\infty \frac{[t]-t}{t^{z+1}}dt.$$

The integral on the right is uniformly convergent when $0 < x_1 \le x \le x_2$ and since $|[t]-t| < 1$, it represents a holomorphic function on $x > 0$. Hence the above integral representation is valid on $x > 0$. When $x \ge \frac{1}{2}$ and $|z| > 2$, it follows from this integral representation that

$$\zeta(z) \le \frac{|z|}{z-1} + |z|\int_1^\infty \frac{[t]-t}{t^{x+1}} < |z| + |z|\int_1^\infty \frac{dt}{t^{\frac{3}{2}}} = O(|z|).$$

Now

$$|\Gamma(z)| \le \int_0^\infty e^{-t}t^{x-1}dt < \int_0^1 e^{-t}t^{-\frac{1}{2}}dt + \int_1^\infty e^{-t}t^{[x]}dt = O(1) + \int_0^\infty e^{-t}t^{[x]}dt$$
$$= O(1) + [x]! \le O(1) + [x]^{[x]} \le O(1) + [|z|]^{[|z|]} \le O(1) + |z|^{|z|}$$
$$= O(1) + e^{|z|\log|z|} = O(e^{|z|^{1+\varepsilon}})$$

where $\varepsilon > 0$.
Since $\pi^{-\frac{1}{2}z}$ is an entire function of order 1, $\xi(z) = O((exp(|z|^{1+\varepsilon}))$ whenever $x \ge \frac{1}{2}$. Since $\xi(z) = \xi(1-z)$, $\xi(z) = O(exp(|z|^{1+\varepsilon}))$ holds throughout the complex plane. Thus $\xi(z)$ is of order at most 1.
Next, if z is a real number > 2, then

$$\xi(z) = 2(z-1)\pi^{-\frac{1}{2}z}\zeta(z)\Gamma(\frac{1}{2}z+1) > 2\Gamma(\frac{1}{2}z+1) = 2\int_0^\infty e^{-t}t^{\frac{1}{2}z}dt$$

$$> 2\int_z^\infty e^{-t}t^{\frac{1}{2}z}dt > 2z^{\frac{1}{2}z}\int_z^\infty e^{-t}dt = 2z^{\frac{1}{2}z}e^{-z} = e^{\frac{1}{2}z\log z - z + \log 2.}$$

If $\varepsilon > 0$ is arbitrary, then for large z,

$$\frac{1}{2}\log z - 1 + \frac{1}{z}\log 2 > z^{-\varepsilon}$$

and so $\xi(z) > exp(z^{1-\varepsilon})$ for large real values z.
Consequently, the order of $\xi(z)$ is 1.

Theorem 4.5. *The function $\xi(z)$ is of infinite type.*

Since $\log M(r) \sim \frac{1}{2}r\log r$ as $r \longrightarrow \infty$, $\frac{\log M(r)}{r} \sim \frac{1}{2}\log r$ as $r \longrightarrow \infty$. Since the order of $\xi(z)$ is 1, the type τ is

$$\limsup_{r\longrightarrow\infty} \frac{\log M(r)}{r} = \infty.$$

Theorem 4.6. *The function $\xi(z)$ has infinitely many zeros.*

Proof. This follows easily now by this author's previous result that an entire function of order 1 and infinite type must have infinitely many zeros [2].

Euler Product Formula shows that every zero z_0 of ξ has $Re(z_0) \leq 1$ while the Functional Equation $\xi(1-z) = \xi(z)$ shows that all zeros are in the critical strip. With some additional work one can show that all zeros of ξ lie inside the critical strip.

Theorem 4.7. *If the Riemann Hypothesis is false, then the zeros of the ξ function in the critical strip that are not on the critical line would occur in quadruples as vertices of rectangles.*

Proof. For real values of $z, \xi(z)$ is also real. Thus $\xi(\overline{z}) = \overline{\xi(z)}$. Therefore if s is a zero of ξ, then $\overline{s}, 1-s, 1-\overline{s}$ are also zeros of ξ. As a result, zeros on the critical line occur in conjugate pairs and zeros off the critical line occur in quadruples of the above form.

