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Strict and Uniform Convexity of the Measurable Section Spaces

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Abstract. We consider the measurable section spaces $E(\mathcal{X})$, where $\mathcal{X}$ is a measurable bundle of Banach spaces and E is Köthe function spaces. We show that $E(\mathcal{X})$ space is uniformly (strict) convex if the space E and almost every all fibers $\mathcal{X}_t$ are uniformly (strict) convex. If $E(\mathcal{X})$ is uniformly (strict) convex so is a E, moreover if a fiber $\mathcal{X}_t$ have a density $r(\mathcal{X}_t) > 0$ then $\mathcal{X}_t$ is uniformly (strict) convex.

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1. Introduction

Today the theory of Banach bundles is a vast area of Functional Analysis. Continuous and measurable Banach bundles are often used for representing various functional-analytical objects [4, 5, 6, 7, 11, 16]. In this article we investigate a geometric structure of the measurable sections spaces and give a necessary and sufficient condition for this spaces to be a strict convex and uniformly convex.

2. Preliminaries

The goal of this section is to introduce some basic definitions and facts. General information on Banach spaces and measurable bundles of Banach spaces reader can find in the books [3, 10, 14, 15]. All our linear spaces are real. Let (Ω, Σ, μ) be a complete σ-finite measure space. A Banach space E consisting of equivalence classes modulo equality almost everywhere integrable real-valued functions on Ω is called *Köthe function space* if E following conditions hold.

i) If $f \in L_0(\mu)$ and $|f| \leq |g|\,\mu$-a.e. for some $g \in E$, then $f \in E$ and $\|f\|_E \leq \|g\|_E$.

ii) For every $A \in \Sigma$ with $\mu(A) < \infty$ the characteristic function $\mathbf{1}_A$ belongs to E.

Definition 2.1. Let Ω be a nonempty set. A *bundle of Banach spaces over* Ω is a mapping $\mathcal{X}$ defined on Ω and associating a Banach space $\mathcal{X}_t := \mathcal{X}(t) := (\mathcal{X}(t), \|\cdot\|_{\mathcal{X}(t)})$ with every point $t \in \Omega$. The value $\mathcal{X}_t$ of bundle is called its *fiber* over t. A mapping s defined on a nonempty set $\mathrm{dom}(s) \subset \Omega$ is called a *section* over $\mathrm{dom}(s)$ if $s(t) \in \mathcal{X}_t$ for every $t \in \mathrm{dom}(s)$. A section over Ω is called *global.* Let $S(\Omega, \mathcal{X})$ stands for the set of all global sections of $\mathcal{X}$ endowed with the structure of vector space by letting $(\alpha u + \beta v)(t) = \alpha u(t) + \beta v(t)$, $(t \in \Omega)$, where $\alpha, \beta \in \mathbb{R}$ and $u, v \in S(\Omega, \mathcal{X})$. For each section $s \in S(\Omega, \mathcal{X})$ we define its point-wise norm by $|\|s\|| : t \mapsto \|s(t)\|_{\mathcal{X}(t)}, (t \in \Omega)$. A set of sections $\mathcal{D}$ is called *fiberwise dense* in $\mathcal{X}$ if the set $\{s(t) : \ s \in \mathcal{D}\}$ is dense in $\mathcal{X}_t$ for every $t \in \Omega$.

Definition 2.2. Now consider a nonzero σ-finite measure space (Ω, Σ, μ). Let $\mathcal{X}$ be a bundle of Banach spaces over Ω. A set of sections $\mathcal{I} \subset S(\Omega, \mathcal{X})$ is called a *measurability structure* on $\mathcal{X}$ if it satisfies the following conditions:

(1) $\mathcal{I}$ is a vector space, i.e. $\lambda v + \mu u \in \mathcal{I}\, (\lambda, \mu \in \mathbb{R},\, u, v \in \mathcal{I})$;

(2) $|\|s\|| : \Omega \to \mathbb{R}$ is measurable for $s \in \mathcal{I}$;

(3) the set $\mathcal{I}$ is fiberwise dense in $\mathcal{X}$. If $\mathcal{I}$ is a measurability structure in $\mathcal{X}$ then we call the pair $(\mathcal{X}, \mathcal{I})$ a *measurable bundle of Banach spaces* over (Ω, Σ, μ). We shall write simply $\mathcal{X}$ instead $(\mathcal{X}, \mathcal{I})$.

