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A Study of the Variational Iteration Method for Solving Three Species Food Web Model

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Abstract

This paper applies the variational iteration method to the three species prey predator food web model. This method is based on the use of Lagrange multipliers for identification of optimal values of parameters in a correction functional. It is shown that the successive approximation of the solution of the correction functional will be readily obtained upon using the determined Lagrange multiplier and the first approximation of the solution.

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1. Introduction

Many real world physical problems are modeled as non linear differential equations. Recently, many researchers are concentrating on solving non linear differential equations by analytic or numerical methods. The perturbation method is one of the well known techniques to solve nonlinear equations. This method was studied in (3) using Adomian decomposition method and

Homotopy perturbation method. The variational iteration method is useful to eliminate the small parameters that arise in the perturbation technique and gives exact solution of the problem.

In this paper, the three species food web model is solved by the variational iteration method using a Lagrange multiplier and a numerical comparison is made between the Adomian decomposition and variational iteration method. The main feature of this method is that the solution of a mathematical problem with linearization assumption is used as initial approximation and then a more precise approximation at some special point can be obtained. This approximation converges rapidly to an accurate solution.

2. Variational Iteration Method

To illustrate the basic concept of the technique, we consider the following general differential equation

$$Lu + Nu = g(x) \tag{2.1}$$

where, L is a linear operator, N is a non linear operator and $g(x)$ is a forcing term. Using the variational iteration method, we construct a correction functional as follows.

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda \left(Lu_n(s) + N\tilde{u_n}(s) - g(s)\right)ds \tag{2.2}$$

where, λ is a Lagrangian multiplier which can be identified optimally via the variational iteration method. The subscript n denotes the nth approximation. $\tilde{u}_n$ is considered as a restricted variation i.e. $\delta\tilde{u}_n = 0$.

Equation (2.2) is called is called a correction functional. The solution of the linear problem can be solved in a single iteration due to the exact identification of the Lagrange multiplier. In this method, it is required to first determine the Langrange multiplier, λ optimally. The successive approximation, $u_{n+1},\ n \geq 0$ of the solution of the correction functional will be readily obtained upon using the determined Lagrange multiplier and the first approximation, u_0. Consequently, the solution is given by $u = \lim_{n\to\infty} u_n$.

3. Application of Variational Iteration Method

Consider the three species food web model. Let u, v, w be the densities of the three species forming a food web in a closed environment. Then, their evolution is governed by the set of equations

$$u' = u(a_1 - b_1u - c_1v)$$

$$v' = v(-a_2 + b_2u - c_2v - d_2w)$$

$$w' = w(-a_3 + c_3 v - d_3 w) \tag{3.1}$$

where a_i, b_i, c_i $(i = 1, 2, 3)$, d_i $(i = 2, 3)$ are positive constants. Now applying (2.2) to (3.1) in order to obtain $u_{n+1}(t)$, $v_{n+1}(t)$ and $w_{n+1}(t)$,

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda_1(s)\left(\frac{d}{ds}u_n(s) - a_1 u_n(s) + b_1 \tilde{u}_n^2(s) + c_1 \tilde{u}_n(s)\tilde{v}_n(s)\right)ds \tag{3.2}$$

$$v_{n+1}(t) = v_n(t)$$
$$+ \int_0^t \lambda_2(s)\left(\frac{d}{ds}v_n(s) + a_2 v_n(s) - b_2 \tilde{u}_n(s)\tilde{v}_n(s) + c_2 \tilde{v}_n^2(s) + d_2 \tilde{v}_n(s)\tilde{w}_n(s)\right)ds \tag{3.3}$$

$$w_{n+1}(t) = w_n(t) + \int_0^t \lambda_3(s)\left(\frac{d}{ds}w_n(s) + a_3 w_n(s) - c_3 \tilde{v}_n(s)\tilde{w}_n(s) + d_3 \tilde{w}_n^2(s)\right)d\,s \tag{3.4}$$

where $\tilde{u}_n$, $\tilde{v}_n$ and $\tilde{w}_n$ are considered restricted variations. This means that $\delta\tilde{u}_n$, $\delta\tilde{v}_n$, $\delta\tilde{w}_n$ are all zero. To find the optimal value of λ, we make the above equations stationary with respect to u_n , v_n, w_n i.e.

