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On Integral Operators of $(p + \alpha)$-valent Analytic Functions

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Abstract

In the present paper, two integral operators $F_{p.g}(\lambda, \alpha; z)$ and $G_{p.g}(\lambda, \alpha; z)$, defined using a differential operator, for $f \in \Sigma_{p,\alpha}$, are introduced. Certain classes defined by these operators are introduced and sufficient conditions for functions belonging to these classes are obtained. Some subclasses of $f \in \Sigma_{p,\alpha}$ are introduced and some of the properties of the integral operators $F_{p.g}(\lambda, \alpha; z)$ and $G_{p.g}(\lambda, \alpha; z)$ on these classes are discussed. Also subordination results for functions $f \in \Sigma_{p,\alpha}$ associated with the differential operator are obtained.

Keywords: Multivalent analytic functions, Hadamard product, differential operator, integral operator, starlike and convex functions, differential subordination

1 Introduction, Definitions and Preliminaries

Let $\Sigma_{p,\alpha}$ denote the class of functions of the form

$$f(z) = z^{p+\alpha} + \sum_{n=2}^{\infty} a_n z^{n+p+\alpha}, \quad p \in \mathbb{N}, 0 \leq \alpha < 1, z \in E \tag{1.1}$$

which are analytic in the unit disk $E = \{z \in \mathbb{C} / |z| < 1\}$.

For functions $f(z)$ given by (1.1) and $g(z)$ given by

$$g(z) = z^{p+\alpha} + \sum_{n=2}^{\infty} b_n z^{n+p+\alpha} \tag{1.2}$$

the Hadamard product or convolution of f and g is defined by

$$(f * g)(z) = z^{p+\alpha} + \sum_{n=2}^{\infty} a_n b_n z^{n+p+\alpha} \tag{1.3}$$

Let $T_{p,\alpha}$ be the subclass of $\Sigma_{p,\alpha}$ consisting of functions of the form

$$f(z) = z^{p+\alpha} - \sum_{n=2}^{\infty} |a_n| z^{n+p+\alpha}, \quad 0 \leq \alpha < 1, p \in \mathbb{N}, z \in E \tag{1.4}$$

which are analytic in the unit disk E.

The Hadamard product or convolution of $f(z)$ given by (1.4) and

$$g(z) = z^{p+\alpha} - \sum_{n=2}^{\infty} |b_n| z^{n+p+\alpha}, \quad 0 \leq \alpha < 1, p \in \mathbb{N}, z \in E$$

is defined by

$$f(z) * g(z) = z^{p+\alpha} - \sum_{n=2}^{\infty} |a_n||b_n| z^{n+p+\alpha}, \quad 0 \leq \alpha < 1, p \in \mathbb{N}, z \in E$$

These classes were studied by Ibrahim and Darus in [4]. Note that the class $T_{p,\alpha}$ reduces to the class of functions with negative coefficients $T_{1,0} = T$ introduced and studied by Silverman [14] when $\alpha = 0$, $p = 1$ and $|a_n| = a_n \geq 0$. The classes $S_{p,\alpha}(\delta)$, $C_{p,\alpha}(\delta)$ and $P_\alpha(p, \delta)$ for functions $f \in \Sigma_{p,\alpha}$ were also studied by the authors Ibrahim and Darus in [4].

Analogous to the operator defined by Selvaraj and Selvakumaran [13], we define an operator $D_{\lambda,g}^{k,p,\alpha}$ on $\Sigma_{p,\alpha}$ as follows:

For a fixed function $g \in \Sigma_{p,\alpha}$ given by (1.2), the operator $D_{\lambda,g}^{k,p,\alpha}$ is defined by

$$\begin{aligned}
D_{\lambda,g}^{0,p,\alpha} f(z) &= z^{p+\alpha} + \sum_{n=2}^{\infty} a_n b_n z^{n+p+\alpha} = (f * g)(\alpha) \\
D_{\lambda,g}^{1,p,\alpha} f(z) &= (1-\lambda)(f * g)(z) + \frac{\lambda z}{p+\alpha}((f * g)(z))' \\
&\vdots \\
D_{\lambda,g}^{k,p,\alpha} f(z) &= D_{\lambda,g}^{p,\alpha}(D_{\lambda,g}^{k-1,p,\alpha} f(z)), \quad k \in \mathbb{N}.
\end{aligned}$$