Definition 2.3. Let $(\mathcal{X}, \mathcal{I})$ be a measurable bundle of Banach spaces over Ω. Denote by $S_\sim(\Omega, \mathcal{X})$ the set of all section of $\mathcal{X}$ defined almost everywhere on Ω. We say that $s \in S_\sim(\Omega, \mathcal{X})$ is a *step-section*, if $s = \sum\limits_{i=1}^{n} \mathbf{1}_{A_i} c_i$ for some $n \in \mathbb{N}$, $A_1, \dots, A_n \in \Sigma$, $c_1, \dots, c_n \in \mathcal{I}$. A section $u \in S_\sim(\Omega, \mathcal{X})$ is called *measurable* if for every $D \in \Sigma$, $\mu(D) < \infty$ there is a sequence $(s)_{n=1}^{\infty}$ of step-sections such that $s(t) \to u(t)$ for almost all $t \in D$. The set of all measurable sections of $\mathcal{X}$ is denoted by $\mathcal{L}_0(\Omega, \Sigma, \mu, \mathcal{X})$ or $\mathcal{L}_0(\mu, \mathcal{X})$ for simplicity.

Suppose that $\mathcal{X}$ is a measurable bundle of Banach spaces over Ω. Consider the equivalence relation $\sim$ in the set $\mathcal{L}_0(\mu, \mathcal{X}) : v \sim u$ means that $u(t) = v(t)$ for almost all $t \in \Omega$. The coset containing $v \in \mathcal{L}_0(\mu, \mathcal{X})$ is denoted by $v^\sim$. The quotient set $L_0(\mu, \mathcal{X}) := L_0(\Omega, \Sigma, \mu, \mathcal{X}) := \mathcal{L}_0(\mu, \mathcal{X}) \setminus \sim$ is a vector space under the operations $\lambda u^\sim + \xi v^\sim = (\lambda u + \xi v)^\sim, (\xi, \lambda \in \mathbb{R})$.

Let $\mathcal{X}_{t_0}$ be a fiber of the measurable bundle of the Banach spaces $\mathcal{X}$. A *density* of the $\mathcal{X}_{t_0}$ is a nonnegative number $\rho(\mathcal{X}_{t_0})$ such that

$$\rho(\mathcal{X}_{t_0}) := \mu\{t \in \Omega : \ \mathcal{X}_t \text{ is isomorphic to } \mathcal{X}_{t_0}\}.$$

Let x be a element of the $\mathcal{X}_{t_0}$ and $\rho(\mathcal{X}_{t_0}) > 0$. By $\widetilde{x}$ we denote the section such that

$$\widetilde{x}(t) = \begin{cases} x, & \text{if } \mathcal{X}_t \text{ is isomorphic } \mathcal{X}_{t_0} \\ 0 & \text{otherwise} \end{cases}$$

Measurable bundle of Banach spaces is called *locally constant* if for every fibers $\mathcal{X}_t$ with $\rho(\mathcal{X}_t) > 0$ and for every $x \in \mathcal{X}_t$ the sections $\widetilde{x}$ are measurable. All measurable bundles of Banach spaces we consider below are locally constant.

