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Cauchy Problem for One Class of Ordinary Differential Equations

A. Tungatarov

Department of Mechanics and Mathematics
Al-Farabi Kazakh National University
av. al-Farabi 71, 050012 Almaty, Kazakhstan
tun-mat@list.ru

D. K. Akhmed-Zaki

Department of Mechanics and Mathematics
Al-Farabi Kazakh National University
av. al-Farabi 71, 050012 Almaty, Kazakhstan
mdina84@mail.ru

Abstract

In this article the general solution of one class of second order ordinary differential equations is fined. The Cauchy problem for this class are solved.

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1 Introduction

Let us choose $0 < R < \infty$. By $S[0,R]$ we denote the class of measurable functions on $[0,R]$ which are essential bounded there. The norm of an element from $S[0,R]$ is defined by the formula $\|f\|_{S[0,R]} = \text{supvrai}_{x\in[0,R]}|f(x)| = \lim\limits_{p\to\infty} \|f\|_{L_p[0,R]}$.

In this note we consider the equation

$$\frac{\partial^2 u}{\partial x^2} + a(x)u = f(x),\ x \in [0,R], \tag{1}$$

with coefficient $a = a(x)$ and right-hand side $f = f(x)$ from $S[0, R]$. The equation (1) is called Mathieu's equation for $a(x) = d\cos(2x) + c,\ f(x) \equiv 0$, or Hill's equation for $a(x)$ is a periodic function, $f(x) \equiv 0$ or Lame's equation for $a(x) = -Ap(x) + B,\ f(x) \equiv 0$. There d, c, A, B are given real numbers, $p = p(x)$ is a periodic function. Many problems from physics and techniques are related with the above equation (see [1], [2]). Asymptotically properties of solutions of equation (1) with continuous coefficient and right-hand side are discovered in [2]. In the present note the general solution for equation(1) and for the Cauchy problem for this equation are constructed from the class

$$W^2_\infty[0, R] \cap C^1[0, R]. \tag{2}$$

Here $W^2_\infty[0,\ R]$ is a functions class $f(x)$, such that

$$\frac{\partial^2 f}{\partial x^2} \in S[0,\ R].$$

If $a(x)$, $f(x) \in C[0,\ R]$, then general solution was being found by us in this article belong the class $C^2[0,\ R]$. Note that in Kamke's monograph [1] one can not find any representation of solutions to equation (1).

2 Construction of general solution to equation (1)

Integrating twice the equation (1) we get

$$u(x) = -(Bu)(x) + g(x) + c_1 x + c_2, \tag{3}$$

where c_1, c_2 are arbitrary real numbers,

$$(Bu)(x) = -\int_0^x \int_0^y a(t)u(t)dtdy,\ \ g(x) = -\int_0^x \int_0^y f(t)dtdy.$$

Applying the operator B to equation (3) we get

$$(Bu)(x) = (B^2u)(x) + (Bg)(x) + c_1 a_1(x) + c_2 b_1(x), \tag{4}$$

where

$$(B^2u)(x) = (B(Bu)(x))(x),\ \ a_1(x) = -\int_0^x \int_0^y ta(t)dtdy,\ \ b_1(x) = -\int_0^x \int_0^y a(t)dtdy.$$

From (3) and (4) it follows

$$u(x) = (B^2u) + c_1(x + a_1(x)) + c_2(1 + b_1(x)) + g(x) + (Bg)(x). \tag{5}$$

In the following we use the formulas

$$(B^n u)(x) = (B(B^{n-1}u)(x))(x),\ (n = 2, 3, ...),\ (B^1 u)(x) = (Bu)(x),$$

$$a_k(x) = -\int_0^x \int_0^y a(t)a_{k-1}(t)dtdy,\ \ b_k(x) = -\int_0^x \int_0^y a(t)b_{k-1}(t)dtdy, (k = 2, 3, ...).$$

Applying the operator B to both sides of equation (5) and using the previous formulas implies

$$(Bu)(x) = (B^3 u) + c_1(a_1(x) + a_2(x)) + c_2(b_1(x) + b_2(x)) + (Bg)(x) + (B^2 g)(x). \tag{6}$$

From (3) and (6) it follows

$$u(x) = (B^3 u) + c_1(x + a_1(x) + a_2(x)) + c_2(1 + b_1(x) + b_2(x)) + g(x) + (Bg)(x) + (B^2 g)(x).$$

Continuing this procedure n times we obtain the following integral equation for the solutions of equation (1):

$$u(x) = (B^n u) + c_1(x + \sum_{k=1}^{n-1} a_k(x)) + c_2(1 + \sum_{k=1}^{n-1} b_k(x)) + \sum_{k=0}^{n-1}(B^k g)(x), \tag{7}$$

where $(B^0 g)(x) = g(x)$.