If $f(z) \in \Sigma_{p,\alpha}$ then

$$D_{\lambda,g}^{k,p,\alpha} f(z) = z^{p+\alpha} + \sum_{n=2}^{\infty} \left(1 + \frac{\lambda n}{p+\alpha}\right)^k a_n b_n z^{n+p+\alpha} \tag{1.5}$$

where $k \in \mathbb{N}_0$, $\lambda \geq 0$, $p \in \mathbb{N}$ and $0 \leq \alpha < 1$.

It follows from the definition of the operator $D^{k,p,\alpha}_{\lambda,g}$ that

$$\frac{\lambda z}{p+\alpha}(D^{k,p,\alpha}_{\lambda,g} f(z))' = D^{k+1,p,\alpha}_{\lambda,g} f(z) - (1-\lambda) D^{k,p,\alpha}_{\lambda,g} f(z) \tag{1.6}$$

For functions $f_i \in \Sigma_{p,\alpha}$ given by (1.1), $\lambda \geq 0$, $p \in \mathbb{N}$, $0 \leq \alpha < 1$, $\beta_i \in \mathbb{R}$, $\beta_i > 0$, $i \in \{1, 2, \ldots, m\}$. We define the integral operators $F_{p,g}(\lambda, \alpha; z)$ and $G_{p,g}(\lambda, \alpha; z)$ as follows:

$$F_{p,g}(\lambda, \alpha; z) = (p+\alpha) \int_0^z t^{p+\alpha-1} \left(\frac{D^{k,p,\alpha}_{\lambda,g} f_1(z)}{t^{p+\alpha}} \right)^{\beta_1} \cdots \left(\frac{D^{k,p,\alpha}_{\lambda,g} f_m(z)}{t^{p+\alpha}} \right)^{\beta_m} dt \tag{1.7}$$

and

$$G_{p,g}(\lambda, \alpha; z) = (p+\alpha) \int_0^z t^{p+\alpha-1} \left(\frac{(D^{k,p,\alpha}_{\lambda,g} f_1(z))'}{(p+\alpha) t^{p+\alpha-1}} \right)^{\beta_1} \cdots \left(\frac{(D^{k,p,\alpha}_{\lambda,g} f_m(z))'}{(p+\alpha) t^{p+\alpha-1}} \right)^{\beta_m} dt$$

It is clear from (1.7) that

$$f \in \Sigma_{p,\alpha} \Rightarrow F_{p,g}(\lambda, \alpha; z) \in \Sigma_{p,\alpha} \quad \text{and} \quad f \in \Sigma_{p,\alpha} \Rightarrow G_{p,g}(\lambda, \alpha; z) \in \Sigma_{p,\alpha}.$$

Remark 1.1. *(i) By taking $\alpha = 0$, $k = 0$, $p = 1$ and $b_{n,i} = 1 \ \forall \ n \geq 2$ and $i \in \{1, 2, \ldots, m\}$, the integral operators $F_{p,g}(\lambda, \alpha; z)$ and $G_{p,g}(\lambda, \alpha; z)$ reduces to the operators*

$$F_\beta(z) = \int_0^z \left(\frac{f_1(t)}{t} \right)^{\beta_1} \cdots \left(\frac{f_m(t)}{t} \right)^{\beta_m} dt$$

and

$$G_\beta(z) = \int_0^z (f_1'(t))^{\beta_1} \cdots (f_m'(t))^{\beta_m} dt,$$

introduced and studied by Breaz and Breaz [1] and Breaz et al. [2].