Let (Ω, Σ, μ) be a σ-finite measure space, E is Köthe function space over (Ω, Σ, μ) and $\mathcal{X}$ is a measurable bundle of Banach spaces over (Ω, Σ, μ). Define

$$E(\mathcal{X}) := \{f \in L_0(\mu, \mathcal{X}) : \|f(\cdot)\|_{\mathcal{X}(\cdot)} \in E\}.$$

Then $E(\mathcal{X})$ is also a Banach space with respect to the norm

$$|||f||| := \|\|f(\cdot)\|_{\mathcal{X}(\cdot)}\|_E.$$

If $\mathcal{X}$ is a constant measurable bundle of Banach spaces, where $\mathcal{X}(t)$ isomorphic $\mathcal{X}(s)$ for every $t_1, t_2 \in \Omega$ we denote it by X and $E(X)$ is a well-known in the literature as the Köthe-Bochner function space. The most important class of Köthe-Bochner function spaces $E(X)$ are the Lebesgue-Bochner spaces $L_p(X)$, $(1 \leq p < \infty)$ and their generalization the Orlicz-Bochner spaces $L_\Phi(X)$. The geometric properties of the Köthe-Bochner function spaces have been studied by many authors [1, 2, 8, 9, 12, 17].

A Banach space X is said to be *strictly convex* or *rotund* if for any two distinct vectors x, y in the unit ball $B(X)$ of X $\|x + y\| < 2$. Let E be a Banach lattice. Than for every $x, y \in E$, $|x| \leq |y|$ implies $\|x\| \leq \|y\|$. The Banach lattice E is said to be *strictly monotone* if for every $x, y \in E$ $|x| > |y|$ implies $\|x\| > \|y\|$. It is known that every strictly convex Banach lattice is strictly monotone. A Banach space is said to be *uniformly convex* or *uniformly rotund* if for any two sequences $\{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty$ in the unit ball $B(X)$ of X, $\lim\limits_{n\to\infty} \|x_n + y_n\| = 2$ implies $\lim\limits_{n\to\infty} \|x_n - y_n\| = 0$. Moreover this definition is equivalent to the following ([14], p.441), for every $\varepsilon > 0$ there is a unique $\delta > 0$, such that for all x, y in Y the conditions $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$ imply $\frac{\|x+y\|}{2} \leq 1 - \delta$. Clearly, every uniformly convex Banach space is strictly convex. For a Banach space X we denote by δ_X the modules of convexity

$$\delta_X(\varepsilon) := \inf\{1 - \frac{1}{2}\|x + y\| : x, y \in X;\ \|x\| = \|y\| = 1;\ \|x - y\| > \varepsilon\}$$

for any $\varepsilon \in [0, 2]$. Note that X is uniformly convex if and only if $\delta_X(\varepsilon) > 0$ whenever $\varepsilon > 0$. If X uniformly convex, we define the characteristic of convexity by

$$\chi_r(X) := \sup\{\varepsilon \in [0, 2];\ \delta_X(\varepsilon) \leq r\}.$$

Let $\mathcal{X}$ be a measurable bundle of Banach spaces and almost every fiber is uniformly convex. For $\mathcal{X}$ we define the characteristic of convexity by

$$\chi_r(\mathcal{X}) := \sup\{\chi_r(\mathcal{X}_t) : \ \mathcal{X}_t \text{ is uniformly convex}\}.$$

Let $\Omega_0 \subset \Omega$ be a measurable set such that $\mu(\Omega \backslash \Omega_0) = 0$ and Banach spaces $\mathcal{X}_t$ are uniformly convex for all $t \in \Omega_0$. Let $\preccurlyeq$ be a complete order on Ω_0. For all measurable bundles of Banach spaces we consider below there exists a function

$$\begin{aligned} &\phi : \Omega_0 \times \mathbb{R}_+ \to \mathbb{R}_+ \text{ such that } \phi(t, r) = \chi_r(\mathcal{X}_t) \\ &\text{and } \phi(t, \cdot) \le \phi(t', \cdot) \text{ for every } t, t' \in \Omega, t \preccurlyeq t'. \end{aligned} \tag{1}$$

3. Strict and uniform convexity

For the rest of the text (Ω, Σ, μ) is assumed to be a finite measure space, E is a Köthe function space over (Ω, Σ, μ). Let us prove some preliminary results which will allow us in Theorem 3.5 to obtain a characterization of the uniform convexity of measurable section space $E(\mathcal{X})$.

