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Hyers-Ulam Stability of an Additive Functional Inequality

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Abstract. In this paper, we prove the generalized Hyers-Ulam stability of the additive functional inequality

$$\|f(x+y)+f(x-y)+f(z)\| \le \|f(2x+z)\|$$

in Banach spaces.

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1. Introduction and preliminaries

In 1940, S.M. Ulam [4] suggested the stability problem of functional equations concerning the stability of group homomorphisms as follows: *Let* $(\mathcal{G}, \circ)$ *be a group and let* $(\mathcal{H}, \star, d)$ *be a metric group with the metric* $d(\cdot,\cdot)$. *Given* $\varepsilon > 0$, *does there exist a* $\delta = \delta(\varepsilon) > 0$ *such that if a mapping* $f : \mathcal{G} \to \mathcal{H}$ *satisfies the inequality*

$$d\big(f(x \circ y), f(x) \star f(y)\big) < \delta$$

for all $x, y \in \mathcal{G}$, *then a homomorphism* $F : \mathcal{G} \to \mathcal{H}$ *exits with*

$$d\big(f(x), F(x)\big) < \varepsilon$$

for all $x \in \mathcal{G}$*?*

In the next year, D.H. Hyers [1] gave a first (partial) affirmative answer to the question of Ulam for Banach spaces as follows: *If* $\delta > 0$ *and if* $f : \mathcal{E} \to \mathcal{F}$ *is a mapping between Banach spaces* $\mathcal{E}$ *and* $\mathcal{F}$ *satisfying*

$$\big\|f(x+y) - f(x) - f(y)\big\| \le \delta$$

for all $x, y \in \mathcal{E}$*, then there is a unique additive mapping* $A : \mathcal{E} \to \mathcal{F}$ *such that*

$$\|f(x) - A(x)\| \le \delta$$

for all $x, y \in \mathcal{E}$*.*

Thereafter, we call that type the Hyers-Ulam stability.

2. Hyers-Ulam stability in Banach spaces

Throughout this paper, let $\mathcal{X}$ be a normed linear space and $\mathcal{Y}$ a Banach space. In 2007, C. Park, Y. S. Cho and M.-H. Han [3] proved the generalized Hyers-Ulam stability of the additive functional inequality

$$\|f(x) + f(y) + f(z)\| \le \|f(x + y + z)\|$$

in Banach spaces. In 2011, J. R. Lee, C. Park and D. Y. Shin [2] prove the generalized Hyers-Ulam stability of the additive functional inequality

$$\|f(2x) + f(2y) + 2f(z)\| \le \|2f(x + y + z)\|$$

in Banach spaces. In this paper, we prove the generalized Hyers-Ulam stability of the additive functional inequality

$$\|f(x + y) + f(x - y) + f(z)\| \le \|f(2x) + f(z)\|$$

in Banach spaces.

Lemma 2.1. *Let* $f : \mathcal{X} \to \mathcal{Y}$ *be a mapping. Then it is additive if and only if it satisfies*

$$\|f(x + y) + f(x - y) + f(z)\| \le \|f(2x + z)\| \tag{1}$$

for all $x, y, z \in \mathcal{X}$*.*

Proof. If f is additive, then clearly

$$\|f(x + y) + f(x - y) + f(z)\| = \|f(2x + z)\|$$

for all $x, y, z \in \mathcal{X}$.

Assume that f satisfies (1). Letting $x = y = z = 0$ in (1), we gain $\|3f(0)\| \le \|f(0)\|$ and so $f(0) = 0$. Putting $x = z = 0$ in (1), we get

$$\|f(y) + f(-y)\| \le \|f(0)\| = 0$$

and so $f(-y) = -f(y)$ for all $y \in \mathcal{X}$. Replacing x and y by $\frac{x+y}{2}$ and $\frac{x-y}{2}$, respectively, and setting $z = -x - y$ in (1), we have

$$\|f(x) + f(y) - f(x + y)\| \le \|f(0)\| = 0$$

for all $x, y \in \mathcal{X}$. Thus we obtain

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in \mathcal{X}$. □

Theorem 2.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping with $f(0) = 0$. If there is a function $\varphi : X^3 \to [0, \infty)$ satisfying*

$$\|f(x+y) + f(x-y) + f(z)\| \le \|f(2x+z)\| + \varphi(x,y,z) \tag{2}$$

and

$$\widetilde{\varphi}(x,y,z) := \sum_{j=0}^{\infty} \frac{1}{2^j}\varphi\big((-2)^j x, (-2)^j y, (-2)^j z\big) < \infty \tag{3}$$

for all $x, y, z \in \mathcal{X}$, then there exists a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - A(x)\| \le \frac{1}{2}\widetilde{\varphi}(x, 0, -2x) \tag{4}$$

for all $x \in \mathcal{X}$.