The Riemann Hypothesis can therefore be stated as:
All zeros of $\xi(z)$ are on the critical line $Re(z) = \frac{1}{2}$.

Remark 4.8. *As of the time of this writing, no double zero of the function ξ has been found on the critical line.*

5 Connections with the Riemann Hypothesis

To reach this connection we use the following Lemma whose proof employs complex analysis:

Lemma 5.1. *(a) z_0 is a pole of order m of a function f if and only if there is a unique $m-$tuple $(b_1, b_2, \cdots, b_m)$ of complex numbers such that $b_m \neq 0$ and the function $h(z) = f(z) - \sum_{j=1}^{m} \frac{b_j}{(z-z_0)^j}$ has a removable singularity at z_0.*
(b) If z_0 is a pole of order m of a function f, then z_0 is a simple pole of $\frac{f'}{f}$ with $Res(\frac{f'}{f}, z_0) = -m$.

Theorem 5.2. *Let S denote the set of zeros of the zeta function in the critical strip whose real part is greater than $\frac{1}{2}$. Then $\frac{\zeta'(z)}{\zeta(z)} + \frac{1}{z-1}$ is holomorphic on S if and only if the Riemann hypothesis is true.*

Proof. Recall that the zeta function ζ is holomorphic everywhere in the complex plane except for a simple pole at $z = 1$. By Lemma 5.1(b) $z = 1$ is a simple pole of $\frac{\zeta'(z)}{\zeta(z)} + \frac{1}{z-1}$ with $Res(\frac{\zeta'}{\zeta}, 1) = -1$. Moreover, by Lemma 5.1(a), there is a unique nonzero complex number b such that the function $h(z) = \frac{\zeta'(z)}{\zeta(z)} - \frac{b}{z-1}$ has a removable singularity at 1. Furthermore, by the definition of the residue of a function at z_0 we must have $b = -1$. As a result the function

$$H(z) = \begin{cases} \frac{\zeta'(z)}{\zeta(z)} + \frac{1}{z-1} & \text{if} \quad z \neq 1; \\ L & \text{if} \quad z = 1, \end{cases}$$

where $L = \lim_{z\to 1} h(z)$, is an entire function.
($\Rightarrow$) Suppose that $\frac{\zeta'(z)}{\zeta(z)} + \frac{1}{z-1}$ is holomorphic on S. In addition, suppose that z_0 is a non-trivial zero of ζ; that is, $\zeta(z_0) = 0$ with $z_0 \neq 0$ to which we can also add that $0 < Re(z_0) < 1$. It is enough to obtain a contradiction if we assume either the case $0 < Re(z_0) < \frac{1}{2}$ or $Re(z_0) > \frac{1}{2}$. First, assume that $0 < Re(z_0) < \frac{1}{2}$. Then, since ζ is holomorphic at $z_0, \lim_{t\to 0} \frac{\zeta(z_0+t)-\zeta(z_0)}{t}$ exists. Thus $\zeta'(z_0)$ exists. In addition, $\zeta'(z_0) \neq 0$ for otherwise $\frac{\zeta'(z)}{\zeta(z)} + \frac{1}{z-1}$ won't exist and so won't be holomorphic on S. Thus z_0 would be a pole of $\frac{\zeta'(z)}{\zeta(z)} + \frac{1}{z-1}$ other than 1 which is a contradiction. Next, assume that $Re(z_0) > \frac{1}{2}$. Then $\frac{\zeta'(z)}{\zeta(z)} + \frac{1}{z-1}$ is holomorphic at z_0. Then the derivative of $\frac{\zeta'(z)}{\zeta(z)} + \frac{1}{z-1}$ exists at z_0; that is $\frac{\zeta''(z_0)\zeta(z_0)-[\zeta'(z_0)]^2}{[\zeta(z_0)]^2} - \frac{1}{(z_0-1)^2}$ exists which is the desired contradiction.
($\Leftarrow$) If the Riemann Hypothesis is true, then ζ has no zeros on $\{z : Re(z) > 1/2\}$ and therefore $\frac{\zeta'(z)}{\zeta(z)} + \frac{1}{z-1}$ is holomorphic there. In particular, $\frac{\zeta'(z)}{\zeta(z)} + \frac{1}{z-1}$ is holomorphic on S.