Lemma 3.1. *If $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ are sequences in the $E(\mathcal{X})$ with $|||f_n||| = |||g_n||| = 1$ for every $n \in \mathbb{N}$ and $|||f_n + g_n||| \to 2$ then*

$$\|\|f_n(\cdot)\|_{\mathcal{X}(\cdot)} + \|g_n(\cdot)\|_{\mathcal{X}(\cdot)}\|_E \to 2$$

Proof. Using the following inequalities we get the required result

$$\begin{aligned} |||f_n + g_n||| &= \|\|f_n(\cdot) + g_n(\cdot)\|_{\mathcal{X}(\cdot)}\|_E \le \\ &\le \|\|f_n(\cdot)\|_{\mathcal{X}(\cdot)} + \|g_n(\cdot)\|_{\mathcal{X}(\cdot)}\|_E \le \\ &\le \|\|f_n(\cdot)\|_{\mathcal{X}(\cdot)}\|_E + \|\|g_n(\cdot)\|_{\mathcal{X}(\cdot)}\|_E = |||f_n||| + |||g_n||| = 2. \end{aligned}$$

□

Lemma 3.2. *Let $\mathcal{X}$ be a measurable bundle of Banach spaces. If $(E, \|\cdot\|_E)$ is uniformly convex Köthe space and $(f_n)_{n=1}^{\infty}$, $(g_n)_{n=1}^{\infty}$ are sequences in the $E(\mathcal{X})$ with $|||f_n||| = |||g_n||| = 1$ for every $n \in \mathbb{N}$ and $|||f_n + g_n||| \to 2$ then,*

$$\lim_{n\to\infty} \|\|f_n(\cdot)\|_{\mathcal{X}(\cdot)} - \|g_n(\cdot)\|_{\mathcal{X}(\cdot)}\|_E = 0; \tag{2}$$

$$\lim_{n\to\infty} \|\|f_n(\cdot)\|_{\mathcal{X}(\cdot)} + \|g_n(\cdot)\|_{\mathcal{X}(\cdot)} - \|f_n(\cdot) + g_n(\cdot)\|_{\mathcal{X}(\cdot)}\|_E = 0 \tag{3}$$

Proof. Note that $(\|f_n(\cdot)\|_{\mathcal{X}(\cdot)})_{n=1}^{\infty}$ and $(\|g_n(\cdot)\|_{\mathcal{X}(\cdot)})_{n=1}^{\infty}$ are sequences in E with

$$\|\|f_n(\cdot)\|_{\mathcal{X}(\cdot)}\|_E = \|\|g_n(\cdot)\|_{\mathcal{X}(\cdot)}\|_E = 1.$$

Using lemma 3.1 and the fact that E is uniformly convex Banach space, it is straightforward to get (3.1). To prove (3.2) we note the inequalities

$$\begin{aligned} 2|||f_n + g_n||| &= 2\|\|f_n(\cdot) + g_n(\cdot)\|_{\mathcal{X}(\cdot)}\|_E \le \\ &\le \|\|f_n(\cdot)\|_{\mathcal{X}(\cdot)} + \|g_n(\cdot)\|_{\mathcal{X}(\cdot)} + \|f_n(\cdot) + g_n(\cdot)\|_{\mathcal{X}(\cdot)}\|_E \le \\ &\le 2(\|\|f_n(\cdot)\|_{\mathcal{X}(\cdot)}\|_E + \|\|g_n(\cdot)\|_{\mathcal{X}(\cdot)}\|_E) = 2(|||f_n||| + |||g_n|||) = 4. \end{aligned}$$

In this line, we get

$$\Big\|\frac{\|f_n(\cdot)\|_{\mathcal{X}(\cdot)}}{2} + \frac{\|g_n(\cdot)\|_{\mathcal{X}(\cdot)}}{2} + \frac{\|f_n(\cdot) + g_n(\cdot)\|_{\mathcal{X}(\cdot)}}{2}\Big\|_E \to 2.$$

