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# $n$-Tuples and Chaoticity

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**Abstract**

In this paper we characterize the Condition for Chaoticity of Tuples of operators on a Frechet space.

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**Keywords:** Hypercyclic vector, Hypercyclicity Criterion, Periodic vector, Chaotic $n$-tuple, $n$-Tuple of operators

## 1 Introduction

Let $T_1, T_2, ..., T_n$ be commutative bounded linear operators on a Banach space $\mathcal{X}$. For $n$-Tuple $\mathcal{T} = (T_1, T_2, ..., T_n)$, put

$$\Gamma = \{T_1^{m_1} T_2^{m_2} ... T_n^{m_n} : m_1, m_2, ..., m_n \geq 0\}$$

the semigroup generated by $\mathcal{T}$. For $x \in \mathcal{X}$, the orbit of $x$ under $\mathcal{T}$ is the set $Orb(\mathcal{T}, x) = \{S(x) : S \in \Gamma\}$, that is

$$Orb(\mathcal{T}, x) = \{T_1^{m_1} T_2^{m_2} ... T_n^{m_n}(x) : m_1, m_2, ..., m_n \geq 0\}$$

The vector $x$ is called Hypercyclic vector for $\mathcal{T}$ and $n$-Tuple $\mathcal{T}$ is called Hypercyclic $n$-Tuple, if the set $Orb(\mathcal{T}, x)$ is dense in $\mathcal{X}$, that is

$$\overline{Orb(\mathcal{T}, x)} = \overline{\{T_1^{m_1} T_2^{m_2} ... T_2^{m_n}(x) : m_1, m_2, ..., m_n \geq 0\}} = \mathcal{X}$$

The vector $x$ in $\mathcal{X}$ is called a Periodic vector for the $n$-Tuple $\mathcal{T} = (T_1, T_2, ..., T_n)$, if there exist some numbers $\mu_1, \mu_2, ..., \mu_n \in \mathcal{N}$ such that

$$T_1^{\mu_1} T_2^{\mu_2} ... T_n^{\mu_n}(x) = x.$$

Also the $n$-Tuple $\mathcal{T} = (T_1, T_2, ..., T_n)$, is called chaotic tuple, if we have tree below conditions together,
(1). It is topologically transitive, that is, if for any given open sets $\mathcal{U}$ and $\mathcal{V}$, there exist positive integer numbers $\alpha_1, \alpha_2, ..., \alpha_n \in \mathcal{N}$ such that

$$T_1^{\alpha_1} T_2^{\alpha_2} ... T_n^{\alpha_n}(\mathcal{U}) \cap \mathcal{V} \neq \phi$$

(2). It has a dense set of periodic points, in other word, there is a set $\mathcal{X}$ such that for each $x \in \mathcal{X}$, there exist some numbers $\beta_1, \beta_2, ..., \beta_n \in \mathcal{N}$ such that

$$T_1^{\beta_1} T_2^{\beta_2} ... T_n^{\beta_n}(x) = x$$

(3). It has a certain property called sensitive dependence on initial conditions.

Notice that, all operators in this paper are commutative. For some topics see [1–12].

# 2 Main Results

**Theorem 2.1.[The Hypercyclicity Criterion]** Let $\mathcal{X}$ be a separable Banach space and $\mathcal{T} = (T_1, T_2, ..., T_n)$ is an $n$-tuple of continuous linear mappings on $\mathcal{X}$. If there exist two dense subsets $\mathcal{Y}$ and $\mathcal{Z}$ in $\mathcal{X}$, and $n$ strictly increasing sequences $\{m_{j,1}\}$, $\{m_{j,2}\}$, ...,$\{m_{j,n}\}$ such that :
1. $T_1^{m_{j,1}} T_2^{m_{j,2}} ... T_n^{m_{j,n}} \to 0$ on $\mathcal{Y}$ as $j \to \infty$,
2. There exist function $\{S_j : Z \to \mathcal{X}\}$ such that for every $z \in Z$, $S_j z \to 0$, and $T_1^{m_{j,1}} T_2^{m_{j,2}} ... T_n^{m_{j,n}} S_j z \to \ z$,
then $\mathcal{T}$ is a Hypercyclic $n$-tuple.
If the tuple $T$ satisfying the hypothesis of previous theorem then we say that T satisfying the Hypercyclicity criterion.

