Int. Journal of Math. Analysis, Vol. 6, 2012, no. 15, 701 - 707

Beurling's Theorem for Vector-Valued Hardy Spaces

H. Rezaei[1], S. Talebzadeh[2] and D.Y. Shin[3*]

[1] Department of Mathematics, College of Sciences
Yasouj University, Yasouj, 75918-74831, Iran
rezaei@mail.yu.ac.ir

[2] Department of Mathematics, Islamic Azad University
Firooz Abad Branch, Firooz Abad, Iran
s.talebzadeh@iauf.ac.ir

[3] Department of Mathematics, College of Natural Science,
University of Seoul, Korea
dyshin@uos.ac.kr
[*] Corresponding author

Mathematics Subject Classification: Primary: 47A15

Keywords: Beurling's theorem, vector-valued Hardy space

1 Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane. The Hardy space $H^2 := H^2(\mathbb{D})$ is the space of all holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ for which

$$\|f\|_{H^2} := \lim_{r\to 1}\Big(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta}|^2 d\theta\Big)^{1/2} < +\infty$$

A closed linear subspace M of $H^2(\mathbb{D})$ is said to be shift invariant if M is invariant under the multiplication operator by z(shift operator) on $H^2(\mathbb{D})$. Beurling's theorem states that every shift invariant subspace of $H^2(\mathbb{D})$ is either zero subspace or of the form $\varphi H^2(\mathbb{D})$, where φ is an inner function in $H^2(\mathbb{D})$, a bounded analytic function on $\mathbb{D}$ with non-tangential boundary values of modulus 1 almost everywhere with respect to the Lebesgue measure on the unit circle $\partial\mathbb{D}$ [1]. Beurling's theorem is viewed as one of the most celebrated theorems in operator theory and it has been extended to many directions [1, 6]. In this paper, we study the invariant space problem for the shift operator on vector-valued Hardy space $H^2(\mathbb{D}, E)$. Recall that a holomorphic function $f : \mathbb{D} \to E$ belongs to the vector-valued Hardy space $H^2(E) := H^2(\mathbb{D}, E)$, if

$$\|f\|_{H^2(E)} := \lim_{r\to 1}\Big(\frac{1}{2\pi}\int_0^{2\pi} \|f(re^{i\theta})\|_E^2 d\theta\Big)^{1/2} < +\infty.$$

In fact, $H^2(E)$ becomes a Hilbert space with the following inner product

$$< f, g >_{H^2(E)} := \lim_{r\to 1}\Big(\frac{1}{2\pi}\int_0^{2\pi} < f(re^{i\theta}), g(re^{i\theta}) >_E\Big) d\theta.$$

A more detailed discussion of vector-valued analytic functions and Hardy spaces can be found in Hille and Philips [5], Rosenblum and Rovnyak [7], Hensgen [4] and a convenient reference for classical Hardy spaces is Duren [2]. It is not to hard to see the shift operator S on $H^2(E)$ is well defined and bounded. The main result of this paper gives a characterization of shift invariant subspaces of vector-valued Hardy space $H^2(E)$. It involves with the well known Hadamard product on Hilbert space. Recall that for given Hilbert space E with an orthonormal basis $\{e_n : n \in I\}$ the Hadamard product of two vectors x and y in E is defined by

$$x * y = \sum_{n\in I} < x, e_n >< e_n, y >$$

An application of Cauchy-Schwarz inequality with the Parseval identity implies that the Hadamard product is well defined and $\|x * y\| \leq \|x\|\|y\|$. Moreover, the Hadamard product of two analytic function f and g is determined by

$$(f * g)(z) := f(z) * g(z) = \sum_{n=1}^{+\infty} < f(z), e_n >< g(z), e_n > e_n.$$

For a separable Hilbert space E and a nonzero shift invariant subspace $\mathcal{M}$ of $H^2(E)$ we show there exist a vector-valued function $\Phi : \mathbb{D} \to E$ such that $\|\Phi(z)\| = 1$ almost every where on $\partial\mathbb{D}$ and

$$\mathcal{M} = \Phi * H^2(E) = \{\Phi * F : F \in H^2(E)\},$$

provided that $E * \mathcal{M} \subseteq \mathcal{M}$. Here $E * \mathcal{M}$ is the collection of all functions $a * f$ mapping $z \to f(z)a$ for $a \in E$ and $f \in \mathcal{M}$. In particular, when E is of finite dimensional the condition $E * \mathcal{M} \subseteq \mathcal{M}$ is hold and the shift invariant subspaces of $H^2(E)$ are represented as above.