(ii) For $p = 1$, $\alpha = 0$, $m = 1$, $\beta_1 = \beta \in [0, 1]$, $k = 0$, $f_1 = f$ and $b_{n,i} = 1 \ \forall$ $n \geq 2$, $i \in \{1, 2, \ldots, m\}$, the integral operator $F_{p,g}(\lambda, \alpha; z)$ reduces to the operator $I(z) = \int_0^z \left(\frac{f(t)}{t} \right)^\beta dt$ which was introduced by Miller et al. in [7] and when $p = 1 = m$, $k = 0$, $\alpha = 0$, $\beta_1 = \beta \in \mathbb{C}$, $|\beta| \leq 1/4$, $f_1 = f$ and $b_n = 1 \ \forall \ n \geq 2$, the operator $G_{p,g}(\lambda, \alpha; z)$ reduces to the integral operator $\int_0^z (f'(t))^\beta dt$ studied by Kim and Merkes, Pascu and Pescar in [5, 10].

(iii) *For* $\alpha = 0$, $b_{n,i} = 1 \ \forall \ n \geq 2$, $i \in \{1, 2, \ldots, m\}$, $k = 0$, *the operators* $F_{p,g}(\lambda, \alpha; z)$ *and* $G_{p,g}(\lambda, \alpha; z)$ *reduces to the operators*

$$F_p(z) = \int_0^z pt^{p-1} \left(\frac{f_1(t)}{t^p}\right)^{\beta_1} \ldots \left(\frac{f_m(t)}{t^p}\right)^{\beta_m} dt$$

and

$$G_p(z) = \int_0^z pt^{p-1} \left(\frac{f_1'(t)}{pt^{p-1}}\right)^{\beta_1} \ldots \left(\frac{f_m'(t)}{pt^{p-1}}\right)^{\beta_m} dt,$$

which were defined and studied by Frasin in [3].

By using the differential operator $D_{\lambda,g}^{k,p,\alpha} f$ and the integral operators $F_{p,g}(\lambda, \alpha; z)$ and $G_{p,g}(\lambda, \alpha; z)$, we introduce some subclasses of analytic functions $f \in \Sigma_{p,\alpha}$:

Definition 1.1. *A function* $f \in \Sigma_{p,\alpha}$ *is said to be in the class* $M_{p,\alpha}(\delta)$ *if it satisfies the inequality* $Re\left(\frac{z(D_{\lambda,g}^{k,p,\alpha} f(z))'}{D_{\lambda,g}^{k,p,\alpha} f(z)}\right) < \delta$ *where* $z \in E$ *and* $\delta > p + \alpha$.

Definition 1.2. *A function* $f \in \Sigma_{p,\alpha}$ *is said to be in the class* $N_{p,\alpha}(\delta)$ *if it satisfies the inequality* $Re\left(1 + \frac{z(D_{\lambda,g}^{k,p,\alpha} f(z))''}{(D_{\lambda,g}^{k,p,\alpha} f(z))'}\right) < \delta$, $z \in E$ *and* $\delta > p + \alpha$.

Definition 1.3. *A function* $f \in MT_{p,\alpha}(\gamma, \mu)$ *for* $0 \leq \gamma < p + \alpha$ *and* $0 < \mu \leq p + \alpha$ *if it satisfies the analytic characterization,*

$$\left|\frac{z(D_{\lambda,g}^{k,p,\alpha} f(z))'}{D_{\lambda,g}^{k,p,\alpha} f(z)} - (p+\alpha)\right| < \mu \left|\gamma \frac{z(D_{\lambda,g}^{k,p,\alpha} f(z))'}{D_{\lambda,g}^{k,p,\alpha} f(z)} + (p+\alpha)\right|.$$

Definition 1.4. *A function* $f \in NT_{p,\alpha}(\gamma, \mu)$ *for* $0 \leq \gamma < p + \alpha$ *and* $0 < \mu \leq p + \alpha$ *if it satisfies the inequality,*

$$\left|1 + \frac{z(D_{\lambda,g}^{k,p,\alpha} f(z))''}{(D_{\lambda,g}^{k,p,\alpha} f(z))'} - (p+\alpha)\right| < \mu \left|\gamma \left(1 + \frac{z(D_{\lambda,g}^{k,p,\alpha} f(z))''}{(D_{\lambda,g}^{k,p,\alpha} f(z))'}\right) + (p+\alpha)\right|.$$

Note that for $k = 0$, $b_n = 1 \ \forall \ n \geq 2$ and $\alpha = 0$, the classes $M_{p,\alpha}(\delta)$,$N_{p,\alpha}(\delta)$, $MT_{p,\alpha}(\gamma, \mu)$ and $NT_{p,\alpha}(\gamma, \mu)$ reduces to the classes $M_p(\delta)$, $N_p(\delta)$, $MT_p(\gamma, \mu)$ and $NT_p(\gamma, \mu)$ respectively, which were introduced and studied by Mohammed et al. [8].