Taking into consideration the definition of the iterated operators $(B^n u)(x)$ and the iterated functions $a_k(x),\ b_k(x)$ the following estimates are obtained without any difficulties:

$$|(B^n u)(x)| \le |u|_0 \cdot \frac{(|a|_0 \cdot x^2)^n}{n!} \cdot \frac{1}{(n+1)...2n}, \tag{8}$$

$$|a_k(x)| \le |a|_0^k \cdot \frac{x^{2k+1}}{(2k+1)!},\ \ |b_k(x)| \le |a|_0^k \cdot \frac{x^{2k}}{(2k)!},$$

where $|f|_0 = \text{supvrai}_{x \in [0,\infty)} |f(x)|,\ |f|_1 = \max\limits_{x \in [0,R]} |f(x)|$.

Passing to the limit $n \to \infty$ in the representation(7), by virtue of (8), we conclude

$$u(x) = c_1 I(x) + c_2 J(x) + F(x), \tag{9}$$

where

$$I(x) = 1 + \sum_{k=1}^{\infty} b_k(x),\ \ J(x) = x + \sum_{k=1}^{\infty} a_k(x),\ \ F(x) = \sum_{k=0}^{\infty}(B^k g)(x).$$

Using the estimates (8) we get

$$|I(x)| \le \cosh(\sqrt{|a|_0} \cdot x),\ \ |J(x)| \le x + \frac{1}{\sqrt{|a|_0}} \sinh(\sqrt{|a|_0} \cdot x), \tag{10}$$

$$|F(x)| \le |g|_1 \cosh(\sqrt{|a|_0} \cdot x).$$

The following relations for the functions $I(x),\ J(x)$, and $F(x)$ are of importance:

$$I'(x) = 1 - \int_0^x a(t)I(t)dt,\ \ J'(x) = -\int_0^x a(t)J(t)dt, \tag{11}$$

$$F'(x) = \int_0^x f(t)dt - \int_0^x a(t)F(t)dt,$$

$$I''(x) = -a(x)I(x),\ \ J''(x) = -a(x)J(x), \tag{12}$$

$$F''(x) = f(x) - a(x)F(x), \tag{13}$$

$$I(0) = F(0) = F'(0) = J'(0) = 0,\ \ J(0) = I'(0) = 1. \tag{14}$$

The formulas (12) and (13) tell us, that the functions $I(x),\ J(x)$ are particular solutions from the class (2) of the homogeneous equation

$$\frac{\partial^2 u}{\partial x^2} + a(x)u = 0,$$

and $F(x)$ is a particular solution of the inhomogeneous equation (1). From (11) and the above relations for the functions $I(x)$ and $J(x)$ we see that the Wronskian $W(x)$ is equal -1 in $x = 0$. Therefore the functions $I(x)$ and $J(x)$ are linear independent on $[0, R]$ and the general solution to equation (1) is determined by the formula (9).

Summarizing we proved the following theorem.

Theorem 1. *The general solution of equation (1) from the class (2) is given by the formula (9).*

3 Cauchy problem

Let us devote to the Cauchy problem.

Cauchy problem. *Find a solution of equation (1) from the class (2) satisfying the Cauchy conditions*

$$u(0) = \alpha,\ \ u'(0) = \beta, \tag{15}$$

where α, β are given real numbers.

Solution of the Cauchy problem.

To solve the Cauchy problem we use the general solution of the equation (1) which is given by formula (9). Substituting (9) into the initial conditions and taking into account (14) gives $c_2 = \alpha, \ \ c_1 = \beta$.

Thus, the solution of the above Cauchy problem is given by

$$u(x) = \beta I(x) + \alpha J(x) + F(x). \tag{16}$$

Then, the following theorem holds:

Theorem 2.

The Cauchy problem has a unique solution. This solution is given by the formula (16).

Remark.

If the coefficient $a(x)$ and the right-hand side $f(x)$ belong to $C[0, R]$, then solution of the Cauchy problem given by formula (16) and general solution given by formula (9) belong to $C^2[0,\ R]$.

References

[1] E. Kamke, Differentialgleichungen. Lösungsmethoden und Lösungen, Teil I, Akad. Verlag, Leipzig, 1959.

[2] M.F. Fedoryk, The asymptotically methods for lineary ordinary differential equations, Nauka, Moscow, 1983.

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