Using the uniform convexity of E and the facts that

$$\Big\|\frac{\|f_n(\cdot)\|_{\mathcal{X}(\cdot)}}{2} + \frac{\|g_n(\cdot)\|_{\mathcal{X}(\cdot)}}{2}\Big\|_E \le 1$$

and

$$\Big\|\frac{\|f_n(\cdot)+g_n(\cdot)\|_{\mathcal{X}(\cdot)}}{2}\Big\|_E \le 1$$

result (3.2) is completely proved. □

Lemma 3.3. *Let $\mathcal{X}$ be a measurable bundle of Banach spaces and almost every all fibers $\mathcal{X}_t$ are uniformly convex. If $f, g \in E(\mathcal{X})$ then for every $\varepsilon \in (0,1)$*

$$(4)\qquad |||f-g||| \le \chi_\varepsilon(\mathcal{X})(|||f|||+|||g|||)+\frac{2}{\varepsilon}(\|\|f(\cdot)\|_{\mathcal{X}(\cdot)}-\|g(\cdot)\|_{\mathcal{X}(\cdot)}\|_E + \|\|f(\cdot)\|_{\mathcal{X}(\cdot)}+\|g(\cdot)\|_{\mathcal{X}(\cdot)}-\|f(\cdot)+g(\cdot)\|_{\mathcal{X}(\cdot)}\|_E)$$

Proof. Holding $t \in \Omega$ fixed and letting $\varepsilon > 0$, we have two inequalities

$$(5)\qquad \Big|\, \|f(t)\|_{\mathcal{X}(t)} - \|g(t)\|_{\mathcal{X}(t)} \Big| \le \varepsilon \max\{\|f(t)\|_{\mathcal{X}(t)},\, \|g(t)\|_{\mathcal{X}(t)}\};$$

$$(6)\qquad \|f(t)\|_{\mathcal{X}(t)} + \|g(t)\|_{\mathcal{X}(t)} - \|f(t)+g(t)\|_{\mathcal{X}(t)} \le \varepsilon \max\{\|f(t)\|_{\mathcal{X}(t)},\, \|g(t)\|_{\mathcal{X}(t)}\}.$$

Let us consider three cases. The case 1, $t \in \Omega_1$, (3.4) and (3.5) are true. Assuming that $\|f(t)\|_{\mathcal{X}_t} \ge \|g(t)\|_{\mathcal{X}_t}$ and letting $\overline{f}(t) = \frac{f(t)}{\|f(t)\|_{\mathcal{X}_t}}$ $\overline{g}(t) = \frac{g(t)}{\|f(t)\|_{\mathcal{X}_t}}$, we find that $\|\overline{g}(t)\|_{\mathcal{X}_t} \le \|\overline{f}(t)\|_{\mathcal{X}_t} = 1$. Furthemore, it is straightforward to verify that

$$\|\overline{g}(t)+\overline{f}(t)\|_{\mathcal{X}_t} \ge \frac{\|f(t)\|_{\mathcal{X}(t)} + \|g(t)\|_{\mathcal{X}(t)} - \varepsilon\|f(t)\|_{\mathcal{X}_t}}{\|f(t)\|_{\mathcal{X}_t}} =$$

$$= 1 + \frac{\|g(t)\|_{\mathcal{X}_t}}{\|f(t)\|_{\mathcal{X}_t}} - \varepsilon \ge 1 + \frac{\|f(t)\|_{\mathcal{X}_t} - \varepsilon\|f(t)\|_{\mathcal{X}_t}}{\|f(t)\|_{\mathcal{X}_t}} - \varepsilon = 2 - 2\varepsilon$$

and, since $\mathcal{X}_t$ is uniformly convex and $f(t), g(t) \in \mathcal{X}_t$ find that

$$\|\overline{g}(t) - \overline{f}(t)\|_{\mathcal{X}_t} \le \chi_\varepsilon(\mathcal{X}_t) \le \chi_\varepsilon(\mathcal{X}).$$

