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# Quasi-Periodic Solutions of $2k$ Order Wave Equations

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**Abstract**

In this paper, one-dimensional (1D) nonlinear $2k$ order wave equations $u_{tt}+\sum_{\bar{r}=1}^{2k}(-1)^{\bar{r}}\frac{\partial^{2\bar{r}}u}{\partial x^{2\bar{r}}}+mu=f(u)$ with Dirichlet boundary conditions are considered, where the nonlinearity $f$ is an analytic, odd function and $f(u)=O(u^3)$. It is proved that for almost all real parameters $m>0$, the above equations admit small-amplitude quasi-periodic solutions corresponding to finite dimensional invariant tori for an associated infinite dimensional dynamical system. The proof is based on an infinite dimensional KAM theory and a partial Birkhoff normal form technique.

**Mathematics Subject Classification:** 37K55

**Keywords:** $2k$ order wave equations; KAM theorem; Quasi-periodic solutions; Partial Birkhoff normal form

## 1 Introduction and Main Result

We are going to study the nonlinear $2k$ order wave equations

$$u_{tt}+\sum_{\bar{r}=1}^{2k}(-1)^{\bar{r}}\frac{\partial^{2\bar{r}}u}{\partial x^{2\bar{r}}}+mu=f(u) \tag{1}$$

on the finite $x$- interval $[0,\pi]$ with Dirichlet boundary conditions

$$u(t,0)=u(t,\pi)=\frac{\partial^2u(t,0)}{\partial x^2}=\frac{\partial^2u(t,\pi)}{\partial x^2}=\cdots=\frac{\partial^{2k-2}u(t,0)}{\partial x^{2k-2}}=\frac{\partial^{2k-2}u(t,\pi)}{\partial x^{2k-2}}=0. \tag{2}$$

The parameter $m$ is real and positive, the integer $k \geq 1$ and the nonlinearity $f$ is assumed to be real analytic in $u$ and of the form

$$f(u) = au^3 + \sum_{n\geq 5} f_n u^n, \quad a \neq 0. \tag{3}$$

We study equations (1) as a Hamiltonian system on $\mathcal{P} = H_0^1([0,\pi]) \times L^2([0,\pi])$ with coordinates $u$ and $v = u_t$. The Hamiltonian is then

$$H = \frac{1}{2}\langle v, v\rangle + \frac{1}{2}\langle Au, u\rangle + \int_0^\pi g(u)dx, \tag{4}$$

where $A = \sum_{\bar{r}=1}^{2k}(-1)^{\bar{r}} d^{2\bar{r}}/dx^{2\bar{r}} + m$, $g = \int_0 -f(s)ds$ and $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $L^2$. Then, equations (1) can be written in the form

$$u_t = \frac{\partial H}{\partial v} = v, \quad v_t = -\frac{\partial H}{\partial u} = -Au - f(u). \tag{5}$$

Let

$$\phi_j(x) = \sqrt{\frac{2}{\pi}} \sin jx, \quad \lambda_j = \sqrt{j^{2k} + j^{2k-2} + \cdots + j^2 + m}, \quad j = 1, 2, \cdots$$

be the basic modes and frequencies of the linear equations $u_{tt} + \sum_{\bar{r}=1}^{2k}(-1)^{\bar{r}} \frac{\partial^{2\bar{r}} u}{\partial x^{2\bar{r}}} + mu = 0$ with Dirichlet boundary conditions (2). Then every solution of the linear equations are the superposition of their harmonic oscillations and of the form

$$u(t,x) = \sum_{j\geq 1} q_j(t)\phi_j(x), \quad q_j(t) = \sqrt{I_j}\cos(\lambda_j t + \theta_j)$$

with amplitudes $I_j \geq 0$ and initial phases $\theta_j$. The motions are periodic or quasi-periodic, respectively, depending on whether one or finitely many eigenfunctions are excited. In particular, for every choice $J = \{j_1 < j_2 < \cdots < j_n\} \subset \mathbf{N}$ of finitely many modes there is an invariant $2n$- dimensional linear subspace $E_J$ that is completely foliated into rotational tori with frequencies $\lambda_{j_1}, \cdots, \lambda_{j_n}$ :

$$E_J = \{(u,v) = (q_1\phi_{j_1} + \cdots + q_n\phi_{j_n}, \quad p_1\phi_{j_1} + \cdots + p_n\phi_{j_n})\} = \bigcup_{I\in\overline{\mathbf{P}^n}} \mathcal{T}_J(I),$$

where $\mathbf{P}^n = \{I \in \mathbf{R}^n : I_j > 0, \text{ for } 1 \leq j \leq n\}$ is the positive quadrant in $\mathbf{R}^n$ and

$$\mathcal{T}_J(I) = \{(u,v) : q_j^2 + \lambda_j^{-2}p_j^2 = I_j, \text{ for } 1 \leq j \leq n\},$$

using the above representation of $u$ and $v$. In addition, such torus is linearly stable, and all solutions have zero Lyapunov exponents.

Upon restoration of the nonlinearity $f$, we will show that there exists a Cantor set $\mathcal{O} \subset \mathbf{P}^n$, a family of $n$- tori $\mathcal{T}_J[\mathcal{O}] = \bigcup_{I \in \mathcal{O}} \mathcal{T}_J(I) \subset E_J$ over $\mathcal{O}$, and a Whitney smooth embedding $\Phi : \mathcal{T}_J[\mathcal{O}] \to \mathcal{E}_J \subset \mathcal{P}$, such that the restriction of $\Phi$ to each $\mathcal{T}_J(I)$ in the family is an embedding of a rotational $n$-torus for the nonlinear equations. The image $\mathcal{E}$ of $\mathcal{T}_J[\mathcal{O}]$ is called the Cantor manifold of rotational $d$- tori in [9].