$$\delta u_{n+1}(t) = \delta u_n(t) + \delta \int_0^t \lambda_1(s)\left(\frac{d}{ds}u_n(s) - a_1 u_n(s) + b_1 \tilde{u}_n^2(s) + c_1 \tilde{u}_n(s)\tilde{v}_n(s)\right)ds \tag{3.5}$$

$$\delta v_{n+1}(t) = \delta v_n(t) + \delta \int_0^t \lambda_2(s)\left(\frac{d}{ds}v_n(s) + a_2 v_n(s) - b_2 \tilde{u}_n(s)\tilde{v}_n(s) + c_2 \tilde{v}_n^2(s) + \right.$$
$$\left. d_2 \tilde{v}_n(s)\tilde{w}_n(s)\right)ds \tag{3.6}$$

$$\delta w_{n+1}(t) = \delta w_n(t) + \delta \int_0^t \lambda_3(s)\left(\frac{d}{ds}w_n(s) + a_3 w_n(s) - c_3 \tilde{v}_n(s)\tilde{w}_n(s) + d_3 \tilde{w}_n^2(s)\right)ds \tag{3.7}$$

where $\delta\tilde{u}_n = 0$,$\delta\tilde{v}_n = 0$, $\delta\tilde{w}_n = 0$ which implies

$$\delta u_{n+1}(t) = \delta u_n(t) + \delta \int_0^t \lambda_1(s)(\frac{d}{ds}u_n(s) - a_1 u_n(s)ds = 0$$

$$\Rightarrow \delta u_{n+1}(t) = \delta\big(\lambda_1 u_n(s)\big)_0^t - \int_0^t \lambda_1' u_n(s)ds - a_1 \int_0^t \lambda_1 u_n(s)ds) = 0$$

$$\Rightarrow u_n(s)\big(1 + \lambda_1(s)\big)\big|_{s=t} = 0 \text{ and}$$

$$u_n(s)(\lambda_1' + a_1\lambda(s))|_{s=t} = 0$$

$$\Rightarrow \lambda_1 = -1 \; or \; \lambda_1' + a_1\lambda = 0 \; at \; s = t$$

Similarly, we can obtain

$\lambda_2 = -1 \ or \lambda_2' + a_2\lambda = 0 \ at \ s = t$ and

$\lambda_3 = -1 \ or \ \lambda_3' + a_3\lambda = 0 \ at \ s = t.$

Choosing Lagrange multipliers as

$\lambda_1 = -1 \, , \lambda_2 = -1 \, , \lambda_3 = -1$ in (3.2) (3.3) and (3.4), we obtain

$$u_{n+1}(t) = u_n(t) + \int_0^t (-1)(\frac{d}{ds} u_n(s) - a_1 u_n(s) + b_1 u_n^2(s) + c_1 u_n(s) v_n(s))ds \tag{3.8}$$

$$v_{n+1}(t) = v_n(t) + \int_0^t (-1)\left(\frac{d}{ds} v_n(s) + a_2 v_n(s) - b_2(u_n(s) v_n(s) + c_2 v_n^2(s) + d_2 v_n(s) w_n(s)\right) ds \tag{3.9}$$

$$w_{n+1}(t) = w_n(t) + \int_0^t \lambda_3(s)(\frac{d}{ds} w_n(s) + a_3 w_n(s) - c_3 v_n(s) w_n(s) + d_3 w_n^2(s)))ds \tag{3.10}$$

For n = 0, (3.8), (3.9), (3.10) become

$$u_1(t) = u_0(t) - \int_0^t \left(\frac{d}{ds} u_0(s) - a_1 u_0(s) + b_1 u_0^2(s) + c_1 u_0(s) v_0(s)\right) ds \tag{3.11}$$

$$v_1(t) = v_0(t) - \int_0^t \left(\frac{d}{ds} v_0(s) + a_2 v_0(s\,) - b_2 u_0(s) v_0(s) + c_2 v_0^2(s) + d_2 v_0(s) w_0(s)\right) ds \tag{3.12}$$

$$w_1(t) = w_0(t) - \int_0^t \left(\frac{d}{ds} w_0(s) + a_3 w_0(s) - c_3 v_0(s) w_0(s) + d_3 w_0^2(s)\right) ds \tag{3.13}$$

Taking linearized solutions of (3.1), we have

$u(t) = k_1 e^{a_1 t},$

$v(t) = e^{-a_2 t},$

$w(t) = e^{-a_3 t}$

where k_1, k_2 and k_3 are constants.

As an initial approximation, choosing u_0(t) $=k_1 = 1$, v_0 (t) $= k_2 = 2, w_0(t) = k_3 = 3$, we can compute u_1(t), $v_1(t)$, $w_1(t)$ respectively. In a similar way the rest of the components of iterated formulae can be computed from (3.8), (3.9) and (3.10) respectively.