Therefore we deduce that

$$\|f(t)-g(t)\|_{\mathcal{X}_t} \le \chi_\varepsilon(\mathcal{X}) \max\{\|f(t)\|_{\mathcal{X}(t)},\, \|g(t)\|_{\mathcal{X}(t)}\} \le$$
$$\le \chi_\varepsilon(\mathcal{X})(\|f(t)\|_{\mathcal{X}(t)} + \|g(t)\|_{\mathcal{X}(t)}).$$

The case 2, $t \in \Omega_2$, (3.4) is not true. We imply that if

$$|\,\|f(t)\|_{\mathcal{X}(t)} - \|g(t)\|_{\mathcal{X}(t)}| > \varepsilon \max\{\|f(t)\|_{\mathcal{X}(t)},\, \|g(t)\|_{\mathcal{X}(t)}\}.$$

And so

$$\|f(t)-g(t)\|_{\mathcal{X}_t} \le 2\max\{\|f(t)\|_{\mathcal{X}(t)},\, \|g(t)\|_{\mathcal{X}(t)}\} <$$
$$< \frac{2}{\varepsilon}\|f(t)\|_{\mathcal{X}_t} - \|g(t)\|_{\mathcal{X}_t}.$$

The case 3, $t \in \Omega_3$, (3.5) is not true. We can get that

$$\|f(t)-g(t)\|_{\mathcal{X}_t} \le 2\max\{\|f(t)\|_{\mathcal{X}(t)},\, \|g(t)\|_{\mathcal{X}(t)}\} <$$
$$< \frac{2}{\varepsilon}(\|f(t)\|_{\mathcal{X}_t} + \|g(t)\|_{\mathcal{X}_t} - \|f(t)-g(t)\|_{\mathcal{X}_t}).$$

It is clear that $\Omega = \bigcup_{i=1}^{3} \Omega_i$. Then we have

$$|||f-g||| = \|\|f(\cdot)-g(\cdot)\|_{\mathcal{X}(\cdot)}\|_E =$$
$$= \|\mathbf{1}_{\Omega}\|f(\cdot)-g(\cdot)\|_{\mathcal{X}(\cdot)}\|_E \leq \|\mathbf{1}_{\Omega_1}\|f(\cdot)-g(\cdot)\|_{\mathcal{X}(\cdot)}\|_E +$$
$$+\|\mathbf{1}_{\Omega_2}\|f(\cdot)-g(\cdot)\|_{\mathcal{X}(\cdot)}\|_E + \|\mathbf{1}_{\Omega_3}\|f(\cdot)-g(\cdot)\|_{\mathcal{X}(\cdot)}\|_E \leq$$
$$\leq \|\mathbf{1}_{\Omega_1}\chi_\varepsilon(\mathcal{X})(\|f(\cdot)\|_{\mathcal{X}(\cdot)} + \|g(\cdot)\|_{\mathcal{X}(\cdot)})\|_E + \|\mathbf{1}_{\Omega_2}\frac{2}{\varepsilon}\|f(\cdot)\|_{\mathcal{X}(\cdot)} - \|g(\cdot)\|_{\mathcal{X}(\cdot)}\|_E +$$
$$+\|\mathbf{1}_{\Omega_3}\frac{2}{\varepsilon}(\|f(\cdot)\|_{\mathcal{X}(\cdot)} + \|g(\cdot)\|_{\mathcal{X}(\cdot)} - \|f(\cdot)-g(\cdot)\|_{\mathcal{X}(\cdot)})\|_E \leq$$
$$\chi_\varepsilon(\mathcal{X})(\|\|f(\cdot)\|_{\mathcal{X}(\cdot)} + \|g(\cdot)\|_{\mathcal{X}(\cdot)}\|_E) + \frac{2}{\varepsilon}\|\|f(\cdot)\|_{\mathcal{X}(\cdot)} - \|g(\cdot)\|_{\mathcal{X}(\cdot)}\|_E +$$
$$\frac{2}{\varepsilon}(\|\|f(\cdot)\|_{\mathcal{X}(\cdot)} + \|g(\cdot)\|_{\mathcal{X}(\cdot)} - \|f(\cdot)-g(\cdot)\|_{\mathcal{X}(\cdot)}\|_E) =$$
$$= \chi_\varepsilon(\mathcal{X})(|||f+g|||) + \frac{2}{\varepsilon}(\|\|f(\cdot)\|_{\mathcal{X}(\cdot)} - \|g(\cdot)\|_{\mathcal{X}(\cdot)}\|_E +$$
$$+\|\|f(\cdot)\|_{\mathcal{X}(\cdot)} + \|g(\cdot)\|_{\mathcal{X}(\cdot)} - \|f(\cdot)-g(\cdot)\|_{\mathcal{X}(\cdot)}\|_E).$$