**Theorem 1.1** (**Main Theorem**) *Suppose the nonlinearity $f$ is real analytic and of the form (3). Then for almost all $m > 0$, we can find index set $J = \{j_1 < \cdots < j_n\}$ with $j_1$ large enough and $\min_{1 \le i \le n} j_{i+1}^k - j_i^k \le n-1$ to confirm that there exists a Cantor manifold $\mathcal{E}_J$ given by a Whitney smooth embedding $\Phi : \mathcal{T}_J[\mathcal{O}] \to \mathcal{E}_J$, which is a higher order perturbation of the inclusion map $\Phi_0 : E_J \to \mathcal{P}$ restricted to $\mathcal{T}_J[\mathcal{O}]$. Moreover, the Cantor manifold $\mathcal{E}_J$ is constructed by real analytic, linearly stable, $n$- dimensional invariant tori carrying quasi-periodic solutions.*

When $k = 1$, there have been many results on the existence of quasi-periodic solutions for the equations (1)-(2). The first result in this direction is due to Bobenko and Kuksin [1], they get the quasi-periodic solutions corresponding to finite dimensional invariant tori of (1)-(2). Their starting point is to take the equations (1) as a perturbed sine-Gordon equations. This result is regained by Pöschel [9] by the infinite KAM theory and the normal form technique. Later, the existence of quasi-periodic solutions of the Hamiltonian partial differential equations have been studied by many authors, see [2, 3, 4, 5, 6, 12] and the references there in.

In this paper, using the KAM approach originating from Kuksin [7], Wayne [13] and Pöschel [10], we can obtain that equations (1) admit small-amplitude quasi-periodic solutions for any $k \ge 1$ and almost $m > 0$. The rest of the paper is organized as follows. In Section 2, we formulate an infinite-dimensional KAM theorem for nonlinear partial differential equations, which is given by Kuksin and Pöschel [8]. In Section 3, the Hamiltonian function is written in infinitely many coordinates, which is then put into the partial normal form. In Section 4, we complete the proof of the Main Theorem by applying the infinite-dimensional KAM theorem.

# 2 An Infinite-dimensional KAM Theorem

Let $l^2$ be the Hilbert space of bi-infinite square summable sequences with complex coefficients. For $a \ge 0$, $s > 0$, let the subspace $l^{a,s} \subset l^2$ consists, by definition, of all real sequences $q = (q_1, q_2, \cdots)$ with finite norm $\| q \|_{a,s}^2 = \sum_{j \ge 1} |q_j|^2 j^{2s} e^{2ja}$. We now consider a Hamiltonian $H = \Lambda + Q + R$ in a neighborhood of the origin in $l^{a,s}$, where $R$ represents some higher order perturbation of an integrable normal form $\Lambda + Q$. More precisely, set the complex coordinates $q = (\tilde{q}, \hat{q})$ on $l^{a,s}$, where $\tilde{q} = (q_1, \cdots, q_n)$ and $\hat{q} = (q_{n+1}, q_{n+2}, \cdots)$,

$$I = \frac{1}{2}(|q_1|^2, \cdots, |q_n|^2), \quad Z = \frac{1}{2}(|q_{n+1}|^2, |q_{n+2}|^2, \cdots),$$

and the normal form consists of the terms

$$\Lambda = \langle \alpha, I \rangle + \langle \beta, Z \rangle, \quad Q = \frac{1}{2}\langle AI, I \rangle + \langle BI, Z \rangle,$$

where $\alpha, \beta, A, B$ denote vectors and matrices with constant coefficients, respectively. The equations of motion of Hamiltonian $\Lambda + Q$ are

$$\dot{\tilde{q}} = \mathrm{i}\ \mathrm{diag}(\alpha + AI + B^T)\tilde{q}, \quad \dot{\hat{q}} = \mathrm{i}\ \mathrm{diag}(\beta + BI)\hat{q},$$

where $T$ means the transpose of the matrix. There exists a complex $n$- dimensional invariant manifold $E = \{\hat{q} = 0\}$, which is completely filled, up to the origin, by the invariant tori

$$\mathcal{T}(I) = \{\tilde{q} : |\tilde{q}_i|^2 = 2I_i \text{ for } 1 \leq i \leq n\}, \quad I \in \overline{\mathbf{P}^n},$$

where $\mathbf{P}^n = \{I : I_j > 0 \text{ for } j \in \mathbf{N}\}$ is the positive quadrant in $R^n$. On $\mathcal{T}(I)$ and in its normal space, respectively, the flows are given by

$$\dot{\tilde{q}} = \mathrm{i}\ \mathrm{diag}(\omega(I))\tilde{q}, \qquad \omega(I) = \alpha + AI,$$

$$\dot{\hat{q}} = \mathrm{i}\ \mathrm{diag}(\Omega(I))\hat{q}, \qquad \Omega(I) = \beta + BI.$$

Since $\Omega(I)$ is real and $\hat{q} = 0$ is an elliptic fixed point, all the tori are linearly stable, and all their orbits have zero Lyapunov exponents. We call $\mathcal{T}(I)$ an elliptic rotational torus with frequencies $\omega(I)$.

Due to resonances, the manifold $E$ does in general not persist in its entirety under the inclusion of the higher order term $R$. Instead, our aim is to prove the persistence of a large portion of $E$ forming an invariant Cantor manifold $\mathcal{E}$ for the Hamiltonian $H = \Lambda + Q + R$. That is, there exists a family of $n$- tori $\mathcal{T}[\mathcal{C}] = \bigcup_{I \in \mathcal{C}} \mathcal{T}(I) \subset E$ over a Cantor set $\mathcal{C} \subset \mathbf{P}^n$ and a Lipschitz continuous embedding $\Psi : \mathcal{T}[\mathcal{C}] \hookrightarrow l^{a,s}$, such that the restriction of $\Psi$ to each torus $\mathcal{T}(I)$ in the family is an embedding of an elliptic rotational $n$- torus for the Hamiltonian $H$. For the image $\mathcal{E}$ of $\mathcal{T}[\mathcal{C}]$ we call a Cantor manifold of elliptic rotational $n$- tori given by the embedding $\Psi : \mathcal{T}[\mathcal{C}] \to \mathcal{E}$. In addition, the Cantor set $\mathcal{C}$ has full density at the origin, the embedding $\Psi$ is close to the inclusion map $\Psi_0 : E \hookrightarrow l^{a,s}$, and the Cantor manifold $\mathcal{E}$ is tangent to $E$ at the origin. For the existence of $\mathcal{E}$ the following assumptions are made.