2 vector-valued hardy space

In order to state our main result we have to introduce some notation. Unless otherwise stated we assume that E is a Hilbert space with the orthonormal basis $\{e_i : i \in I\}$ for some index set I. For a vector $x \in E$ and a vector-valued function $f : \mathbb{D} \to E$, the scalar function $x \otimes f : \mathbb{D} \to \mathbb{C}$ is defined by

$$(x \otimes f)(z) = < f(z), x > .$$

It is easy to see, f is holomorphic if and only if $x \otimes f$ is holomorphic for every $x \in E$. By this, we can present any holomorphic vector-valued function f by

$$f = \sum_{i \in I} (e_i \otimes f) e_i.$$

Also for a subspace $\mathcal{M}$ of vector-valued holomorphic functions, the subspace $x \otimes M$, is defined by

$$x \otimes M = \{x \otimes f : f \in M\}.$$

in a natural way. If $\{H_i : i \in I\}$ is a collection of Hilbert spaces, then $H := (\bigoplus_{i \in I} H_i)_{\ell^2}$ is defined as the collection of square summable nets; that is, nets of the form $h = \{h_i\}_{i \in I}$ such that $h_i \in H_i$ for all i and $\sum_{i \in I} \|h_i\|^2 < +\infty$. Moreover, H is a Hilbert space by the following inner product:

$$< f, g >_H = \sum_{i \in I} < f_i, g_i >_{H_i}$$

for all $f = \{f_i\}_{i \in I}, g = \{g_i\}_{i \in I} \in H$. Of course, if a net of operators T_i acting on H_i is such that $\{\|T_i\|\}_{i \in I}$, is bounded, then $T := \bigoplus_{i \in I} T_i$ defined in a natural way, is a bounded linear operator on $(\bigoplus_{i \in I} H_i)_{\ell^2}$.

In the following the structure of vector-valued Hardy spaces has been stated as a direct sum of scalar-valued copies:

Theorem 2.1. Let E be a Hilbert space then $H^2(E)$ is isometrically isomorphic to $(\bigoplus_{i \in I} H^2)_{\ell^2}$, for some index set I.

Proof. Let $\{e_i : i \in I\}$ be an orthonormal basis for Hilbert space E, for some index set I. Put $\mathcal{H} = (\bigoplus_{i \in I} H^2)_{\ell^2}$ and define $\Lambda : H^2(E) \to \mathcal{H}^2$ by $\Lambda(f) = \{e_i \otimes f\}_{i \in I}$, then

$$\begin{aligned}
< \Lambda(f), \Lambda(g) >_{\mathcal{H}} &= < \{e_i \otimes f\}_{i \in I}, \{e_i \otimes g\}_{i \in I} >_{\mathcal{H}} \\
&= \sum_{i \in I} < e_i \otimes f, e_i \otimes g >_{H^2} \\
&= \sum_{i \in I} \lim_{r \to 1} \Big(\frac{1}{2\pi} \int_0^{2\pi} (e_i \otimes f)(re^{i\theta}) \overline{(e_i \otimes g)(re^{i\theta})} \Big) dm(t)
\end{aligned}$$

$$= \lim_{r\to 1}\Big(\frac{1}{2\pi}\int_0^{2\pi}\Big(\sum_{i\in I} < f(re^{i\theta}), e_i > \overline{< g(re^{i\theta}), e_i >} dm(t)\Big)\Big)$$
$$= \lim_{r\to 1}\Big(\frac{1}{2\pi}\int_0^{2\pi} < f(re^{i\theta}), g(re^{i\theta}) > dm(t)\Big) =< f, g >_{H^2(E)} .$$

It remains to prove Λ is a surjection. Let $g = \{f_i\}_i \in \mathcal{H}$, then by the Parseval identity,

$$f(t) = \sum_{i\in I} f_i(t)e_i$$

defines a holomorphic function in $H^2(E)$ and $\Lambda(f) = g$.

Corollary 2.2. The vector-valued function f belongs to $H^2(E)$ if and only if

$$\sum_{i\in I} \|e_i \otimes f\|^2_{H^2} < +\infty$$

Conversely, if $\{f_i\}$ is a net in H^2, such that $\sum_{i\in I} \|f_i\|^2_{H^2} < +\infty$ then

$$f(z) := \sum_{i\in I} f_i(z)e_i$$

defines a holomorphic function in $H^2(E)$ and $\|f\|_{H^2(E)} = \sum_{i\in I} \|f_i\|^2_{H^2}$.