and lemma is completely proved. $\square$

Lemma 3.4. *Let $\mathcal{X}$ — measurable bundle of Banach spaces and almost every all fibers are uniformly convex. Then for every $\varepsilon > 0$ there exist $r_0 > 0$, such that $\chi_r(\mathcal{X}) < \varepsilon$ for every $r \leq r_0$.*

Proof. Assume the converse. Then there exist a $\varepsilon > 0$ such that $\chi_r(\mathcal{X}) > \varepsilon$ for every $r > 0$. But this contradicts 2.1. $\square$

In this line, we are able to introduce the following theorem about the uniform convexity of the measurable section space $E(\mathcal{X})$.

Theorem 3.5. *Let $\mathcal{X}$ be a measurable bundle of Banach spaces. If E is a uniformly convex and almost every all fibers $\mathcal{X}_t$ are uniformly convex then $E(\mathcal{X})$ is uniformly convex. If $E(\mathcal{X})$ is uniformly convex then E is uniformly convex and every fiber $\mathcal{X}_t$ such that $\rho(\mathcal{X}_t) > 0$ is uniformly convex.*

Proof. Suppose almost every all $\mathcal{X}_t$ and E are uniformly convex spaces. Let (f_n) and (g_n) be sequences in $E(\mathcal{X})$ with $|||f_n||| = |||g_n||| = 1$ for every $n \in \mathbb{N}$ and $|||f_n + g_n||| \to \infty$, then by Lemma 3.2 it follows that

$$\|\|f_n(\cdot)\|_{\mathcal{X}(\cdot)} - \|g_n(\cdot)\|_{\mathcal{X}(\cdot)}\|_E \to 0,\ n \to \infty;$$
$$\|f_n(\cdot)\|_{\mathcal{X}(\cdot)} + \|g_n(\cdot)\|_{\mathcal{X}(\cdot)} - \|f_n(\cdot) + g_n(\cdot)\|_{\mathcal{X}(\cdot)}\|_E \to 0,\ n \to \infty.$$

Take $\varepsilon > 0$ and using Lemma 3.4 we can choose $r > 0$ such that $\chi_r(\mathcal{X}) \leq \frac{\varepsilon}{4}$. Choose $n_0 \in \mathbb{N}$ sufficiently large so that for all $n > n_0$

$$\|\|f_n(\cdot)\|_{\mathcal{X}(\cdot)} - \|g_n(\cdot)\|_{\mathcal{X}(\cdot)}\|_E + \|\|f_n(\cdot)\|_{\mathcal{X}(\cdot)} +$$
$$+\|g_n(\cdot)\|_{\mathcal{X}(\cdot)} - \|f_n(\cdot) + g_n(\cdot)\|_{\mathcal{X}(\cdot)}\|_E < \frac{\varepsilon^2}{4}$$

From Lemma 3.3 it follows that

$$|||f_n - g_n||| \leq \chi_r(\mathcal{X})(|||f_n||| + |||g_n|||) + \frac{2}{\varepsilon}(\|\|\|f_n(\cdot)\|_{\mathcal{X}(\cdot)} - \|g_n(\cdot)\|_{\mathcal{X}(\cdot)}\|_E +$$

$$\|\|\|f_n(\cdot)\|_{\mathcal{X}(\cdot)} + \|g_n(\cdot)\|_{\mathcal{X}(\cdot)} - \|f_n(\cdot) + g_n(\cdot)\|_{\mathcal{X}(\cdot)}\|_E) < 2\chi_r(\mathcal{X}) + \frac{2}{\varepsilon}\frac{\varepsilon^2}{4} \leq$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Consequently $|||f_n - g_n||| \to 0$, which means that $E(\mathcal{X})$ is a uniformly convex Banach space.