*A. Non-degeneracy.* The normal form $\Lambda + Q$ is non-degenerate in the sense that

$$\det A \neq 0; \quad \langle l, \beta \rangle \neq 0; \quad \langle k, \omega(I) \rangle + \langle l, \Omega(I) \rangle \neq 0,$$

for all $(k,l) \in \mathbf{Z}^n \times \mathbf{Z}^\infty$ with $1 \le |l| \le 2$, where $\omega = \alpha + AI$ and $\Omega = \beta + BI$.
*B. Spectral Asymptotics.* There exist $d \ge 1$ and $\delta < d-1$ such that

$$\beta_j = j^d + \cdots + O(j^\delta),$$

where the dots stands for fixed lower order terms in $j$, allowing also negative exponents.
*C. Regularity.*

$$X_Q,\ X_R \in \mathcal{A}(l^{a,s},\ l^{a,\bar{s}}),\quad \begin{cases} \bar{s} \ge s, & d > 1; \\ \bar{s} > s, & d = 1, \end{cases}$$

where $\mathcal{A}(l^{a,s},\ l^{a,\bar{s}})$ denotes the class of maps from some neighborhood of the origin in $l^{a,s}$ into $l^{a,\bar{s}}$, which are real analytic in the real and imaginary parts of the complex coordinates $q$.

**Theorem 2.1** *(***The Cantor Manifold Theorem, Pöschel** [**8**]*). Suppose the Hamiltonian $H = \Lambda + Q + R$ satisfies assumptions A, B and C, and*

$$|R| = O(\|q\|_{a,s}^g) + O(\|\hat{q}\|_{a,s}^4)$$

*with*

$$g > 4 + \frac{4 - \triangle}{\kappa},\quad \triangle = \min(\bar{s} - s, 1).$$

*Then there exists a Cantor manifold $\mathcal{E}$ of real analytic, elliptic diophantine n-tori given by a Lipschitz continuous embedding $\Psi : \mathcal{T}[\mathcal{C}] \to \mathcal{E}$, where $\mathcal{C}$ has full density at the origin, and $\Psi$ is close to the inclusion map $\Psi_0$:*

$$\|\Psi - \Psi_0\|_{a,\bar{s},B_r \cap \mathcal{T}[\mathcal{C}]} = O(r^\sigma),\quad \sigma = \frac{g}{2} - \frac{\kappa + 1 - \triangle/4}{\kappa} > 1.$$

*Consequently, $\mathcal{E}$ is tangent to $E$ at the origin.*

# 3 The Hamiltonian for 2k Order Wave Equations

We recall that the Hamiltonian of our nonlinear $2k$ order wave equations are

$$H = \frac{1}{2}\langle v, v\rangle + \frac{1}{2}\langle Au, u\rangle + \int_0^\pi g(u)dx. \tag{6}$$

As in [9], we introduce coordinates $q = (q_1, q_2, \cdots), p = (p_1, p_2, \cdots)$ through the relations

$$u = \sum_{j\ge 1} \frac{q_j}{\sqrt{\lambda_j}}\phi_j, \qquad v = \sum_{j \ge 1} \sqrt{\lambda_j} p_j \phi_j, \tag{7}$$

where $\phi_j = \sqrt{\frac{2}{\pi}} \sin jx$, for $j = 1, 2, \cdots$, are the normalized Dirichlet eigenfunctions of the operator $A$ with eigenvalues $\lambda_j^2 = j^{2k} + j^{2k-2} + \cdots + j^2 + m$, the coordinates $q$ and $p$ are taken from some Hilbert space $l^{a,s}$. We obtain the Hamiltonian

$$H = \Lambda + G = \frac{1}{2} \sum_{j \geq 1} \lambda_j (p_j^2 + q_j^2) + \int_0^\pi g(\sum_{j \geq 1} \frac{q_j}{\sqrt{\lambda_j}} \phi_j) dx \tag{8}$$

with the lattice Hamiltonian equation

$$\dot{q}_j = \frac{\partial H}{\partial p_j} = \lambda_j p_j, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j} = -\lambda_j q_j - \frac{\partial G}{\partial q_j}. \tag{9}$$

But instead of discussing its validity, we just take the latter Hamiltonian as our new starting point and make the following simple observation.

**Lemma 3.1** *Let $a \geq 0$ and $s > 0$, $I$ be an interval, and $t \in I \to (q(t), p(t))$ be a real analytic solution of (9) such that*

$$\sup_{t \in I} \sum_{j \geq 1} (|q_j(t)|^2 + |p_j(t)|^2) j^{2s} e^{2ja} < \infty.$$

*Then $u = \sum\limits_{j \geq 1} \frac{q_j}{\sqrt{\lambda_j}} \phi_j$ is an analytic solution of (1).*

**Proof.** For $a \geq 0$, and $s > 0$, the sum in question and its termwise $t$-derivative of first and second order are absolutely convergent in some complex neighborhood of $x$- interval $[0, \pi]$ and some complex disc around a given $t$ in $I$. Hence, we find that

$$\begin{aligned} u_{tt} &= \sum_{j \geq 1} \frac{\ddot{q}_j}{\sqrt{\lambda_j}} \phi_j \\ &= \sum_{j \geq 1} \frac{1}{\sqrt{\lambda_j}} (-\lambda_j^2 q_j - \sqrt{\lambda_j} \langle f(u), \phi_j \rangle) \phi_j \\ &= -\sum_{j \geq 1} \frac{q_j}{\sqrt{\lambda_j}} A\phi_j - \sum_{j \geq 1} \langle f(u), \phi_j \rangle \phi_j \\ &= -Au - f(u) \end{aligned}$$

by the orthogonality and completeness of the $\phi_j$.