**Theorem 2.2.** Suppose $\mathcal{X}$ be an F-sequence space whit the unconditional basis $\{e_\kappa\}_{\kappa \in \mathcal{N}}$. Let $T_1, T_2, ..., T_n$ are unilateral weighted backward shifts with weight sequence $\{a_{i,1} : i \in \mathcal{N}\}$, $\{a_{i,2} : i \in \mathcal{N}\}$,..., $\{a_{i,n} : i \in \mathcal{N}\}$ and $\mathcal{T} = (T_1, T_2, ..., T_n)$ be an $n$-tuple of operators $T_1, T_2, ..., T_n$. Then the following assertions are equivalent:
(1). $\mathcal{T}$ is chaotic,
(2). $\mathcal{T}$ is Hypercyclic and has a non-trivial periodic point,
(3). $\mathcal{T}$ has a non-trivial periodic point,
(4). the series $\Sigma_{m=1}^{\infty}(\prod_{k=1}^{m}(a_{k,i})^{-1} e_m)$ convergence in $\mathcal{X}$ for $i = 1, 2, ..., n$.

*Proof.* Proof of the cases (1) $\to$ (2) and (2) $\to$ (3) are trivial, so we just proof (3) $\to$ (4) and (4) $\to$ (1). First we proof (3) $\to$ (4), for this, Suppose that $\mathcal{T}$ has a non-trivial periodic point, and $x = \{x_n\} \in \mathcal{X}$ be a non-trivial periodic point for $\mathcal{T}$ , that is there are $\mu_1, \mu_2, ..., \mu_n \in \mathcal{N}$ such that,

$$T_1^{\mu_1} T_2^{\mu_2} ... T_n^{\mu_n}(x) = x.$$

Comparing the entries at positions $j + kN$, and $j + kN$, $k \in \mathcal{N} \cup \{0\}$, of $x$ and $T_1^M T_2^N(x)$ we find that

$$x_{j+kM_1} = (\prod_{t=1}^{M}(a_{j+kN+t}))x_{j+(k+1)}$$

$$x_{j+kM_2} = (\prod_{t=1}^{N}(b_{j+kN+t}))x_{j+(k+1)}$$

$$...$$

$$x_{j+kM_n} = (\prod_{t=1}^{N}(b_{j+kN+t}))x_{j+(k+1)}$$

so that we have,

$$x_{j+kM_1} = (\prod_{t=j+1}^{j+kM_1}(a_t))^{-1} x_j = c_1(\prod_{t=1}^{j+kM_1}(a_t))^{-1}, k \in \mathcal{N} \cup \{0\}$$

$$x_{j+kM_2} = (\prod_{t=j+1}^{j+kM_2}(a_t))^{-1} x_j = c_2(\prod_{t=1}^{j+kM_2}(a_t))^{-1}, k \in \mathcal{N} \cup \{0\}$$

$$...$$

$$x_{j+kM_n} = (\prod_{t=j+1}^{j+kM_n}(a_t))^{-1} x_j = c_n(\prod_{t=1}^{j+kM_n}(a_t))^{-1}, k \in \mathcal{N} \cup \{0\}$$

with

$$c_1 = (\prod_{t=1}^{j}(m_{j,1}))x_j$$

$$c_2 = (\prod_{t=1}^{j}(m_{j,2}))x_j$$

$$...$$

$$c_n = (\prod_{t=1}^{j}(m_{j,n}))x_j.$$

Since $\{e_\kappa\}$ is an unconditional basis and $x \in \mathcal{X}$ it follows from [...] that

$$\sum_{k=0}^{\infty}(\frac{1}{\prod_{t=1}^{j+kM_1}(m_{j,1})})e_{j+kM_1} = \frac{1}{c_1}\sum_{k=0}^{\infty} x_{j+kM_1}.e_{j+kM_1}$$

$$\sum_{k=0}^{\infty}(\frac{1}{\prod_{t=1}^{j+kM_2}(m_{j,2})})e_{j+kM_2} = \frac{1}{c_2}\sum_{k=0}^{\infty} x_{j+kM_2}.e_{j+kM_2}$$
$$...$$
$$\sum_{k=0}^{\infty}(\frac{1}{\prod_{t=1}^{j+kM_n}(m_{j,n})})e_{j+kM_n} = \frac{1}{c_n}\sum_{k=0}^{\infty} x_{j+kM_n}.e_{j+kM_n}$$

