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Semi-Periodic Vector for n-Tuples of Operators

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Abstract

In this paper, we introduce fix points for a tuple of operators on a

Banach space and give some Conditions for a vector to be a Fix point for the Tuple.

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1 Introduction

Let $\mathcal{X}$ be a Banach space and $T_1, T_2, ..., T_n$ are commutative bounded linear operators on $\mathcal{X}$. By an n-tuple we mean the n-component $\mathcal{T} = (T_1, T_2, ..., T_n)$ so that "n" is a finite positive integer number. For the tuple $\mathcal{T} = (T_1, T_2, ..., T_n)$ the set

$$\mathcal{F} = \{{T_1}^{k_1}{T_2}^{k_2}...{T_n}^{k_n} : k_i \geq 0, i = 1, 2, ..., n\}$$

is the semigroup generated by $\mathcal{T}$. For $x \in \mathcal{X}$ take

$$Orb(\mathcal{T}, x) = \{Sx : S \in \mathcal{F}\}.$$

In other hand

$$Orb(\mathcal{T}, x) = \{{T_1}^{k_1}{T_2}^{k_2}...{T_n}^{k_n}(x) : k_i \geq 0, i = 1, 2, ..., n\}.$$

2 Preliminary Notes

Definition 2.1 *The set $Orb(\mathcal{T}, x)$ is called, orbit of vector x under $\mathcal{T}$ and Tuple $\mathcal{T} = (T_1, T_2, ..., T_n)$ is called hypercyclic tuple, if there is a vector $x \in \mathcal{X}$ such that, the set $Orb(\mathcal{T}, x)$ is dense in $\mathcal{X}$, that is*

$$\overline{Orb(\mathcal{T}, x)} = \overline{\{{T_1}^{k_1}{T_2}^{k_2}...{T_n}^{k_n}(x) : k_i \geq 0, i = 1, 2, ..., n\}} = \mathcal{X}.$$

In this case, the vector x is called a hypercyclic vector for the tuple $\mathcal{T}$.

Definition 2.2 *Let $\mathcal{X}$ is a metric space with metric d. Take*

$$\eta(\mathcal{X}) = P(\mathcal{X}) - \{\phi\}$$

and

$$Cl(\mathcal{X}) = \{x : x \subset \mathcal{X}, x = closed, x \neq \phi\}$$

Consider an n-tuple $T = (T_1, T_2, ..., T_n)$. An element $x \in \mathcal{X}$ is called a fixed point for T if there exist non-negative integer numbers $m_1, m_2, ..., m_n$ such that

$$T_1^{m_1} T_2^{m_2} ... T_n^{m_n}(x) = x$$

Fixed point for a functional is defining similarly.

Definition 2.3 *Let $\mathcal{X}$ is a Banach space and $x \in X$, the vector x is called a semi-periodic vector for tuple $T = (T_1, T_2, ..., T_n)$ if the sequence*

$$\{T_1^{m_1} T_2^{m_2} ... T_n^{m_n}(x)\}$$

be semi-compact. In this case $\mathcal{T}$ is called semi-periodic tuple.

Definition 2.4 *Let $\mathcal{X}$ is a Banach space and $\mathcal{T} = (T_1, T_2, ..., T_n)$ is a Tuple, then the dense generalized kernel of $\mathcal{T}$ is the set*

$$\bigcup_{k>0} (Ker(T_1^{m_{1,k}} T_2^{m_{2,k}} ... T_n^{m_{n,k}})).$$

Note 2.5 *All of operators in this paper are commutative bounded linear operators on a Banach space. Also, note that by $\{j, i\}$ we mean a number, that was showed by this mark and related with this indexes, not a pair of numbers.*

Readers can see [1-9] for some information.