Next we consider the regularity of the vector field of $G$. Let $l^2$ be the Hilbert space of bi-infinite square summable sequences with complex coefficients. For $a \geq 0$, $s > 0$, let the subspace $l^{a,s} \subset l^2$ consist, by definition, all bi-infinite sequences with finite norm

$$\| q \|_{a,s}^2 = |q_0|^2 + \sum_j |q_j|^2 |j|^{2s} e^{2|j|a}.$$

Let $\mathcal{F} : l^{a,s} \to L^2, \quad q \mapsto \mathcal{F}q = \frac{1}{\sqrt{2\pi}} \sum_j q_j e^{\mathrm{i}jx}$ be the inverse discrete Fourier transform, which defines an isometry between the two spaces, where $L^2$ is all square-integrable complex valued functions on $[-\pi, \pi]$. Through $\mathcal{F}$ they define subspaces $W^{a,s} \subset L^2$ that are normed by setting $\|\mathcal{F}q\|_{a,s} = \|q\|_{a,s}$.

**Lemma 3.2** *For $a \geq 0, s > \frac{1}{2}$, the space $l^{a,s}$ is a Hilbert algebra with respect to convolution of sequences and*

$$\| q * p \|_{a,s} \leq C \| q \|_{a,s} \| p \|_{a,s}$$

*with a constant $C$ depending on $s$. Consequently, $W^{a,s}$ is a Hilbert algebra with respect to multiplication of functions.*

**Proof.** Let $[j] = \max(|j|, 1)$, $\gamma_{jr} = \frac{[j-r][r]}{[j]}$. By the Schwarz inequality

$$|\sum_r x_r|^2 = |\sum_r \frac{\gamma_{jr}^s x_r}{\gamma_{jr}^s}|^2 \leq c_{jr}^2 \sum_r \gamma_{jr}^{2s} |x_r|^2, \quad c_{jr}^2 = \sum_r \frac{1}{\gamma_{jr}^{2s}},$$

for all $j$. We have

$$\frac{1}{\gamma_{jr}} \leq \frac{[j-r]+[r]}{[j-r][r]} = \frac{1}{[j-r]} + \frac{1}{[r]},$$

so that

$$c_{jk}^2 \leq \sum_k (\frac{1}{[j-r]} + \frac{1}{[r]})^{2s} \leq 4^s \sum_r \frac{1}{[r]^{2s}} = C^2 < \infty.$$

It follows that

$$\begin{aligned} \|q * p\|^2 &= \sum_j [j]^{2s} |\sum_r q_{j-r} p_k|^2 e^{2|j|a} \\ &\leq C^2 \sum_j [j]^{2s} \sum_r \gamma_{jr}^{2s} |q_{j-r} p_r|^2 e^{2(|j-r|+|r|)a} \\ &= C^2 \sum_j [j-r]^{2s} |q_{j-r}|^2 e^{2[j-r]a} \sum_r [r]^{2s} |p_r|^2 e^{2[r]a} \\ &= C^2 \|q\|_{a,s}^2 \|p\|_{a,s}^2. \end{aligned}$$

**Lemma 3.3** *For $a \geq 0$ and $s > 0$, the vector field $X_G$ is a map from some neighborhood of the origin in $l^{a,s}$ into $l^{a,s+k}$, with $\| X_G \|_{a,s+k} = O(\| q \|_{a,s}^3)$.*

**Proof.** In a sufficient small neighborhood of the origin, we can consider the nonlinearity $f = u^3$. Due to

$$G = \frac{1}{4} \int_0^{\pi} |u(x)|^4 dx = \frac{1}{4} \sum_{i,j,r,l} G_{ijrl} q_i q_j q_r q_l,$$

we have $\frac{\partial G}{\partial q_l} = \sum_{i,j,r} G_{ijrl} q_i q_j q_r$. Hence

$$\begin{aligned}
\|G_q\|^2_{a,s+k} &= \sum_{l\geq 1} |G_{q_l}|^2 l^{2(s+k)} e^{2al} \\
&\leq c\sum_{l\geq 1}\sum_{\pm i\pm j\pm r=l}\left(\frac{1}{\sqrt{\lambda_i\lambda_j\lambda_r\lambda_l}}|q_iq_jq_r|\right)^2 l^{2(s+k)}e^{2al} \\
&\leq c\sum_{l\geq 1}\left(\frac{1}{l^{k/2}}\right)^2\sum_{\pm i\pm j\pm r=l}\left(\frac{|q_iq_jq_r|}{|i|^{k/2}|j|^{k/2}|r|^{k/2}}\right)^2 l^{2(s+k)}e^{2al} \\
&\leq c\sum_{l\geq 1}\frac{1}{l^k}(\tilde{q}*\tilde{q}*\tilde{q})^2 l^{2(s+k)}e^{2al} \\
&\leq c\sum_{l\geq 1}(\tilde{q}*\tilde{q}*\tilde{q})^2 l^{2(s+k/2)}e^{2al} \\
&\leq c\|\tilde{q}*\tilde{q}*\tilde{q}\|^2_{a,s+k/2} \\
&\leq c(\|\tilde{q}\|^2_{a,s+k/2})^3 \\
&\leq c(\|q\|^2_{a,s})^3,
\end{aligned}$$

with $\tilde{q}_j = \frac{|q_j|}{j^{k/2}}$, where the constant $c$ above may be different. Hence $\|G_q\|_{a,s+k} \leq c(\|q\|_{a,s})^3$. The regularity of $X_G$ follows from the regularity of its components.