3 Beurling's Theorem

In what follows E is a sparable Hilbert space with an orthonormal basis $\{e_n : n \in \mathbb{N}\}$. Recall that the shift operator $S : H^2(E) \to H^2(E)$ defined by $S(f)(z) = zf(z)$ is well defined and bounded. From now on, we call the invariant subspaces $H^2(E)$ under S as a shift invariant subspace. By an scalar-valued inner function we mean a bounded analytic function φ such that $|\varphi(e^{i\theta})| = 1$ almost everywhere.

Theorem 3.1 (Beurling). Let M be a nonzero shift invariant subspace of H^2 then there is an inner function φ such that $M = \varphi H^2$.

Definition 3.2. For any two E-valued function f, g in $H^2(E)$, the Hadmard product f and g is defined by the following:

$$(f * g)(z) = \sum_{n=1}^{+\infty} < f(z), e_n >< g(z), e_n > e_n.$$

In fact,

$$f * g = \sum_{n=1}^{+\infty} (e_n \otimes f)(e_n \otimes g) e_n.$$

By Cauchy-Schwarz inequality, $f * g$ is a holomorphic E-valued function on $\mathbb{D}$. In particular, for any $a \in E$ the Hadamard product $a * g$ means the Hadamard product of the constant vector- valued function $f \equiv a$ with g. More generally, for a nonzero subspace $\mathcal{M}$ of $H^2(E)$ the Hadamard product E and $\mathcal{M}$ is denoted by $E * \mathcal{M}$ and defined by

$$E * \mathcal{M} := \{a * f : a \in E \text{ and } f \in \mathcal{M}\}$$

Theorem 3.3. Let $\mathcal{M}$ be a nonzero shift invariant subspace of $H^2(E)$ then $E * \mathcal{M} \subseteq \mathcal{M}$ if and only if then there is a vector-valued bounded analytic $\Phi : \mathbb{D} \to E$ such that $\|\Phi(e^{i\theta})\|_E = 1$ almost everywhere on $\partial\mathbb{D}$ and $\mathcal{M} = \Phi * H^2(E)$.

Proof. Assume that $\mathcal{M} = \Phi * H^2(E)$ then it is easy to see that $E * \mathcal{M} \subseteq \mathcal{M}$. For the converse let $\mathcal{M}$ be a shift invariant subspace such that $E * \mathcal{M} \subseteq \mathcal{M}$. Since $\mathcal{M}$ is shift invariant subspace of $H^2(E)$, every close subspace $e_n \otimes \mathcal{M}$ in H^2 is also shift invariant for any integer $n \geq 1$. Let $A := \{n \in \mathbb{N} : e_n \otimes \mathcal{M} \neq \{0\}\}$. Consider the scalar sequence $\{\beta_n\}$ defined in terms of the cardinality of A:

$$\beta_n = \begin{cases} |A|^{1/2} & A \text{ is finite and } n \in A \\ 2^{n/2} & A \text{ is infinite and } n \in A \end{cases}$$

where $|A|$ means the cardinality of A. In fact, the sequence $\{\beta_n\}$ is chosen in such a way that

$$\sum_{n \in A} \beta_n^2 = 1.$$

By using the Beurling's Theorem in scalar-valued case we obtain an inner function φ_n such that

$$\beta_n^{-1} e_n \otimes \mathcal{M} = \varphi_n H^2$$

for any $n \in A$. If the complement of A is nonempty we set $\varphi_n = 0$ for every $n \notin A$. Put

$$\Phi = \sum_{n=1}^{+\infty} \beta_n \varphi_n e_n = \sum_{n \in A} \beta_n \varphi_n e_n.$$

Then $\Phi : \mathbb{D} \to E$ is a holomorphic function and we claim $\mathcal{M} = \Phi * H^2(E)$. Assume that $f \in \mathcal{M}$, and $n \in A$. Then $e_n \otimes f = \beta_n^{-1} \varphi_n f_n$ for some $f_n \in H^2$.