3 Main Results

Theorem 3.1 (The Hypercyclicity Criterion) *Let $\mathcal{X}$ be a separable Banach space and $\mathcal{T} = (T_1, T_2, ..., T_n)$ is an n-tuple of continuous linear mappings on $\mathcal{X}$. If there exist two dense subsets $\mathcal{Y}$ and $\mathcal{Z}$ in $\mathcal{X}$, and strictly increasing sequences $\{m_{j,1}\}_{j=1}^{\infty}$, $\{m_{j,2}\}_{j=1}^{\infty}$, ...,$\{m_{j,n}\}_{j=1}^{\infty}$ such that :*
1. $T_1^{m_{j,1}} T_2^{m_{j,2}} ... T_n^{m_{j,n}} \to 0$ on $\mathcal{Y}$ as $j \to \infty$,
2. There exist functions $\{S_j : \mathcal{Z} \to \mathcal{X}\}$ such that for every $z \in \mathcal{Z}$, $S_j z \to 0$, and $T_1^{m_{j,1}} T_2^{m_{j,2}} ... T_n^{m_{j,n}} S_j z \to z$, on $\mathcal{Z}$ as $j \to \infty$,
then $\mathcal{T}$ is a hypercyclic n-tuple.

Theorem 3.2 *Let $\mathcal{X}$ be a separable Banach space and $\mathcal{T} = (T_1, T_2, ..., T_n)$ is an hypercyclic n-tuple of commutative continuous linear mappings on $\mathcal{X}$. the tuple $\mathcal{T}$ satisfying the hypothesis of The Hypercyclicity Criterion, if there is a subset $\mathcal{S}$ of $\mathcal{X}$ such that, all elements of $\mathcal{S}$ are semi-periodic vectors for tuple $\mathcal{T}$.*

proof. Since $\mathcal{T}$ is hypercyclic tuple, then we can choice a vector $x \in \mathcal{X}$ such that $Orb(\mathcal{T}, x)$ is dense in $\mathcal{X}$. Take

$$U_k = (0, \frac{1}{k})$$

$$V_k = \{x + u : u \in U_k\}.$$

Since the set of hypercyclic vectors for any tuple is dense in $\mathcal{X}$, so let $u_1 \in U_1$ and the natural numbers $m_{1,1}, m_{2,1}, ..., m_{n,1}$ with property

$$T_1^{m_{1,1}} T_2^{m_{2,1}} ... T_n^{m_{n,1}}(u_1) \in V_1.$$

Now, take $u_2 \in U_2$. Since

$$\overline{\{T_1^{m_{1,1}+1} T_2^{m_{2,1}+1} ... T_n^{m_{n,1}+1}(u_2), T_1^{m_{1,1}+2} T_2^{m_{2,1}+2} ... T_n^{m_{n,1}+2}(u_2), ...\}} = \mathcal{X}$$

so there are $m_{1,2}, m_{2,2}, ..., m_{n,2}$ such that

$$T_1^{m_{1,2}} T_2^{m_{2,2}} ... T_n^{m_{n,2}}(u_2) \in V_2.$$

Similarly, there are $m_{1,t}, m_{2,t}, ..., m_{n,t}$ such that

$$T_1^{m_{1,t}} T_2^{m_{2,t}} ... T_n^{m_{n,t}}(u_t) \in V_k.$$

There are sequence $\{u_t\}_{k=1}^{\infty}$ of hypercyclic vectors and subsequences $\{m_{1,t}\}, \{m_{2,t}\}, ..., \{m_{n,t}\}$ of natural numbers, such that

$$lim_{k \to \infty}(u_k) = 0$$

and

$$T_1^{m_{1,t}} T_2^{m_{2,t}} ... T_n^{m_{n,t}}(u_t) \in V_t.$$

Now we try to find subsequence

$$\{m'_{1,t}\}, \{m'_{2,t}\}, ..., \{m'_{n,t}\}$$

of

$$\{m_{1,t}\}, \{m_{2,t}\}, ..., \{m_{n,t}\}$$

such that

$$T_1^{m'_{1,t}} T_2^{m'_{2,t}} ... T_n^{m'_{n,t}}(u_2)(V_t) \bigcap (U_t) \neq \phi.$$

