Int. Journal of Math. Analysis, Vol. 6, 2012, no. 13, 609 - 617

Generators of Diagram Groups over Graphical Presentations of Integers with Four Initial Generators Using Lifting Method

Yousof Gheisari

Islamic Azad University, Sciences and Research Branch
Department of Mathematics, Bushehr, Iran
yousof.gheisari@iaubushehr.ac.ir

S. Seddighi Chaharborj

Islamic Azad University, Sciences and Research Branch
Department of Mathematics, Bushehr, Iran
sarkhosh@math.upm.edu.my

Abstract

In this study, we investigate to determine the generators of diagram groups over the graphical presentations with four initial generators by using lifting method. We consider the graphs $\Gamma_n, n \in N$ obtained from these kinds of graphical presentation. The determinations were made by systematically applying a lifting method to a graph according to length of words. We determined the general formula for the total number of lifts of generators and the lifts of generators at any given word in general for the 2-complex of diagram groups of the graphical presentation with four initial generators.

Mathematics Subject Classification: 20M05, 14L40

Keywords: Lift of generators, Diagram groups, graphical presentation, Generators

1 Introduction

In our previous researches we described the component of graphs, the spanning tree, and the generators for diagram group over the graphical presentation ${}^3S = \langle x_1, x_2, x_3 \,|\, x_1 = x_2,\, x_1 = x_3,\, x_2 = x_3 \rangle$ using lifting method. We also discussed construction of graphical presentations and class of diagram groups (refer to [8], [9] [10], [11], and [12]).
In this study we discuss the determination of lift of generators for diagram groups, particularly of the graphical with the presentation of integers with four initial generators ${}^4S = \langle x_1, x_2, x_3, x_4 \,|\, x_i = x_j;\ 1 \le i < j \le 4 \rangle$. Let $S = \langle X \,|\, R \rangle$ be a graphical presentation. Then we may obtain the diagram group $D(S,W)$ which w is a word on X as defined by Guba and Sapir (please see [5], [6], and [7]).

The 2-complex, associated with presentation S is denoted by $K(S)$. As the complex we have fundamental group and we denoted by $\pi_1(K(S),W)$. Kilibarda has shown that the fundamental group is isomorphic to diagram group $D(S,W)$ (refer to [3], and [4]).
We will consider the fundamental group $\pi_1(K(S),W)$ constructed from graphical presentation ${}^3S = \langle x_1, x_2, x_3 \,|\, x_1 = x_2,\, x_1 = x_3,\, x_2 = x_3 \rangle$.

For our presentation, the 2-complex consists of infinitely connected component $\Gamma_1, \Gamma_2, \Gamma_3, \ldots, \Gamma_n$. Note that all vertices in Γ_i are words of length i. Thus if length (u) = length (v) then $\pi_1(K(S),u) = \pi_1(K(S),v)$ (see [1]).

As a group, it is sufficient to determine its generators and relations. The generators of this group can be determined from the 2-complex $K(S)$ by identifying the spanning tree T. Fix a vertex $v \in K(S)$ and the edge $e \notin T$. Then $\gamma_{\iota(e)} e \gamma^{-1}_{\tau(e)}$ is the generator, where $\gamma_{\iota(e)}, \gamma_{\tau(e)}$ are paths in T from $v \in K(S)$, to the initial and terminal of e respectively.
Let u_i be a word of length i. We will show that the generator for $\pi_1(K(S),u_{i+1})$ can be obtained from generator of $\pi_1(K(S),u_i)$. This is a lifting method. Hence it is sufficient to determine the generator for $\pi_1(K(S),x_1)$. Lifting method can determine all generators for the whole fundamental groups $\pi_1(K(S),u_i)$.

In section 2 we briefly explain diagram groups for presentations of graphical and in section 3, we give some definitions about graphs, paths and lifting of paths. Then the determination of lifts of generators will be shown in section 4. This section includes determining the lift of the generator g at a word W, the general formula for the number of lifts of generators and also the total number of lifts of generators of fundamental groups in the specific graphs $\Gamma_n, n \in N$.

2 Determining the graphs Γ_n, n∈N and Preliminaries

Now we want to determine the graphs $\Gamma_n, n \in N$ and the generators of diagram groups in these graphs.