For the nonlinearity $u^3$ we find

$$G = \frac{1}{4}\int_0^\pi |u(x)|^4 dx = \frac{1}{4}\sum_{i,j,r,l} G_{ijrl} q_i q_j q_r q_l \tag{10}$$

with $G_{ijrl} = \frac{1}{\sqrt{\lambda_i\lambda_j\lambda_r\lambda_l}}\int_0^\pi \phi_i\phi_j\phi_r\phi_l dx$. It is not difficult to verify that $G_{ijrl} = 0$ unless $\pm i \pm j \pm r \pm l = 0$, for some combination of plus and minus signs. Particularly, we have

$$G_{iijj} = \frac{1}{2\pi}\frac{2+\delta_i^j}{\lambda_i\lambda_j} \tag{11}$$

by elementary calculation. In the following, we focus on the nonlinearity $u^3$, since a non-zero coefficient in front of $u^3$ and all terms of order five or more will make no difference.

Next we transform the Hamiltonian (8) into some partial Birkhoff form of order four so that it may serve as a small perturbation of some nonlinear integrable system in a sufficiently small neighborhood of the origin. we introduce the complex coordinate $z_j = \frac{1}{\sqrt{2}}(q_j + \mathrm{i}p_j)$, $\bar{z}_j = \frac{1}{\sqrt{2}}(q_j - \mathrm{i}p_j)$. Then Hamiltonian is given by

$$H = \Lambda + G = \sum_j \lambda_j |z_j|^2 + \int_0^\pi g\left(\sum_j \frac{z_j + \bar{z}_j}{\sqrt{2\lambda_j}}\phi_j\right)dx, \tag{12}$$

with symplectic structure $\mathrm{i}\sum_j dz_j \wedge d\bar{z}_j$.

In the following, we always suppose $m \in \mathcal{I} = (0, M^*]$, where $M^*$ is a fixed large number.

**Lemma 3.4** *If $\{i, j, r, l\}$ are nonzero integers, such that $i \pm j \pm r \pm l = 0$, but $(i, j, r, l) \neq (p, -p, q, -q)$, then for almost all $m \in \mathcal{I}$ we have*

$$|\lambda_i \pm \lambda_j \pm \lambda_r \pm \lambda_l| \geq c,$$

*where $c$ is a constant depending on $k$ and $m$.*

**Proof.** Let

$$f(m) = k_1\lambda_{i_1} + k_2\lambda_{i_2} + k_3\lambda_{i_3} + k_4\lambda_{i_4},$$

where $k_{\bar{j}} \in \{1, -1\}$, $0 \leq |k_1+k_2+k_3+k_4| \leq 4, \lambda_{i_{\bar{j}}} = \sqrt{i_{\bar{j}}^{2k} + i_{\bar{j}}^{2k-2} + \cdots + i_{\bar{j}}^2 + m}$, $i_{\bar{j}} \in \{i, j, r, l\}$, $\bar{j} = 1, 2, 3, 4$. Obviously, we have

$$f^{(n)}(m) = c_n(k_1\lambda_{i_1}^{(\frac{1}{2}-n)} + k_2\lambda_{i_2}^{(\frac{1}{2}-n)} + k_3\lambda_{i_3}^{(\frac{1}{2}-n)} + k_4\lambda_{i_4}^{(\frac{1}{2}-n)}),$$

where $c_n = (-1)^{n-1}\frac{(2n-1)!!}{2^n}$. For convenience, we let $i_1 = \min\{i, j, r, l\}$. It is easy to check that,

$$|k_1\lambda_{i_1}| \geq |k_2\lambda_{i_2} + k_3\lambda_{i_3} + k_4\lambda_{i_4}|$$

when $n > N_0 = \frac{\lg 4}{\lg \frac{i_i^{2k}+i_i^{2k-2}+\cdots+i_i^2+M^*}{(i_i+1)^{2k}+(i_i+1)^{2k-2}+\cdots+(i_i+1)^2+M^*}} + \frac{1}{2}$. Therefor, there exists positive integer $n_0$ such that $|f^{(n_0)}(m)| > 0$. For example, we choose $n_0 = [N_0]+1$. So, there is at most $n_0 m'$s such that $f(m) = 0$. Similar as Lemma 2.1 in [11], it is the fact that: if $g(m)$ is $n$-th differentiable function on $\mathcal{I}$, let $I_c = \{m : |g(m)| < c, m \in \mathcal{I}, c > 0\}$, and on $\mathcal{I}$, $g^{(n)}(m) > \bar{d} > 0$, $\bar{d}$ is a constant. Then $I_c \leq \bar{k}c^{\frac{1}{n}}$, where $\bar{k} = 2(2 + 3 + \cdots + n + \bar{d}^{-1})$. By the fact above, we can easily get the conclusion.

**Proposition 3.5** *For the index set $J = \{j_1 < \cdots < j_n\}$ with $\min\limits_{1\leq i\leq n} j_{i+1} - j_i \leq n-1$, and almost all $m \in (0, +\infty)$ there exists a change of coordinates $\Gamma$ in a neighborhood of the origin in $l^{a,s}$ such that the Hamiltonian $H = \Lambda + G$ with the nonlinearity (10) is changed into*

$$H \circ \Gamma = \Lambda + \bar{G} + \hat{G} + K,$$

*where $X_{\bar{G}}, X_{\hat{G}}, X_K : l^{a,s} \to l^{a,s+k}, \bar{G} = \frac{1}{2} \sum\limits_{\text{one of } \{i,j\}\in J} \bar{G}_{ij}z_i^2 z_j^2$ , with coefficient $\bar{G}_{ij} = \frac{6}{\pi}\frac{4-\delta_{ij}}{\lambda_i\lambda_j}$, and $|\hat{G}| = O(\|\hat{z}\|_{a,s}^4), |K| = O(\|z\|_{a,s}^6)$, $\hat{z} = \{z_j\}_{j\notin J}$. Moreover, the dependence of $\Gamma$ on $m$ is real analytic for almost all compact $m$- interval in $(0, +\infty)$ .*