Let $F := \sum_{n\in A}\beta_n^{-1} f_n e_n$, where $(f_n e_n)(z) = f_n(z)e_n$. Note that each φ_n is an inner function, hence

$$\|f_n\|_{H^2} = \|\varphi_n f_n\|_{H^2} = \beta_n\|e_n \otimes f\|_{H^2}.$$

Since $f \in H^2$, $\sum_{n=1}^{+\infty}\|e_n \otimes f\|_{H^2}^2$ is finite, so is $\sum_{n=1}^{+\infty}\beta_n^{-1}\|f_n\|_{H^2}^2$ and hence $F \in H^2(E)$ by Corollary 2.3. Moreover,

$$\begin{aligned} f &= \sum_{n=1}^{+\infty}(e_n \otimes f)e_n = \sum_{n=1}^{+\infty}\beta_n^{-1}(\varphi_n f_n)e_n \\ &= \sum_{n=1}^{+\infty}\beta_n^{-1}(e_n \otimes \Phi)(e_n \otimes F)e_n = \Phi * F \end{aligned}$$

Now suppose $F \in H^2(E)$ and $n \in A$. Then

$$m^{-1}\varphi_n(e_n \otimes F) \in \varphi_n H^2 = e_n \otimes \mathcal{M},$$

whence $m^{-1}\varphi_n(e_n \otimes F) = e_n \otimes f_n$ for some $f_n \in A$. Hence,

$$\begin{aligned} \Phi * F &= \sum_{n=1}^{+\infty}[e_n \otimes (\Phi * F)]e_n \\ &= \sum_{n=1}^{+\infty}(e_n \otimes \Phi)(e_n \otimes F)e_n \\ &= \sum_{n\in A} m^{-1}\varphi_n(e_n \otimes F)e_n = \sum_{n\in A}(e_n \otimes f_n)e_n \end{aligned}$$

Note that $(e_n \otimes f_n)e_n = e_n * f_n \in E * \mathcal{M} \subseteq \mathcal{M}$ for any $n \in A$. Since $\mathcal{M}$ is a close subspace,

$$\Phi * F = \sum_{n\in A}(e_n \otimes f_n)e_n \in \mathcal{M}.$$

This implies that $\Phi * H^2(E) = \mathcal{M}$. It remains to prove $\|\Phi(e^{i\theta})\| = 1$ almost everywhere on $\partial\mathbb{D}$. For this let $B_n = \{e^{i\theta} \in \partial\mathbb{D} : |\varphi_n(z)| \neq 1\}$ and $B = \cup_{n\in A}B_n$. Since all φ_n are inner functions, $\|\varphi_n\|_{H^2} = 1$ and B is of zero measure. Now for any $e^{i\theta} \notin B$,

$$\|\Phi(e^{i\theta})\|_{H^2(E)}^2 = \sum_{n\in A}\beta_n^2\|\varphi_n(e^{i\theta})\|_{H^2}^2 = \sum_{n\in A}\beta_n^2 = 1$$

This ends the proof.

It is easy to see that if E is a finite dimensional Hilbert space then $E * \mathcal{M} \subseteq \mathcal{M}$ and so the following corollary is deduced:

Corollary 3.4. Let E is a finite dimensional Hilbert space and $\mathcal{M}$ be a nonzero shift invariant subspace of $H^2(E)$ then there is a vector-valued bounded analytic $\Phi : \mathbb{D} \to E$ such that $\|\Phi(e^{i\theta})\|_E = 1$ almost everywhere on $\partial\mathbb{D}$ and $\mathcal{M} = \Phi * H^2(E)$.

Acknowledgement

Dong Yun Shin(corresponding author) was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2010-0021792).

References

[1] A. Beurling, On two problems concerning linear transformations in Hilbert spaces, Acta. Math. 81(1949), 239-255.

[2] P. L. Duren, Theory of H^p spaces, New York. Academic Press (1970).

[3] H. Hedenmalm and A. Shields, Invariant subspaces in Banach spaces of analytic functions, Michigan Math. J. 37 (1990) 91-104 .

[4] W. Hensgen, Hardy Raume vecktorwertiger Functionen, Thesis. Munich (1986).

[5] E, Hille and R.S. Phillips, Functional analysis and semi-groups, Amer. Math. Soc. Providence (1957).

[6] S. Richter, Invariant subspaces in Banach spaces of analytic functions, Trans. Amer. Math. Soc. 304 (1987), 585-616.

[7] M. Rosenblum and J. Rovnyak, Hardy classes and operator theory, Oxford University Press (1985).

Received: October, 2011