Let ${}^4S = \langle x_1, x_2, x_3, x_4 \mid x_i = x_j;\ 1 \le i < j \le 4 \rangle$ be a graphical presentation. Note that the graph obtained from S is collections of sub graphs. The graph Γ obtained from S is a union of Γ_n connecting all vertices of length n and respective edges. For example in Figure 1, the graph of Γ_1 obtained for graphical presentation with four initial generators (we called x_1, x_2, x_3 and x_4 initial generators) is in Figure 1.

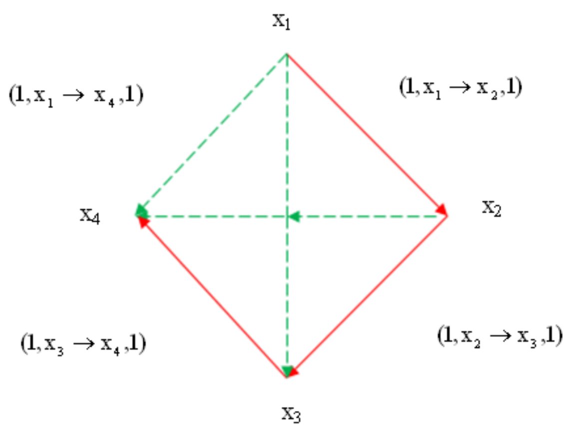

Figure 1: Graph of Γ_1

While in Figure 2, graph of Γ_2 is in Figure 2.

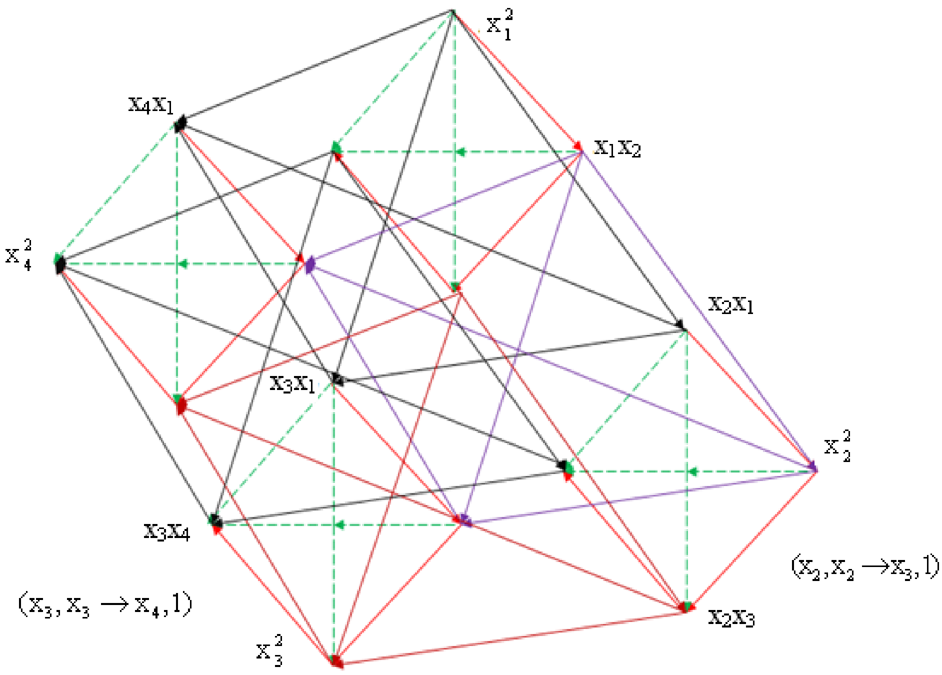

Figure 2 : Graph of Γ_2

Note that Γ_2 is four copies of Γ_1 and each vertex in each copy are joined together respectively. Similarly with four copies of Γ_2, we may obtain Γ_3. Repeat similar procedures for Γ_4 and so on.