convergence in $\mathcal{X}$. Without loss of generality we may assume that $j \geq N$. Applying the operators $T$ , $T^2$, $T^3$,..., $T^{R-1}$, with $R = Min\{M_i : i = 1, 2, ..., n\}$, to this series and note that $T_1(e_n) = a_n e_{n-1}$ and $T_2(e_n) = b_n e_{n-1}$ for $n \geq 2$, we deduce that

$$\sum_{k=0}^{\infty}(\frac{1}{\prod_{t=1}^{j+kM_1-\upsilon_1}(m_{j,1})})e_{j+kM_1-\upsilon_1}$$
$$\sum_{k=0}^{\infty}(\frac{1}{\prod_{t=1}^{j+kM_2-\upsilon_2}(m_{j,2})})e_{j+kM_2-\upsilon_2}$$
$$...$$
$$\sum_{k=0}^{\infty}(\frac{1}{\prod_{t=1}^{j+kM_n-\upsilon_1}(m_{j,n})})e_{j+kM_n-\upsilon_n}$$

convergence in $\mathcal{X}$ for $\gamma = 0, 1, 2, ..., N-1$. By adding these series, we see that condition (4) holds.

Proof of (4) $\Rightarrow$ (1). It follows from theorem (2.1), that under condition (4) the operator $\mathcal{T}$ is Hypercyclic. Hence it remains to show that $\mathcal{T}$ has a dense set of periodic points. Since $\{e_\kappa\}$ is an unconditional basis, condition (4) with proposition 2.3 implies that for each $j \in \mathcal{N}$ and $M, N \in \mathcal{N}$ the series

$$\psi_1(j, M_1) = \sum_{k=0}^{\infty}(\frac{1}{\prod_{t=1}^{j+kM_1}(m_{k,1})})e_{j+kM_1} = (\prod_{t=1}^{j} m_{k,1}).(\sum_{k=0}^{\infty}\frac{1}{\prod_{t=1}^{j+kM_1} m_{k,1}}e_{j+kM_1})$$

$$\psi_2(j, M_2) = \sum_{k=0}^{\infty}(\frac{1}{\prod_{t=1}^{j+kM_2}(m_{k,2})})e_{j+kM_2} = (\prod_{t=1}^{j} m_{k,2}).(\sum_{k=0}^{\infty}\frac{1}{\prod_{t=1}^{j+kM_2} m_{k,2}}e_{j+kM_2})$$
$$...$$
$$\psi_n(j, M_n) = \sum_{k=0}^{\infty}(\frac{1}{\prod_{t=1}^{j+kM_n}(m_{k,n})})e_{j+kM_n} = (\prod_{t=1}^{j} m_{k,n}).(\sum_{k=0}^{\infty}\frac{1}{\prod_{t=1}^{j+kM_n} m_{k,n}}e_{j+kM_n})$$

converges and define $n$ elements in $\mathcal{X}$. Moreover, if $M \geq i$ then

$$T_1^{m_{j,1}}T_2^{m_{j,2}}...T_n^{m_{j,n}} = \psi_1(j, M_1)$$
$$T_1^{m_{j,1}}T_2^{m_{j,2}}...T_n^{m_{j,n}} = \psi_2(j, M_2)$$
$$...$$
$$T_1^{m_{j,1}}T_2^{m_{j,2}}...T_n^{m_{j,n}} = \psi_n(j, M_n)$$
$$T_1^{m_{j,1}}T_2^{m_{j,2}}...T_n^{m_{j,n}} = \psi_1(j, M_1)T_1^{M_1}T_2^{M_2}...T_n^{M_n}\psi_1(j, M_1) = \omega(j, M_1) \quad (1)$$

$$T_1^{m_{j,1}} T_2^{m_{j,2}} ... T_n^{m_{j,n}} = \psi_2(j, M_2) T_1^{M_1} T_2^{M_2} ... T_n^{M_n} \psi_2(j, M_2) = \omega(j, M_2)$$

