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On Duals of Some Rate Spaces

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Abstract

α and β – duals of some rate spaces are found out.

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1. Introduction

In this paper we investigate α – dual of the rate spaces $(cs)_{\frac{1}{\pi}}$, $l_{\frac{1}{\pi}}$, Γ_{π}, $(cs)_{\pi}$ and β – dual of $(c_0)_{\pi}$. A complex sequence, whose k term is x_k, will be denoted by $\{ x_k \}$ or x. Let $\pi = \{ \pi_k \}$ be a sequence of positive numbers. If X is a sequence space, we write $X_\pi = \left\{x : \left\{\frac{x_k}{\pi_k}\right\} \in X\right\}$. We require the following sequence spaces:

c_0 = { all null sequences };

cs = { all those sequences $\{x_k\}$ such that $\sum x_k$ converges };

l = { all those sequences { x_k } such that $\sum |x_k| < \infty$ };

Γ = { all entire sequences };

□ = { all analytical sequences};

l^∞ = { all bounded sequences };

2. Growth Sequences

A sequence t is called a growth sequence for a set S of sequences if $x_k = O(t_k)$ for all x ∈ S.

Theorem 2.1

If X has a growth sequence, then X_π has a growth sequence.

Proof:

Let be a growth sequence for X. Then $|x_k| \le$ M $|t_k|$ for some M > 0.

Let x ∈ X_π. Then $\left\{\frac{x_k}{\pi_k}\right\}$ ∈ X . We have $\left|\frac{x_k}{\pi_k}\right| \le |x_k| \le$ M $|t_k|$ which means that $|x_k| \le$ M $|\pi_k\, t_k|$.

Thus, $\{\pi_k\, t_k\}$ is a growth sequence for X_π . In other words, X_π has the growth sequence πt.

Theorem 2.2

Let X be a BK – space. Then the rate space X_π has a growth sequence.

Proof:

Let x ∈ X_π. Then $\left\{\frac{x_k}{\pi_k}\right\}$ ∈ X.

Put $P_k(x) = \dfrac{x_k}{\pi_k}$ $\forall$ x ∈ X_π .Then P_k is a continuous functional on X_π . Hence $\|P_k\| < \infty$.

Also for every positive integer k, we have

$|x_k| = |P_k(x)\pi_k| = |P_k(x\pi)|$

$\leq \|P_k\| \; \|x\pi\| = \|P_k\pi_k\| \; \|x\|$

Hence , $x_k = O(P_k\pi_k)$.

Thus { $P_k\pi_k$ } is a growth sequence for X_π .

3. α – duals

Theorem 3.1

$$\left\{(cs)_{1/\pi}\right\}^\alpha = l_\pi$$

Proof:

Let x ∈ $\left\{(cs)_{1/\pi}\right\}^\alpha$, but x $\notin$ l_π . Then for each positive integer k , we can find an increasing sequence { n_k } of positive integers such that

$$\sum_{i=n_k+1}^{n_{k+1}} |\pi_i x_i| > 2^k \text{ for k = 1, 2, ...}$$

Take $y_i = \begin{cases} (\frac{-1}{\pi_i})^i \frac{1}{\sqrt{2}}, n_k < i \leq n_{k+1} \\ 0, otherwise \end{cases}$ for k ≥ 1.

Then {y_i} ∈ $(cs)_{1/\pi}$ and $\sum |x_i y_i| = \infty$.

This contradicts the fact that x ∈ $(cs)^\alpha_{\frac{1}{\pi}}$.

This contradiction shows that $(cs)^\alpha_{\frac{1}{\pi}} \subset l_\pi$

But $(cs)_{\frac{1}{\pi}} \subset (c_0)_{\frac{1}{\pi}}$

Consequently,

$l_\pi \subset (c_0)_{\frac{1}{\pi}}^{\alpha} \subset (cs)_{\frac{1}{\pi}}^{\alpha}$.

Therefore $(cs)_{\frac{1}{\pi}}^{\alpha} = l_\pi$.

Theorem 3.2

$l_{\frac{1}{\pi}}^{\alpha} = l_\pi^\infty$

Proof:

Let x ∈ $l_{\frac{1}{\pi}}^{\alpha}$. Assume that x $\notin l_\pi^\infty$. Then there exists an increasing sequence $\{n_i\}$ of positive integers such that $\left|\frac{x_{n_i}}{\pi_i}\right| > i^3$.

Take $y_n = \begin{cases} \frac{1}{\pi_i i^2}, n = n_i, i \geq 1. \\ 0, otherwise \end{cases}$

Then $\{y_n\} \in l_{\frac{1}{\pi}}$. But $\sum |x_i y_i| = \infty$. Thus x $\notin l_{\frac{1}{\pi}}^{\alpha}$.

Accordingly, $l_{\frac{1}{\pi}}^{\alpha} \subset l_\pi^\infty$ ...(1)

On the otherhand let x ∈ l_π^∞. We then have $\left|\frac{x_i}{\pi_i}\right| \leq M$

where M > 0 is a scalar.

Also y $\in l_{\frac{1}{\pi}} \Rightarrow \sum |x_i y_i| < \infty$. Therefore x ∈ $l_{\frac{1}{\pi}}^{\alpha}$.

Consequently, $l_\pi^\infty \subset l_{\frac{1}{\pi}}^{\alpha}$ ...(2)

From (1) and (2) it follows that

$l_{\frac{1}{\pi}}^{\alpha} = l_\pi^\infty$.