Suppose $E(\mathcal{X})$ is a uniformly convex Banach space. Let ξ be a measurable section such that $\|\xi(\cdot)\|_{\mathcal{X}(\cdot)} = \mathbf{1}_\Omega(\cdot)$. Then exists the linear isometry $H : E \to E(\mathcal{X})$ such that $H(e) = e\xi$. Every fiber $\mathcal{X}_{t_0}$ so that $\rho(\mathcal{X}_{t_0}) > 0$ is isometrically embedded into $E(\mathcal{X})$ by map $x \mapsto \widetilde{x}\frac{1}{\|\mathbf{1}_A\|_E}$, where A is a measurable set and for every $t \in A$ the Banach space $\mathcal{X}_t$ is isomorphic to $\mathcal{X}_{t_0}$. Using the fact that uniform convexity inherited by subspaces, we deduce that both spaces E and $\mathcal{X}_{t_0}$ are uniformly convex, and thereby the theorem is completely proved. $\square$

Theorem 3.6. *Let $\mathcal{X}$ be a measurable bundle of Banach spaces. If E is a strictly convex and almost every all fibers $\mathcal{X}_t$ are strictly convex then $E(\mathcal{X})$ is strictly convex. If $E(\mathcal{X})$ is strictly convex then E is strictly convex and every fiber $\mathcal{X}_t$ such that $\rho(\mathcal{X}_t) > 0$ is strictly convex.*

Proof. Suppose $E(\mathcal{X})$ is strictly convex Banach space. Since spaces $\mathcal{X}_t$, $\rho(\mathcal{X}_t) > 0$ and E are embedded isometrically into $E(\mathcal{X})$, and due to the fact that strict convexity inherited by subspaces, we deduce that spaces $\mathcal{X}_t$ and E are strictly convex.

We show that $E(\mathcal{X})$ is strictly convex if E and almost every all fibers $\mathcal{X}_t$ are strictly convex. Let f_1 and f_2 be two unit vectors in $E(\mathcal{X})$ such that $|||f_1 + f_2||| = 2$. Then

$$\|\|f_1(\cdot)\|_{\mathcal{X}(\cdot)}\|_E = \|\|f_2(\cdot)\|_{\mathcal{X}(\cdot)}\|_E = 1$$

and

$$2 = \|\|(f_1 + f_2)(\cdot)\|_{\mathcal{X}(\cdot)}\|_E \leq \|\|f_1(\cdot)\|_{\mathcal{X}(\cdot)} + \|f_2(\cdot)\|_{\mathcal{X}(\cdot)}\|_E \leq$$
$$\leq \|\|f_1(\cdot)\|_{\mathcal{X}(\cdot)}\|_E + \|\|f_2(\cdot)\|_{\mathcal{X}(\cdot)}\|_E = 2.$$

Since E is strictly convex, for almost all $t \in \Omega$

$$\|f_1(t)\|_{\mathcal{X}(t)} = \|f_2(t)\|_{\mathcal{X}(t)}.$$

Using the fact that strictly convex Banach lattice E is a strictly monotone, for almost all $t \in \Omega$ we have

$$\|f_1(t)\|_{\mathcal{X}(t)} = \|f_2(t)\|_{\mathcal{X}(t)} = \left\|\frac{(f_1 + f_2)}{2}(t)\right\|_{\mathcal{X}(t)}.$$

Since almost every all fibers $\mathcal{X}(t)$, $t \in \Omega$ are strictly convex, $f_1(t) = f_2(t)$ for almost every all $t \in \Omega$ and $f_1 = f_2$. $\square$

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