For finding the generators of fundamental groups in graphs, the edges which are not to the spanning tree will be generators. For example we have three generators of fundamental groups in Γ_1 and there are:

- $g_{1_{13}} = (1, x_1 \to x_2, 1)(1, x_2 \to x_3, 1)(1, x_1 \to x_3, 1)^{-1}$
- $g_{1_{14}} = (1, x_1 \to x_2, 1)(1, x_2 \to x_3, 1)(1, x_3 \to x_4, 1)(1, x_1 \to x_4, 1)^{-1}$
- $g_{1_{24}} = (1, x_1 \to x_2, 1)(1, x_2 \to x_3, 1)(1, x_3 \to x_4, 1)\ (1, x_2 \to x_4, 1)^{-1}\ (1, x_1 \to x_2, 1)^{-1}$

Similarly in Γ_2 we may obtain its generators.

Definition 2.1: A path on a graph Γ is either a vertex or a non-empty sequence of edges $e_1e_2e_3...e_n$ such that $\tau(e_i)=\iota(e_{i+1})$ for any $i=1,2,3,...,n-1$ $(e_i \in E)$. If $\alpha = e_1e_2e_3...e_n$ is a path then the inverse path α^{-1} is the path $e_n^{-1}...e_2^{-1}e_1^{-1}$. A path consisting of one vertex is called an empty path. An empty path coincides with its inverse. We define α to be a closed path if $\tau(\alpha)=\iota(\alpha)$. For example in the graph of Γ_1 in Figure 1, $(1,x_1 \to x_2,1)(1,x_2 \to x_3,1)(1,x_1 \to x_3,1)^{-1}$ is a path and $(1,x_1 \to x_2,1)(1,x_2 \to x_3,1)(1,x_3 \to x_4,1)(1,x_1 \to x_4,1)^{-1}$ is a closed path.

Definition 2.2: Let $\phi:\Gamma^* \to \Gamma$ be a mapping of graphs. If v and v^* are the vertices of Γ and Γ^* respectively such that $\phi(v^*) = v$ then v^* is said to lie over v. Let α be a path in Γ with $\iota(\alpha) = v$ and suppose v^* lies over v. A path α^* in Γ^* is said to be a lift of α at v^* if $\phi(\alpha^*) = \alpha$, (see [2]).

Let $\alpha = (1,x_1 \to x_2,1)(1,x_2 \to x_3,1)(1,x_3 \to x_4,1)$ be a path, $\iota(\alpha)=x_1, \tau(\alpha)=x_4$ and suppose x_1x_2 lies over x_1. The lift of α at x_1x_2 is

$\alpha* = (1,x_1 \to x_2,x_2)(1,x_2 \to x_3,x_2)(1,x_3 \to x_4,x_2)$

Example 2.3: We have three generators in $\pi_1({}^4\Gamma_1)$ and $g_{1_{13}}$ is $(1,x_1 \to x_2,1)(1,x_2 \to x_3,1)(1,x_1 \to x_3,1)^{-1}$.

Then lifts of $g_{1_{13}}$ at $x_1^{\,2}$ are $(x_1,x_1 \to x_2,1)(x_1,x_2 \to x_3,1)(x_1,x_1 \to x_3,1)^{-1}$, and $(1,x_1 \to x_2,x_1)(1,x_2 \to x_3,x_1)(1,x_1 \to x_3,x_1)^{-1}$.

Lifts of $g_{1_{13}}$ at $x_1\lambda$ $\lambda \in \{x_2,x_3,x_4\}$

$(x_1,x_1 \to \lambda,1)(1,x_1 \to x_2,\lambda)(1,x_2 \to x_3,\lambda)(1,x_1 \to x_3,\lambda)^{-1}(x_1,x_1 \to \lambda,1)^{-1}$.

Similarly we can show that lifts of all generators $g_{1_{14}}$ and $g_{1_{24}}$ at $x_1^{\,2}$, x_1x_2, x_1x_3, x_1x_4 are the same as the lift of $g_{1_{13}}$ at $x_1^{\,2}$, x_1x_2, x_1x_3, x_1x_4.

Note that the first lifts of $g_{1_{14}}$ and $g_{1_{24}}$, at $x_1^{\,2}$ has no conjugate , but the lifts of $g_{1_{14}}$ and $g_{1_{24}}$ at x_1x_2, $x_1x_3,\dots$, x_1x_n has a conjugate.

3 Lift of generators

Now we will show our technique for computing the general formula for the total number of lifts of generators in Γ_n and the number lifts of generators in Γ_n.