4. Application of Adomian Decomposition Method (ADM)

Solving the system of equations (3.1) by ADM with initial conditions $u(0) = u_0$, $v(0) = v_0$, $w(0)$

= w_0 and the inverse operator $L^{-1}(\,.) = \int_0^t (\,.\,)dt$ and letting

f(u) = u^2 = $\sum_{n=0}^{\infty} A_n$ where $A_n = \sum_{k=0}^{n} u_k\, u_{n-k}$

g(v) = v^2 = $\sum_{n=0}^{\infty} B_n$ where $B_n = \sum_{k=0}^{n} v_k\, v_{n-k}$

h(w) = w^2 = $\sum_{n=0}^{\infty} C_n$ where $C_n = \sum_{k=0}^{n} c_k\, c_{n-k}$ (4.1)

$\Phi_1(u, v) = uv = \sum_{n=0}^{\infty} D_n = (\sum_{n=0}^{\infty} u_n \sum_{n=0}^{\infty} v_n) = \sum_{n=0}^{\infty}(\sum_{k=0}^{n} u_k\, v_{n-k})$

or $D_n = u_k v_{n-k}$, n= 0, 1, 2... (4.2)

$\Phi_2(\, v, w) = vw = \sum_{n=0}^{\infty} E_n = (\sum_{n=0}^{\infty} v_n \sum_{n=0}^{\infty} w_n) = \sum_{n=0}^{\infty}(\sum_{k=0}^{n} v_k\, w_{n-k}\,)$

$E_n = (\sum_{k=0}^{n} v_k\, w_{n-k})$, n=0, 1, 2... (4.3)

where $A_n, B_n\,, C_{n,} D_n$ *and* E_n are Adomian polynomials and applying inverse operator on (3.1), we obtain

u_{n+1} (t) = $\int_0^t (a_1\, u_n$ - $b_1 A_n$ - $c_1 D_n)dt,$ n=0, 1, 2… (4.4)

v_{n+1}(t) = $\int_0^t (-a_2\, v_n + b_2 D_n$ - $c_2 B_n$ - $d_2 E_n)dt,$ n=0, 1, 2… (4.5)

w_{n+1} (t) = $\int_0^t (-a_3\, w_n + c_3 E_n$ - $d_3 C_n)dt,$ n=0, 1, 2…. (4.6)

Taking n= 0, 1, 2 … in (4.4) to (4.6), we can evaluate the components $u_i, v_i\,, w_i$ where i=0,1,2…

5. Example to compare solutions obtained from Adomian decomposition and Variational iteration methods

Letting $a_1 = a_2 = a_3 = 1$, $b_1 = 1$, $b_2 = 2$, $c_1 = \frac{1}{4}$, $c_2 = \frac{1}{2}$, $c_3 = 2$, $d_2 = d_3 = 1$ in (3.1) and u_0 =1 , v_0= 2, w_0 = 3 as initial approximations, the Adomian decomposition iterations are obtained as

u_0 = 1 , v_0= 2 , w_0 = 3

$u_1 = \frac{-1}{2}\, t$, $v_1 = -6t$, w_1 = 0,

$u_2 = \frac{9}{8}\, t^2$,v_2 =11t^2 ,w_2 =-18 t^2 , and so on, so that

$u(t)$ = 1+ $\frac{-1}{2}\, t + \frac{9}{8}\, t^2$ ……

$v(t)$ = 2 $-6t$ + 11t^2 +….

$w(t)$ = 3 - 18 t^2 +…

Whereas, the variational iteration method iterates are obtained as

u_0 =1 , v_0=2 , w_0 =3

$u_1 = 1 + \frac{-1}{2}t$,

$v_1 = 2 - 6t$,

w_1 =3,

$u_2 = 1 + \frac{-1}{2}t + \frac{9}{8}t^2 - \frac{1}{3}t^3$

$v_2 = 2 - 6t + 11t^2 - 4t^3$

$w_2 = 3 - 18t^2$

It shows that the iterations obtained by variational iteration method are more accurate and converge more rapidly when compared to Adomian decomposition method. Similarly, the other iterations can be computed using Mathcad7.

6. Conclusions

In this paper, the variational iteration method is applied to the three species food web model based on the Lagrangian multiplier for identification of optimal values of parameter in a functional. A numerical comparison is made between the variational iteration method and Adomian decomposition method (ADM) and it is shown that variational iteration method is more accurate and convergent than ADM.

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