Suppose that $V = V_k$ and $U = U_k$ for any given k. If $\mathcal{S}$ be the set of all semi-periodic vectors of $\mathcal{T}$ then $\overline{\mathcal{S}} = \mathcal{X}$ so

$$\mathcal{S} \bigcap (x + B(0, \frac{1}{2k})) \neq \phi$$

in other word, we can take

$$\omega \in \mathcal{S} \bigcap (x + B(0, \frac{1}{2k}))$$

indeed the orbit of

$$\{T_1^{m_{1,t}} T_2^{m_{2,t}} ... T_n^{m_{n,t}}(u_t)\}$$

is semi-compact, so there subsequence

$$\{\eta_{1,t}\}, \{\eta_{2,t}\}, ..., \{\eta_{n,t}\}$$

of

$$\{m_{1,j}\}, \{m_{2,j}\}, ..., \{m_{n,j}\}$$

such that

$$\{T_1^{\eta_{1,t}} T_2^{\eta_{2,t}} ... T_n^{\eta_{n,t}}(\omega)\}$$

is a convergence sequence. Suppose $\omega_0 \in \mathcal{X}$ and

$$T_1^{\eta_{1,t}} T_2^{\eta_{2,t}} ... T_n^{\eta_{n,t}}(\omega) \rightarrow \omega_0$$

as $t \rightarrow \infty$. With replace $\epsilon = \frac{1}{2k}$ we have k_0,

$$\|T_1^{\eta_{1,t}} T_2^{\eta_{2,t}} ... T_n^{\eta_{n,t}}(\omega) - \omega_0\| \leq \frac{1}{2k_0}$$

as $\eta_{j,t} > k$ for $j = 1, 2, ..., n$ and $t = 1, 2, ...$ since x be a hypercyclic vector for $\mathcal{S}$ then there are natural numbers

$$\eta_{1,t}, \eta_{2,t}, ..., \eta_{n,t}$$

such that,

$$\|T_1^{\eta_{1,t}} T_2^{\eta_{2,t}} ... T_n^{\eta_{n,t}}(\omega) + \omega\| \leq \frac{1}{2k}.$$

Since $lim(u_i) = 0$ as $i \rightarrow 0$, then there is α such that

$$\| u_\alpha \| \leq \frac{1}{2k_0 \| T_1^{\eta_{1,t}} T_2^{\eta_{2,t}} ... T_n^{\eta_{n,t}} \|}.$$

For k, take $\eta_{i,t}$ with property $\eta_{i,t} > k$ and $\eta_{i,t} > k_0$ for $i = 1, 2, ..., n$ and $t = 1, 2, ...,$ now we have

$$\|T_1^{\eta_{1,k}} T_2^{\eta_{2,k}} ... T_n^{\eta_{n,k}} T_1^{\eta_{1,t}} T_2^{\eta_{2,t}} ... T_n^{\eta_{n,t}}(u_i) + \omega\|$$

$$= \|T_1^{\eta_{1,k}} T_2^{\eta_{2,k}} ... T_n^{\eta_{n,k}} T_1^{\eta_{1,t}} T_2^{\eta_{2,t}} ... T_n^{\eta_{n,t}}(u_i)$$

$$+ T_1^{\eta_{1,k}} T_2^{\eta_{2,k}} ... T_n^{\eta_{n,k}}(x) - T_1^{\eta_{1,k}} T_2^{\eta_{2,k}} ... T_n^{\eta_{n,k}}(x) + \omega\|$$

$$= \|T_1^{\eta_{1,k}} T_2^{\eta_{2,k}} ... T_n^{\eta_{n,k}}(T_1^{\eta_{1,t}} T_2^{\eta_{2,t}} ... T_n^{\eta_{n,t}}(u_i) - x) +$$

$$T_1^{\eta_{1,k}} T_2^{\eta_{2,k}} ... T_n^{\eta_{n,k}}(x) + \omega\|$$

$$\leq \|T_1^{\eta_{1,k}} T_2^{\eta_{2,k}} ... T_n^{\eta_{n,k}}(T_1^{\eta_{1,t}} T_2^{\eta_{2,t}} ... T_n^{\eta_{n,t}}(u_i) - x)\| +$$

$$\|T_1^{\eta_{1,k}} T_2^{\eta_{2,k}} ... T_n^{\eta_{n,k}}(x) + \omega\|$$

$$\leq \|T_1^{\eta_{1,k}} T_2^{\eta_{2,k}} ... T_n^{\eta_{n,k}}\|$$

$$.\|(T_1^{\eta_{1,t}} T_2^{\eta_{2,t}} ... T_n^{\eta_{n,t}}(u_i) - x)\| + \tfrac{1}{2k}$$

$$\leq \|T_1^{\eta_{1,k}} T_2^{\eta_{2,k}} ... T_n^{\eta_{n,k}}\| + \tfrac{1}{2r_i}$$