Lemma 3.1: Let ${}^4S = \langle x_1, x_2, x_3, x_4 \mid x_i = x_j;\ 1 \le i < j \le 4 \rangle$ be a graphical presentation. The general formula for total number of lifts of generators of diagram groups in the graph Γ_{n-1} is $a_n = 4a_{n-1} + 3$. where $a_i\ (i = 1,2,3,\ldots n)$ the number of lift of generators in is Γ_i and $a_1 = 3$.

Proof: By definition, Γ_n is four copies of Γ_{n-1} , then the spanning tree T_n in Γ_n is four copies of the spanning T_{n-1} in Γ_{n-1}. We know that the edges do not belong to spanning tree will be generators. Thus the generators of fundamental group $\pi_1(\Gamma_n)$ is four copies of generators in $\pi_1(\Gamma_{n-1})$. Thus the number of lifts of generators of $\pi_1(\Gamma_n)$ is four copies of the number of lift of generators of $\pi_1(\Gamma_{n-1})$ plus three generators, which obtained from between edges of three spanning trees. Hence the number of lifts of generators of $\pi_1(\Gamma_n)$ is $a_n = 4a_{n-1} + 3$.

Lemma 3.2: Let ${}^4S = \langle x_1, x_2, x_3, x_4 \mid x_i = x_j;\ 1 \le i < j \le 4 \rangle$ be a graphical presentation. The total number of lifts of generators of $\pi_1(\Gamma_{n-1})$ is $a_n = (4^n - 1), (n = 2,3,4,\ldots)$.
Proof: By induction, for n = 2, we have a_2 = 15. Now suppose $a_k = (4^k - 1)$ is the total number lift of generators of diagram groups in Γ_{k-1}, we will prove that $a_{k+1} = (4^{k+1} - 1)$ is the number lift of generators of diagram groups in Γ_k . By Lemma 1, we have $a_{k+1} = 4a_k + 3 = 4(4^k - 1) + 3 = (4^{k+1} - 1)$.

Theorem 3.3: Let ${}^4S = \langle x_1, x_2, x_3, x_4 \mid x_i = x_j;\ 1 \le i < j \le 4 \rangle$ be the graphical presentation and $\rho_{1n} = (u, x_1 \to x_2, v)(u, x_2 \to x_3, v)(u, x_1 \to x_3, v)^{-1}$ where length $(uv) = n - 1$, then:

- Lifts of ρ_{1n} at x_1^{n+1} is:

$(x_1^{n+1-m}, x_1 \to x_2, x_1^{m-1})(x_1^{n+1-m}, x_2 \to x_3, x_1^{m-1})\ (x_1^{n+1-m}, x_1 \to x_3, x_1^{m-1})^{-1}$, $m = 1,2,3,\ldots,n+1$, $m \le n+1$

- Lifts of ρ_{1n} at $a_1a_2a_3...a_{n+1}$, $a_i \in \{x_1, x_2, x_3, x_4\}$, $\exists\,(i = 1, 2, 3, ..., m)$ $a_i = x_1$ from $x_1^{\,n+1}$ is: $s_1s_2...s_{n-m}\rho_{1(n+1)}s_{n-m}^{\;-1}...s_2^{\;-1}s_1^{\;-1}$

where, $\rho_{1(n+1)} = (u, x_1 \to x_2, v)(u, x_2 \to x_3, v)(u, x_1 \to x_3, v)^{-1}$, length $(uv) = n$ and $s_i = (u^*, x_1 \to a, v^*)$, length $(u^*v^*) = n$ and $a \in \{x_2, x_3, x_4\}$.

Note that lift of $\rho_{2n} = (1, x_1 \to x_2, 1)$ $(1, x_2 \to x_4, 1)(1, x_1 \to x_4, 1)^{-1}$ and $\rho_{3n} = (1, x_1 \to x_2, 1)$ $(1, x_2 \to x_3, 1)(1, x_3 \to x_4, 1)(1, x_1 \to x_4, 1)^{-1}$, where length $(uv) = n - 1$, at w the same as this theorem.

Proof: By induction on length of words.

Note that the lift of generator has no conjugate if it is the lift of ρ_n at $x_1^{\,n+1}$, but the lift of ρ_n at wx_1 or x_1w has conjugates and the number of conjugate is $(n+1) - \exp_{x_1}(w)$.