**Proof.** It is convenient to introduce coordinates $(\cdots, w_{-2}, w_{-1}, w_1, w_2, \cdots)$ in $l^{a,s}$ by setting $z_j = w_j,\ \bar{z}_j = w_{-j}$. Let $\lambda_i^{'} = (\text{sgn } i)\lambda_{|i|}$. The Hamiltonian under consideration then reads

$$H = \sum_n \lambda_n w_n w_{-n} + \sum_{i,j,r,l} G_{ijrl} w_i w_j w_r w_l. \tag{13}$$

Consider a Hamiltonian function $F = \sum_{i,j,r,l} F_{ijrl} w_i w_j w_r w_l$ with coefficients

$$\mathrm{i}F_{ijrl} = \begin{cases} \frac{G_{ijrl}}{\lambda_i^{'}+\lambda_j^{'}+\lambda_r^{'}+\lambda_l^{'}}, & \text{for } \{|i|,|j|,|r|,|l|\} \in \mathcal{L}_n \backslash \mathcal{N}_n, \\ 0, & \text{otherwise.} \end{cases}$$

Here $\mathcal{L}_n = \{(i,j,r,l) \in \mathbf{Z}^4 : \text{one of } (|i|,|j|,|r|,|l|) \in J = \{j_1, \cdots, j_n\}\}$, $\mathcal{N}_n = \{(i,j,r,l) \in \mathbf{Z}^4 : (i,j,r,l) = (p,-p,q,-q)\} \subset \mathcal{L}_n$. Let $\Gamma$ be the time-1 map of the flow of the Hamiltonian vector field $F$. Expanding at $t = 0$ and using Taylor's formula, we obtain

$$\begin{aligned} H \circ \Gamma &= H + \{H,F\} + \textstyle\int_0^1 (1-t)\{\{H,F\},F\} \circ X_F^t dt \\ &= \Lambda + \{\Lambda, F\} + G + \{G,F\} + \textstyle\int_0^1 (1-t)\{\{H,F\},F\} \circ X_F^t dt, \end{aligned}$$

where $\{\Lambda, F\} = -\mathrm{i}\sum_{i,j,r,l}(\lambda_i^{'} + \lambda_j^{'} + \lambda_r^{'} + \lambda_l^{'})\, F_{ijrl} w_i w_j w_r w_l$, hence $G + \{\Lambda, F\} = \sum_{(i,j,r,l)\in\mathcal{N}_n} + \sum_{(i,j,r,l)\notin\mathcal{L}_n} G_{ijrl} w_i w_j w_r w_l = \bar{G} + \hat{G}$. Returning to the notations $z_j, \bar{z}_j$, we have

$$\bar{G} = \frac{1}{2} \sum_{\text{one of } (i,j)\in J} \bar{G}_{ij} |z_i|^2 |z_j|^2$$

with

$$\bar{G}_{ij} = \begin{cases} 24G_{iijj} = \frac{24}{\pi} \cdot \frac{1}{\lambda_i \lambda_j} & \text{for } i \neq j, \\ 12G_{iijj} = \frac{18}{\pi} \cdot \frac{1}{\lambda_i \lambda_j} & \text{for } i = j, \end{cases}$$

by (11), where $\hat{G}$ is independent of $\{z_j\}_{j\notin J}$. Hence, formally we have $H \circ \Gamma = \Lambda + \bar{G} + \hat{G} + K$ as claimed.

To prove analyticity and regularity of the preceding transformation we first show that $X_F : l^{a,s} \to l^{a,s+k}$. Indeed, by Lemma3.4 and equation (10) with $\tilde{w}_j = \frac{1}{|j|^{k/2}}(|w_j| + |w_{-j}|)$, we have

$$\begin{aligned} |\frac{\partial F}{\partial w_l}| &\leq \sum_{\pm i \pm j \pm r = l} |F_{ijrl}||w_i w_j w_r| \\ &\leq \frac{c}{|l|^{k/2}} \sum_{\pm i \pm j \pm r = l} \frac{|w_i w_j w_r|}{|ijr|^{k/2}} \\ &\leq \frac{c}{|l|^{k/2}} \sum_{\pm i \pm j \pm r = l} \tilde{w}_i \tilde{w}_j \tilde{w}_r \\ &= \frac{c}{|l|^{k/2}} (\tilde{w} * \tilde{w} * \tilde{w})_l. \end{aligned}$$

By Lemma3.2, we have

$$\|F_w\|_{a,s+k} \leq c\|\tilde{w} * \tilde{w} * \tilde{w}\|_{a,s+k/2} \leq \|w\|_{a,s}^3. \tag{14}$$

The analyticity of $F_w$ follows from the analyticity of each component functions and its local boundedness. Hence in a sufficiently small neighborhood of the origin in $l^{a,s}$ the time-1-map $\Gamma$ is well defined with the estimates

$$\|\Gamma - id\|_{a,s+k} = O(\|w\|_{a,s}^3), \quad \|D\Gamma - I\|_{a,s+k,s} = O(\|w\|_{a,s}^2),$$

where the operator norm $\|\cdot\|_{a,\bar{r},s}$ is defined by $\|A\|_{a,\bar{r},s} = \sup_{w\neq 0} \frac{\|Aw\|_{a,\bar{r}}}{\|w\|_{a,s}}$. Obviously, $\|D\Gamma - I\|_{a,s+k,s+k} \leq \|D\Gamma - I\|_{a,s+k,s}$, while in a sufficiently small neighborhood of the origin, $D\Gamma$ defines an isomorphism of $l^{a,s+k}$. Since $X_H : l^{a,s} \to l^{a,s+k}$, then

$$\Gamma^* X_H = D\Gamma^{-1} X_H \circ \Gamma = X_{H\circ\Gamma} : l^{a,s} \to l^{a,s+k}.$$

The same holds for the Lie bracket: the boundedness of $\|DX_F\|_{a,s+k,s}$ implies that $[X_F, X_H] = X_{\{H,F\}} : l^{a,s} \to l^{a,s+k}$. These two facts show that $X_K : l^{a,s} \to l^{a,s+k}$. The analogous claims for $X_{\bar{G}}$ and $X_{\hat{G}}$ are obvious.