$$\leq \tfrac{1}{2k} + \tfrac{1}{2k}$$

$$= \tfrac{1}{k}$$

Since

$$\|\omega + T_1^{\eta_{1,t}} T_2^{\eta_{2,t}} ... T_n^{\eta_{n,t}}(u_i) - x\| = \|\omega - x + T_1^{\eta_{1,t}} T_2^{\eta_{2,t}} ... T_n^{\eta_{n,t}}(u_i)\|$$

$$\leq \|\omega - x\| + \|T_1^{\eta_{1,t}} T_2^{\eta_{2,t}} ... T_n^{\eta_{n,t}}(u_i)\|$$

$$\leq \tfrac{1}{2k} + \|T_1^{\eta_{1,t}} T_2^{\eta_{2,t}} ... T_n^{\eta_{n,t}}\|.\|(u_i)\|$$

$$\leq \tfrac{1}{2k} + \tfrac{1}{2k}$$

$$= \frac{1}{k}$$

then $h = \omega + T_1^{\eta_{1,t}} T_2^{\eta_{2,t}} ... T_n^{\eta_{n,t}}(u_i) \in V_k$ so

$$\|T_1^{\eta_{1,t}} T_2^{\eta_{2,t}} ... T_n^{\eta_{n,t}}(h)\| = \|T_1^{\eta_{1,t}} T_2^{\eta_{2,t}} ... T_n^{\eta_{n,t}}(\omega + T_1^{\eta_{1,t}} T_2^{\eta_{2,t}} ... T_n^{\eta_{n,t}}(u_i))\|$$

$$= \|T_1^{\eta_{1,t}} T_2^{\eta_{2,t}} ... T_n^{\eta_{n,t}}(\omega) + T_1^{\eta_{1,t}} T_2^{\eta_{2,t}} ... T_n^{\eta_{n,t}}(T_1^{\eta_{1,t}} T_2^{\eta_{2,t}} ... T_n^{\eta_{n,t}}(u_i))\|$$

$$= \|T_1^{\eta_{1,t}} T_2^{\eta_{2,t}} ... T_n^{\eta_{n,t}}(\omega) - \omega_0 + T_1^{\eta_{1,t}} T_2^{\eta_{2,t}} ... T_n^{\eta_{n,t}}(T_1^{\eta_{1,t}} T_2^{\eta_{2,t}} ... T_n^{\eta_{n,t}}(u_i)) + \omega_0\|$$

$$\leq \|T_1^{\eta_{1,t}} T_2^{\eta_{2,t}} ... T_n^{\eta_{n,t}}(\omega) - \omega_0\| + \|T_1^{\eta_{1,t}} T_2^{\eta_{2,t}} ... T_n^{\eta_{n,t}}(T_1^{\eta_{1,t}} T_2^{\eta_{2,t}} ... T_n^{\eta_{n,t}}(u_i)) + \omega_0\|$$

$$\leq \frac{1}{2k} + \frac{1}{2k}$$

$$= \frac{1}{k}$$

that is

$$T_1^{\eta_{1,k}} T_2^{\eta_{2,k}} ... T_n^{\eta_{n,k}}(V_k) \bigcap U_k \neq \phi$$

By this the proof is complete.

Lemma 3.3 *Let $\mathcal{X}$ be a separable Banach space and $\mathcal{T} = (T_1, T_2, ..., T_n)$ is a chaotic n-tuple of commutative continuous linear operators on $\mathcal{X}$. the tuple $\mathcal{T}$ satisfying the hypothesis of The Hypercyclicity Criterion.*

proof. Since the tuple $\mathcal{T}$ is a chaotic tuple, then have a set of semi-periodic vector, so by the Theorem 3.2, the $\mathcal{T}$ satisfying the hypothesis the Hypercyclicity Criterion.

Lemma 3.4 *Let $\mathcal{X}$ be a separable Banach space and $\mathcal{T} = (T_1, T_2, ..., T_n)$ is an n-tuple of commutative bounded linear mapping on $\mathcal{X}$ and $\mathcal{T}$ have a dense generalized kernel, then tuple $\mathcal{T}$ satisfying the hypothesis of The Hypercyclicity Criterion.*

proof. For proof of this lemma, it is sufficient to choice a vector x for the tuple $\mathcal{T}$, by the theorem 3.2 the proof is clear.

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