$$...$$

$$T_1^{m_{j,1}} T_2^{m_{j,2}} ... T_n^{m_{j,n}} = \psi_1(j, M_1) T_1^{M_1} T_2^{M_2} ... T_n^{M_n} \psi_n(j, M_n) = \omega(j, M_n)$$

and, if $N \geq j_i$ then

$$T_1^{m_{j,1}} T_2^{m_{j,2}} ... T_n^{m_{j,n}} \omega((j,i), N) = \omega((j,i), N) \tag{2}$$

for $m_{j,i} \geq N$ and $i = 1, 2, ..., n$. So that each $\psi(j, N)$ for $j \leq N$ is a periodic point for $\mathcal{T}$. We shall show that $\mathcal{T}$ has a dense set of periodic points. Since $\{e_\kappa\}$ is a basis, it suffices to show that for every element $x \in span\{e_\kappa : \kappa \in \mathcal{N}\}$ there is a periodic point $y$ arbitrarily close to it. For this, let $x = \sum_{j=1}^{m} x_j e_j$ and $\varepsilon > 0$. We can assume without lost of generality that

$$| x_i \prod_{t=1}^{i} a_{t,1} | \leq 1 \quad , \quad i = 1, 2, 3, ..., m_1$$

$$| x_j \prod_{t=1}^{i} a_{t,2} | \leq 1 \quad , \quad i = 1, 2, 3, ..., m_2$$

$$...$$

$$| x_i \prod_{t=1}^{i} a_{t,n} | \leq 1 \quad , \quad i = 1, 2, 3, ..., m_n$$

Since $\{e_n\}$ is an unconditional basis, then condition (4) implies that there are an $M, N \geq m$ such that

$$\| \sum_{n=M_1+1}^{\infty} \varepsilon_{\kappa,1} \frac{1}{\prod_{t=1}^{\kappa} a_{t,1}} e_\kappa \| < \frac{\varepsilon}{m_1}$$

$$\| \sum_{n=M_2+1}^{\infty} \varepsilon_{\kappa,2} \frac{1}{\prod_{t=1}^{\kappa} a_{t,2}} e_\kappa \| < \frac{\varepsilon}{m_2}$$

$$...$$

$$\| \sum_{n=M_n+1}^{\infty} \varepsilon_{\kappa,n} \frac{1}{\prod_{t=1}^{\kappa} a_{t,n}} e_\kappa \| < \frac{\varepsilon}{m_n}$$

for every sequences $\{\varepsilon_{\kappa,i}\}$, $i = 1, 2, ..., n$ taking values 0 or 1. By (1) and (2) the elements

$$y_1 = \sum_{i=1}^{m_1} x_i \psi(i, M_1)$$

$$y_2 = \sum_{i=1}^{m_2} x_i \psi(i, M_2)$$

$$...$$

$$y_n = \sum_{i=1}^{m_n} x_i \psi(i, M_n)$$

of $\mathcal{X}$ is a periodic point for $\mathcal{T}$, and we have

$$\begin{aligned}
\|y_\lambda - x\| &= \| \textstyle\sum_{i=1}^{m_\lambda} x_i(\psi(i, M_\lambda) - e_i)\| \\
&= \| \textstyle\sum_{i=1}^{m_\lambda} (x_i \prod_{t=1}^{i} d_{t,M_\lambda})(\sum_{k=1}^{\infty} \frac{1}{\prod_{t=1}^{i+kM_\lambda} a_{t,M_\lambda}} e_{i+kM_\lambda})\| \\
&\leq \textstyle\sum_{i=1}^{m_\lambda} \|(x_i \prod_{t=1}^{i} d_{t,M_\lambda})(\sum_{k=1}^{\infty} \frac{1}{\prod_{t=1}^{i+kM_\lambda} a_{t,M_\lambda}} e_{i+kM_\lambda})\| \\
&\leq \textstyle\sum_{i=1}^{m_\lambda} \|(\sum_{k=1}^{\infty} \frac{1}{\prod_{t=1}^{i+kM_\lambda} a_{t,M_\lambda}} e_{i+kM_\lambda})\| \\
&\leq \epsilon
\end{aligned}$$

as $\lambda = 1, 2, ..., n$ and by this, the proof is complete.

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