Proof. Replacing x, y, z by $(-2)^n x, 0, (-2)^{n+1}x$, respectively, and dividing by 2^{n+1} in (2), since $f(0) = 0$, we get

$$\left\| \frac{f\big((-2)^{n+1}x\big)}{(-2)^{n+1}} - \frac{f\big((-2)^n x\big)}{(-2)^n} \right\| \le \frac{1}{2^{n+1}}\varphi\big((-2)^n x, 0, (-2)^{n+1}x\big)$$

for all $x \in \mathcal{X}$ and all nonnegative integers n. From the above inequality, we have

$$\begin{aligned} \left\| \frac{f\big((-2)^n x\big)}{(-2)^n} - \frac{f\big((-2)^m x\big)}{(-2)^m} \right\| &\le \sum_{j=m}^{n-1} \left\| \frac{f\big((-2)^{j+1}x\big)}{(-2)^{j+1}} - \frac{f\big((-2)^j x\big)}{(-2)^j} \right\| \\ &\le \sum_{j=m}^{n-1} \frac{1}{2^{j+1}}\varphi\big((-2)^j x, 0, (-2)^{j+1}x\big) \end{aligned} \tag{5}$$

for all $x \in \mathcal{X}$ and all nonnegative integers m, n with $m < n$. By the condition (3), the sequence $\left\{\frac{f((-2)^n x)}{(-2)^n}\right\}$ is a Cauchy sequence for all $x \in \mathcal{X}$. Since $\mathcal{Y}$ is complete, the sequence $\left\{\frac{f((-2)^n x)}{(-2)^n}\right\}$ converges for all $x \in \mathcal{X}$. So one can define a mapping $A : \mathcal{X} \to \mathcal{Y}$ by

$$A(x) := \lim_{n\to\infty} \frac{f\big((-2)^n x\big)}{(-2)^n}$$

for all $x \in \mathcal{X}$. Taking $m = 0$ and letting n tend to ∞ in (5), we have the inequality (4).

Replacing x, y, z by $(-2)^n x, (-2)^n y, (-2)^n z$, respectively, and dividing by 2^n in (2), we obtain

$$\begin{aligned} &\left\| \frac{f\big((-2)^n(x+y)\big)}{(-2)^n} + \frac{f\big((-2)^n(x-y)\big)}{(-2)^n} + \frac{f\big((-2)^n z\big)}{(-2)^n} \right\| \\ &\qquad \le \left\| \frac{f\big((-2)^n(2x+z)\big)}{(-2)^n} \right\| + \frac{1}{2^n}\varphi\big((-2)^n x, (-2)^n y, (-2)^n z\big) \end{aligned}$$

for all $x, y, z \in \mathcal{X}$ and all nonnegative integers n. Since (3) gives that

$$\lim_{n\to\infty} \frac{1}{2^n}\varphi\big((-2)^n x, (-2)^n y, (-2)^n z\big) = 0$$

for all $x, y, z \in \mathcal{X}$, letting n tend to ∞ in the above inequality, we see that A satisfies the inequality (1) and so it is additive by Lemma 2.1.

Let $A' : \mathcal{X} \to \mathcal{Y}$ be another additive mapping satisfying (4). Since both A and A' are additive, we have

$$\begin{aligned}
\big\|A(x) - A'(x)\big\| &= \frac{1}{2^n}\big\|A\big((-2)^n x\big) - A'\big((-2)^n x\big)\big\| \\
&\le \frac{1}{2^n}\Big(\big\|A\big((-2)^n x\big) - f\big((-2)^n x\big)\big\| + \big\|f\big((-2)^n x\big) - A'\big((-2)^n x\big)\big\|\Big) \\
&\le \frac{1}{2^n}\widetilde{\varphi}\big((-2)^n x, 0, (-2)^{n+1} x\big) \\
&= \sum_{j=n}^{\infty} \frac{1}{2^j}\varphi\big((-2)^j x, 0, (-2)^{j+1} x\big)
\end{aligned}$$

which goes to zero as $n \to \infty$ for all $x \in \mathcal{X}$ by (3). Therefore, A is a unique additive mapping satisfying (4), as desired. □

Theorem 2.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping. If there is a function $\varphi : X^3 \to [0, \infty)$ satisfying (2) and*

$$\widetilde{\varphi}(x, y, z) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{(-2)^j}, \frac{y}{(-2)^j}, \frac{z}{(-2)^j}\right) < \infty \tag{6}$$

for all $x, y, z \in \mathcal{X}$, then there exists a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\big\|f(x) - A(x)\big\| \le \frac{1}{2}\widetilde{\varphi}(x, 0, -2x) \tag{7}$$

for all $x \in \mathcal{X}$.