Here $\exp_{x_1}(w)$ is the exponent sum of x_1 in a word w defined to be the number of x_1 appeared in w minus the number of x_1^{-1} appeared in w.

Example 3.4: Lift of $\rho_{1_3} = (u, x_1 \to x_2, v)(u, x_2 \to x_3, v)$ $(u, x_1 \to x_3, v)^{-1}$, length $(uv) = 2$ at $x_1x_2x_3^{\;2}$ from $x_1^{\;4}$ is:

$(x_1^{\;3}, x_1 \to x_3, 1)(x_1^{\;3}, x_1 \to x_3, x_3)$ $(x_1, x_1 \to x_2, x_3^{\;2})$ $(1, x_1 \to x_2, x_2x_1^{\;2})(1, x_2 \to x_3, x_2x_1^{\;2})$
$(1, x_1 \to x_3, x_2x_1^{\;2})^{-1}(x_1^{\;3}, x_1 \to x_3, x_3)^{-1}$ $(x_1, x_1 \to x_2, x_3^{\;2})^{-1}(x_1, x_1 \to x_3, 1)^{-1}$

Example 3.5: If $\gamma_6 = (u, x \to y, v)(u, y \to z, v)(u, x \to z, v)^{-1}$, where length $(uv) = 5$, then the lifts of γ_6 at x^2y^2zyx from x^7 is:

$$(x^5, x \to y, x)(x^4, x \to z, yx)(x^3, x \to y, zyx)(x^2, x \to y, yzyx)$$
$$(x, x \to y, y^2zyx)(x, y \to z, y^2zyx)(x, x \to z, y^2zyx)^{-1}$$
$$(x^2, x \to y, yzyx)(x^3, x \to y, zyx)^{-1}(x^4, x \to z, yx)^{-1}(x^5, x \to y, x)^{-1}$$

Because $x^7 \to x^5yx \to x^4zyx \to x^3yzyx \to x^2y^2zyx$ and by using Theorem 3.3**,** we have four conjugates.

References

[1] A.G.B. Ahmad, and A.M. Al-Odhari. The graph of diagram groups constructed from natural numbers semigroup with a repeating generator. J. Inst. Math. Comput. Sci. (Math. Ser.), **17** (2004): 67-69.

[2] J. Rotman, An introduction to the theory of groups, Fourth edition, Springer-Verlag, USA, 1994.

[3] V. Kilibarda, On the algebra of semigroup diagrams. PhD Thesis. University of Nebraska, 1994.

[4] V. Kilibarda. On the algebra of semigroup diagrams, Int. Journal of Alg. and Comput. **7** (1997), 313-338.

[5] V. Guba and M. Sapir. Diagram Groups. Memoirs of the American Mathematical Society, 1997.

[6] V. Guba and M. Sapir. Diagram groups and directed 2-complexes: homotopy and homology. *Journal of Pure and Applied Algebra* **205** (2006): 1- 47.

[7] V. Guba and M. Sapir. Diagram groups are totally orderable. Journal of Pure and Applied Algebra, **205** (2006): 48-73.

[8] Y. Gheisari, and A.G.B. Ahmad. Lift of Spanning trees for diagram groups over semigroup presentation $P=\langle x,y,z \mid x=y, x=z, y=z \rangle$. International Journal of Algebra, **13** (2010): 647-654.

[9] Y. Gheisari and A.G. B. Ahmad. General polynomial for component of graphs from diagram group of semigroup presentation $P=\langle x,y,z \mid x=y, y=z, x=z \rangle$. Journal of International Mathematical Forum, **5** (2010), no. 18, 875 – 880.

[10] Y. Gheisari. Generators of fundamental groups over the constructions of semigroup presentations of integers. Applied Mathematical Sciences, **5** (2010), 3065 – 3071.

[11] Y. Gheisari, A. Ghafur Ahmad. The class of isomorphic diagram groups over semigroup presentations. International Journal of Contemporary Mathematical Sciences, **5** (2010), 2311-2318.

[12] Y. Gheisari, A. Ghafur Ahmad. Component of graphs from diagram groups over the union of two semigroup presentation with three different initial generators by adding new relation. Journal of International Mathematical Forum, **5** (2010): 3053-3058.

Received: September, 2011