Now given our shift from $\zeta(z)$ to $\xi(z)$ we conjecture and prove the following

Theorem 5.3. *Let T denote the set of zeros of the completed zeta function in the critical strip whose real part is greater than $\frac{1}{2}$. Then $\frac{\xi'(z)}{\xi(z)}$ is holomorphic on T if and only if the Riemann hypothesis is true.*

Proof.($\Rightarrow$) Suppose that $\frac{\zeta'(z)}{\zeta(z)}$ is holomorphic on T. In addition, suppose that z_0 is a zero of ξ; that is, $\xi(z_0) = 0$ to which we can also add that $0 < Re(z_0) < 1$. It is enough to obtain a contradiction if we assume either the case $0 < Re(z_0) < \frac{1}{2}$ or $Re(z_0) > \frac{1}{2}$. First, assume that $0 < Re(z_0) < \frac{1}{2}$. Then, since ξ is holomorphic at $z_0, \lim_{t\to 0} \frac{\xi(z_0+t)-\xi(z_0)}{t}$ exists. Thus $\xi'(z_0)$ exists. In addition, $\xi'(z_0) \neq 0$ for otherwise $\frac{\xi'(z)}{\xi(z)}$ won't exist and so won't be holomorphic on T. Thus z_0 would be a pole of $\frac{\xi'(z)}{\xi(z)}$ which is a contradiction. Next, assume that $Re(z_0) > \frac{1}{2}$. Then $\frac{\xi'(z)}{\xi(z)}$ is holomorphic at z_0. Then the derivative of $\frac{\xi'(z)}{\xi(z)}$ exists at z_0; that is $\frac{\xi''(z_0)\xi(z_0)-[\xi'(z_0)]^2}{[\xi(z_0)]^2}$ exists which is the desired contradiction. $(\Leftarrow)$ If the Riemann Hypothesis is true, then ξ has no zeros on $\{z : Re(z) > 1/2\}$ and therefore $\frac{\xi'(z)}{\xi(z)}$ is holomorphic there. In particular, $\frac{\xi'(z)}{\xi(z)}$ is holomorphic on T.

Remark 5.4. *It is known (see for example [7]) that* $Re(\frac{\xi'(z)}{\xi(z)}) > 0$ *on* $\{z : Re(z) > 1/2\}$ *if and only if the Riemann hypothesis is true.*

Remark 5.5. *Recall that, for* $Re(z) > 1$, *the reciprocal of the zeta function* $\frac{1}{\zeta(z)}$ *is*

$$\frac{1}{\zeta(z)} = -\sum_{n=1}^{\infty} \frac{\mu(n)}{n^z}$$

which converges and represents a holomorphic function on $\{z : Re(z) > 1\}$. *If, in addition,* $\zeta(z)$ *has zeros for* $\frac{1}{2} < Re(z) < 1$, *there would not exist a holomorphic function in this region. On the other hand, if the Riemann Hypothesis is true (no such zeros in the region), then the series can actually be shown to converge to a holomorphic function on* $\{z : Re(z) > \frac{1}{2}\}$.

Remark 5.6. *With* $M(x)$ *being the Mertens function defined by* $M(x) = \sum_{0<n\leq x} \mu(n)$, *it is known that for any* $\epsilon > 0, M(x) = O(x^{\frac{1}{2}+\epsilon})$ *if and only if the Riemann hypothesis is true.*

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