# 4 Proof of the Main Theorem

We now prove Theorem 1.1 by applying Theorem 2.1. By Section 3 there exists a real analytic, symplectic change of coordinates $\Gamma$ , which takes $H$ into $H \circ \Gamma = \Lambda + \bar{G} + \hat{G} + K$, with the notation of the previous section,

$$\Lambda = \langle \alpha, I\rangle + \langle \beta, Z\rangle, \quad \bar{G} = \frac{1}{2}\langle AI, I\rangle + \langle BI, Z\rangle, \quad |\hat{G}| = O(\|\hat{z}\|_{a,s}^4), \quad |K| = O(\|z\|_{a,s}^6),$$

where $\alpha = (\lambda_j)_{j\in J}, \beta = (\beta_j)_{j\notin J}, A = (\bar{G}_{ij})_{i,j\in J}, B = (\bar{G}_{ij})_{j\in J, i\notin J}, I = (|z_j|^2)_{j\in J}, Z = (|z_j|^2)_{j\notin J}$. Moreover, the regularity of the nonlinear vector field is preserved. We now verify the assumptions A, B and C of the Cantor Manifold Theorem for the above Hamiltonian. Since $Q = \bar{G}, |R| = \hat{G} + K = O(|\hat{z}|^4) + O(|z|^6)$, so $X_Q, X_R : l^{a,s} \to l^{a,s+k}$, thus the Assumption C holds true. On the other hand, $\lambda_j = \sqrt{j^{2k} + j^{2k-2} + \cdots + j^2 + m} = j^k + \cdots + \frac{m}{2j} + O(\frac{1}{j^3})$, then $\Omega_{j-n} = (\beta + BI)_{j-n} = \lambda_j + \frac{\langle \nu, I\rangle}{\lambda_j}$ with $\nu = \frac{24}{\pi}(\lambda_1^{-1}, \cdots, \lambda_n^{-1})$. This gives the asymptotic expansion

$$\Omega_{j-n} = j^k + \cdots + \frac{m}{2j} + \frac{\langle \nu, I\rangle}{j} + O(\frac{1}{j^3}) = j^k + \cdots + \frac{m_I}{j} + O(\frac{1}{j^3}),$$

with $m_I = \frac{1}{2}m + \langle \nu, I\rangle$. So Assumption B is satisfied with $d = k, \delta = -1$, and $\bar{s} = s + k$. Moreover, since

$$A = (\bar{G}_{ij})_{j_1 \leq i,j \leq j_n} = \frac{6}{\pi} \begin{pmatrix} \frac{3}{\lambda_{j_1}\lambda_{j_1}} & \frac{4}{\lambda_{j_1}\lambda_{j_2}} & \cdots & \frac{4}{\lambda_{j_1}\lambda_{j_n}} \\ \frac{4}{\lambda_{j_2}\lambda_{j_1}} & \frac{3}{\lambda_{j_2}\lambda_{j_2}} & \cdots & \frac{4}{\lambda_{j_2}\lambda_{j_n}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{4}{\lambda_{j_n}\lambda_{j_1}} & \frac{4}{\lambda_{j_n}\lambda_{j_2}} & \cdots & \frac{3}{\lambda_{j_n}\lambda_{j_n}} \end{pmatrix},$$

we have that $\det(\frac{\pi}{6}A) = (4n-1)\prod_{1\le j\le n}\frac{1}{\lambda_j^2} \neq 0$. So the matrix $A$ is non-degenerate.

**Lemma 4.1** *Denote by* $\omega(I) = \alpha + AI, \Omega(I) = \beta + BI$, *for any index set* $J = \{j_1, \cdots, j_n\}$ *with* $j_1$ *large enough and* $\min_{1\le i\le n} j_{i+1}^k - j_i^k \le n-1$, *one has*

$$\langle l, \beta\rangle \neq 0, \quad \langle k, \omega(I)\rangle + \langle l, \Omega(I)\rangle \neq 0,$$

*for all* $(k,l) \in \mathbf{Z}^n \times \mathbf{Z}^\infty$ *with* $1 \le |l| \le 2$.

**Proof.** Clearly $\langle l, \beta\rangle \neq 0$ for $1 \le |l| \le 2$. We have to show that either $\langle \alpha, k\rangle \neq \langle \beta, l\rangle$ or $Ak \neq B^\top l$. Suppose to the contrary $\langle \alpha, k\rangle = \langle \beta, l\rangle$ and $Ak = B^\top l$. Multiplying both $A$ and $B$ by $\frac{\pi}{6}$, we have

$$\frac{k_i}{\lambda_{j_i}} = 4\sum_{i=1}^n \frac{k_i}{\lambda_{j_i}} - 4\sum_p \frac{l_p}{\lambda_{j_p}}, \qquad 1 \le i \le n \quad j_p \notin J, \quad l = (l_p).$$

So $\frac{k_i}{\lambda_{j_i}}$ is independent of $i$, whence $4\sum_{i=1}^n \frac{k_i}{\lambda_{j_i}} = 4n\frac{k_i}{\lambda_{j_i}}$, and thus

$$k_i = \frac{4}{4n-1}(\sum_p \frac{l_p}{\lambda_{j_p}})\lambda_{j_i}, \quad 1 \le i \le n. \tag{15}$$