Theorem 3.3

$$\Gamma^{\alpha}{}_{\pi} = \Lambda_{\frac{1}{\pi}}$$

Proof:

Let x ∈ $\Lambda_{\frac{1}{\pi}}$. Then there exists M > 0 with $|\pi_i x_i| \leq M^i$ ∀ i ≥ 1.Choose $\in > 0$ such that $\in$M < 1.

If $y \in \Gamma_\pi$, we have $\left|\frac{y_i}{\pi_i}\right| \leq \in^i$ for all i $\geq$ i_0 depending on $\in$.

Therefore $\sum|x_i y_i| \leq \sum(M \in)^i < \infty$,

Hence $\Lambda_{\frac{1}{\pi}} \subset \Gamma_\pi^\alpha$ ...(1)

On the other hand, let $x \in \Gamma_\pi^\alpha$. Assume that $x \notin \Lambda_{\frac{1}{\pi}}$. Then there exists an increasing sequence $\{n_i\}$ of positive integers such that

$\left|\pi_{n_i} x_{n_i}\right| > i^{2n_i}$ for all $i \geq i_0$.

Take $y_n = \begin{cases} \pi_n i^{-n_i} \; for (n = n_i) \\ 0 \; for (n \neq n_i) \end{cases}$. Then $\{y_n\} \in \Gamma_\pi$, but $\sum|x_i y_i| = \infty$,

a contradiction.

This contradiction shows that $\Gamma_\pi^\alpha \subset \Lambda_{\frac{1}{\pi}}$ ...(2)

From (1) and (2) it follows that $\Gamma^\alpha{}_\pi = \Lambda_{\frac{1}{\pi}}$.

Theorem 3.4

$$(cs)_\pi^\alpha = l_{\frac{1}{\pi}}$$

Proof:

First we shall show that $(cs)_\pi^\alpha \subset l_{\frac{1}{\pi}}$.

Let $y \notin l_{\frac{1}{\pi}}$. We show that $y \notin (cs)_\pi^\alpha$. Choose an increasing sequence $\{k_i\}$ of positive integers such that $\sum_{k=k_i+1}^{k_{i+1}} \left|\frac{y_k}{\pi_k}\right| > 4^i$.

Take $x_k = \begin{cases} \frac{(-1)^k \pi_k}{2^i} if k_i < k \leq k_{i+1} \\ 0, otherwise \end{cases}$ for k = 1, 2, ...

Then $x \in (cs)_\pi$.

Also $\sum|x_k y_k| \geq \sum \frac{1}{2^i} \sum_{k=k_i+1}^{k_{i+1}} |y_k| \geq \sum 2^i$.

Hence $\sum|x_k y_k|$ does not converge. Therefore $y \notin (cs)_\pi^\alpha$.

Hence $(cs)_\pi^\alpha \subset l_{\frac{1}{\pi}}$ ...(1)

On the other hand, we have always $(cs)_\pi \subset l_\pi^\infty$.

But $(l_\pi^\infty)^\alpha = l_{\frac{1}{\pi}}$. Hence $l_{\frac{1}{\pi}} \subset (cs)_\pi^\alpha$ ...(2)

From (1) and (2) we get

$(cs)_\pi^\alpha = l_{\frac{1}{\pi}}$.

4. β – duals

Theorem 4.1

$c_{0\pi}^\beta = l_{\frac{1}{\pi}}$

Proof:

Let t ∈ $l_{\frac{1}{\pi}}$ and let x ∈ $c_{0\pi}$ be given. Then, writing $\|t\|$ as l – norm and $\|x\|_\infty$ as l^∞ - norm, $\sum_{k=1}^{n} |t_k x_k| \le \|x\|_\infty \|t\|_1 < \infty$ for all n. Consequently, t ∈ $c_{0\pi}^\beta$.

Thus $l_{\frac{1}{\pi}} \subset c_{0\pi}^\beta$ ...(1)

On the other hand we shall show that x ∈ $c_{0\pi}$ for a given t ∉ $l_{\frac{1}{\pi}}$ with tx ∉ cs. Here tx means $\{t_k x_k\}$. This shows that t ∉ $c_{0\pi}^\beta$. Thus $c_{0\pi}^\beta \subset l_{\frac{1}{\pi}}$.

Now t ∉ $l_{\frac{1}{\pi}}$. We may choose an increasing sequence $\{n_i\}$ of positive integers such that $\sum_{k=n_i+1}^{k=n_{i+1}} \pi_k \ |t_k| > 1$.

Define $x_k = \begin{cases} \frac{1}{i}\operatorname{sgn} t_k, n_i + 1 \le k < n_{i+1} \\ 0, otherwise \end{cases}$

Then $\sum_{k=n_i+1}^{n_{i+1}} t_k x_k = \frac{1}{i} \sum_{k=n_i+1}^{n_{i+1}} |t_k| > 1$ which shows that t ∉ c_0^β.

Consequently,

$c_{0\pi}^\beta \subset l_{\frac{1}{\pi}}$ ...(2)

The inclusion (1) and (2) imply that $c_{0\pi}^\beta = l_{\frac{1}{\pi}}$.

5. Conclusion

The previous work on rate spaces includes the papers [1], [2], [3] and [4]. There is scope to continue further investigation on rate spaces.

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