Proof. Replacing x, y, z by $\frac{x}{(-2)^n}, 0, \frac{x}{(-2)^{n-1}}$, respectively, and multiplying by 2^{n-1} in (2), since $f(0) = 0$, we have

$$\left\|(-2)^n f\left(\frac{x}{(-2)^n}\right) - (-2)^{n-1} f\left(\frac{x}{(-2)^{n-1}}\right)\right\| \le 2^{n-1}\varphi\left(\frac{x}{(-2)^n}, 0, \frac{x}{(-2)^{n-1}}\right)$$

for all $x \in \mathcal{X}$ and all $n \in \mathbb{N}$. From the above inequality, we get

$$(8)\quad \begin{aligned} &\left\| (-2)^n f\left(\frac{x}{(-2)^n}\right) - (-2)^m f\left(\frac{x}{(-2)^m}\right) \right\| \\ &\qquad \le \sum_{j=m+1}^{n} \left\| (-2)^j f\left(\frac{x}{(-2)^j}\right) - (-2)^{j-1} f\left(\frac{x}{(-2)^{j-1}}\right) \right\| \\ &\qquad \le \sum_{j=m+1}^{n} 2^{j-1} \varphi\left(\frac{x}{(-2)^j}, 0, \frac{x}{(-2)^{j-1}}\right) \end{aligned}$$

for all $x \in \mathcal{X}$ and all nonnegative integers m, n with $m < n$. From (6), the sequence $\left\{(-2)^n f\left(\frac{x}{(-2)^n}\right)\right\}$ is a Cauchy sequence for all $x \in \mathcal{X}$. Since $\mathcal{Y}$ is complete, the sequence $\left\{(-2)^n f\left(\frac{x}{(-2)^n}\right)\right\}$ converges for all $x \in \mathcal{X}$. So one can define a mapping $A : \mathcal{X} \to \mathcal{Y}$ by

$$A(x) := \lim_{n\to\infty} (-2)^n f\left(\frac{x}{(-2)^n}\right)$$

for all $x \in \mathcal{X}$. To prove that A satisfies (7), putting $m = 0$ and letting $n \to \infty$ in (8), we have

$$\|f(x) - A(x)\| \le \sum_{j=1}^{\infty} 2^{j-1} \varphi\left(\frac{x}{(-2)^j}, 0, \frac{x}{(-2)^{j-1}}\right) = \frac{1}{2}\widetilde{\varphi}(x, 0, -2x)$$

for all $x \in \mathcal{X}$.

Replacing x, y, z by $\frac{x}{(-2)^n}, \frac{y}{(-2)^n}, \frac{z}{(-2)^n}$, respectively, and multiplying by 2^n in (2), we obtain

$$\begin{aligned} &\left\| (-2)^n f\left(\frac{x+y}{(-2)^n}\right) + (-2)^n f\left(\frac{x-y}{(-2)^n}\right) + (-2)^n f\left(\frac{z}{(-2)^n}\right) \right\| \\ &\qquad \le \left\| (-2)^n f\left(\frac{2x+z}{(-2)^n}\right) \right\| + 2^n \varphi\left(\frac{x}{(-2)^n}, \frac{y}{(-2)^n}, \frac{z}{(-2)^n}\right) \end{aligned}$$

for all $x, y, z \in \mathcal{X}$ and all nonnegative integers n. Since (6) gives that

$$\lim_{n\to\infty} 2^n \varphi\left(\frac{x}{(-2)^n}, \frac{y}{(-2)^n}, \frac{z}{(-2)^n}\right) = 0$$

for all $x, y, z \in \mathcal{X}$, if we let $n \to \infty$ in the above inequality, then we have

$$\big\|A(x+y) + A(x-y) + A(z)\big\| \le \big\|A(2x+z)\big\|$$

for all $x, y, z \in \mathcal{X}$. By Lemma 2.1, the mapping A is additive. The rest of the proof is similar to the corresponding part of the proof of Theorem 2.2. □

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