The assumption $\langle \alpha, k\rangle = \langle \beta, l\rangle$ then further implies

$$\frac{4}{4n-1}\sum_{1\le i\le n}\lambda_{j_i}^2 = \frac{\langle \beta, l\rangle}{\sum_p \frac{l_p}{\lambda_{j_p}}}. \tag{16}$$

When $|l| = 1$, $\langle \beta, l\rangle = \pm\lambda_{j_p}$, for any $j_p \notin J$, then we can get

$$k_i^2 = (\frac{4}{4n-1})^2\frac{\lambda_{j_i}^2}{\lambda_{j_p}^2}, \ 1 \le i \le n.$$

According to the equation(16), it implies that $\frac{4}{4n-1}\sum_{1\le i\le n}\lambda_{j_i}^2 = \frac{\langle\beta,l\rangle}{\sum_p \frac{l_p}{\lambda_{j_p}}} = \lambda_{j_p}^2$. Then we can obtain that $k_i^2 == \frac{4}{4n-1}\frac{\lambda_{j_i}}{\sum_{1\le i\le n}\lambda_{j_i}^2}$, $1 \le i \le n$. But this equation can not have an integer solution for any $n \ge 1$ and $1 \le i \le n$. When $|l| = 2$, if $|\sum_p \frac{l_p}{\lambda_{j_p}}| < \frac{4n-1}{4n-4}\frac{1}{C_J}$, where $C_J = \max_{0\le i\le n}\frac{\sum_{0\le\tilde{m}\le 2k-1, 0\le\tilde{r}\le k-1, \tilde{m}+2\tilde{r}\le m-1} j_{i+1}^{2k-1-\tilde{m}-2\tilde{r}} j_i^{\tilde{m}}}{\sqrt{j_{i+1}^{2k}+j_{i+1}^{2k-2}+\cdots+j_{i+1}^2+m+\sqrt{j_i^{2k}+j_i^{2k-2}+\cdots+j_i^2+m}}}$, is a constant depending on $n$ and the index set $J$ , then the equation (15) implies

$$\begin{aligned}
&\min_{1\le i\le n}|k_{i+1}-k_i| = \frac{4}{4n-1}|\lambda_{j_{i+1}}-\lambda_{j_i}|(\sum_p \frac{l_p}{\lambda_{j_p}}) < \min_{1\le i\le n}\frac{\lambda_{j_{i+1}}-\lambda_{j_i}}{(n-1)C_J}\\
= &\min_{1\le i\le n}\frac{\sqrt{j_{i+1}^{2k}+j_{i+1}^{2k-2}+\cdots+j_{i+1}^2+m}-\sqrt{j_i^{2k}+j_i^{2k-2}+\cdots+j_i^2+m}}{(n-1)C_J}\\
= &\min_{1\le i\le n}\frac{j_{i+1}-j_i}{(n-1)C_J}\frac{\sum\limits_{0\le\tilde m\le 2k-1,0\le\tilde r\le k-1,\tilde m+2\tilde r\le m-1} j_{i+1}^{2k-1-\tilde m-2\tilde r}j_i^{\tilde m}}{\sqrt{j_{i+1}^{2k}+j_{i+1}^{2k-2}+\cdots+j_{i+1}^2+m}+\sqrt{j_i^{2k}+j_i^{2k-2}+\cdots+j_i^2+m}}\\
\le &\ 1,
\end{aligned}$$

by assumption, which is not possible. On the other hand, the inequality $|\sum\limits_p \frac{l_p}{\lambda_{j_p}}| \ge \frac{4n-1}{4n-4}\frac{1}{C_J}$ implies $|\sum\limits_p \frac{l_p}{\lambda_{j_p}}| = \frac{1}{\lambda_{j_{p'}}}+\frac{1}{\lambda_{j_p}}$ with $l=(\cdots,l_p,\cdots,l_{p'},\cdots)=(\cdots,1,\cdots,1,\cdots)$ or$(\cdots,-1,\cdots,-1,\cdots)$. But

$$\frac{\langle\beta,l\rangle}{\sum\limits_p\frac{l_p}{\lambda_{j_p}}} = \frac{\lambda_{j'_p}+\lambda_{j_p}}{\frac{1}{\lambda_{j'_p}}+\frac{1}{\lambda_{j_p}}} \le \frac{4n-4}{4n-1}C_J(\lambda_{j'_p}+\lambda_{j_p}) \le 2\frac{4n-4}{4n-1}\lambda_{j_p}C_J \le 2\frac{4n-4}{4n-1}(2k-1)j_n^{k-1},$$

which leads to the equation(16). So we verify the nondegeneracy condition A.

By the Cantor Manifold Theorem, we obtain a Cantor manifold $\mathcal{E}$ of real analytic, elliptic diophantine $n$- tori in $l^{a,s}$ for the Hamiltonian $H=\Lambda+G$ in complex coordinates given by an embedding $\Gamma\circ\Phi : \mathcal{T}[\mathcal{C}] \to \mathcal{E}$. These tori carry the quasi-periodic motions $z(t)=\Gamma\circ\Phi(e^{\mathrm{i}\omega(I)t}\nu_0)$, for $I\in\mathcal{C}$ and $\nu_0\in\mathcal{T}(I)$. Their real and imaginary parts, $q=\mathrm{Re}z$ and $p=\mathrm{Im}z$ , solve the equations of motions for the corresponding hamiltonian (9) in real coordinates. Going back to the space $l^{a,s+k/2}$ by the isomorphism $q\longmapsto u=\sum\limits_{j\ge1}\frac{q_j}{\sqrt{\lambda_j}\phi_j(x)}$, $\mathcal{E}$ is mapped into another Cantor manifold of real analytic diophantine tori in $l^{a,s+k/2}$ , which by Lemma 3.3 carry real analytic, quasi-periodic solutions $u$ of the given nonlinear wave equations. This prove the Main Theorem.

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