Also for $p = 1$, $k = 0$, $b_n = 1 \ \forall \ n \geq 2$ and $\alpha = 0$, the classes $M_{p,\alpha}(\delta)$ and $N_{p,\alpha}(\delta)$ reduces to $M(\delta)$ and $N(\delta)$, which were studied by Uralegaddi et al. [15] and Owa and Srivastava in [9].

Definition 1.5. *A family of functions f_i, $i \in \{1, 2, \ldots, m\}$ is said to be in the class $KDF_{p,g}(\lambda, \alpha, \gamma, \mu, \beta_1, \ldots, \beta_m)$ if it satisfies the inequality*

$$Re\left(1 + \frac{zF''_{p,g}(\lambda, \alpha; z)}{F'_{p,g}(\lambda, \alpha; z)}\right) \geq \gamma \left|1 + \frac{zF''_{p,g}(\lambda, \alpha; z)}{F'_{p,g}(\lambda, \alpha; z)} - (p+\alpha)\right| + \mu$$

for some $\gamma \geq 0$ and $-(p+\alpha) \leq \mu < (p+\alpha)$, $0 \leq \alpha < 1$, $p \in \mathbb{N}$ where $F_{p,g}(\lambda, \alpha; z)$ is defined as in (1.7).

Definition 1.6. *A family of functions f_i, $i \in \{1, 2, \ldots, m\}$ is said to be in the class $KDG_{p,g}(\lambda, \alpha, \gamma, \mu, \beta_1, \ldots, \beta_m)$ if it satisfies the inequality*

$$Re\left(1 + \frac{zG''_{p,g}(\lambda, \alpha; z)}{G'_{p,g}(\lambda, \alpha; z)}\right) \geq \gamma \left|1 + \frac{zG''_{p,g}(\lambda, \alpha; z)}{G'_{p,g}(\lambda, \alpha; z)} - (p+\alpha)\right| + \mu$$

for some $\gamma \geq 0$ and $-(p+\alpha) \leq \mu < (p+\alpha)$, $0 \leq \alpha < 1$, $p \in \mathbb{N}$ where $G_{p,g}(\lambda, \alpha; z)$ is defined as in (1.7).

Lemma 1.1. *For $f_i \in \Sigma_{p,\alpha}$,*

$$1 + \frac{zF''_{p,g}(\lambda, \alpha; z)}{F'_{p,g}(\lambda, \alpha; z)} - (p+\alpha) = -\sum_{i=1}^{m} \beta_i \left[\frac{\sum_{n=2}^{\infty} n\left(1 + \frac{\lambda n}{p+\alpha}\right)^k |a_{n,i}||b_{n,i}|z^n}{1 - \sum_{n=2}^{\infty}\left(1 + \frac{\lambda n}{p+\alpha}\right)^k |a_{n,i}||b_{n,i}|z^n}\right],$$

where $F_{p,g}(\lambda, \alpha; z)$ is the integral operator given by (1.7).

Proof. Let $f_i(z) = z^{p+\alpha} - \sum_{n=2}^{\infty} |a_{n,i}|z^{n+p+\alpha}$ for $i \in \{1, 2, \ldots, m\}$ then

$$(D^{k,p,\alpha}_{\lambda,g} f_i(z))' = (p+\alpha)z^{p+\alpha-1} - \sum_{n=2}^{\infty}\left(1 + \frac{\lambda n}{p+\alpha}\right)^k |a_{n,i}||b_{n,i}|(n+p+\alpha)z^{n+p+\alpha-1}$$

where

$$g_i(z) = z^{p+\alpha} - \sum_{n=2}^{\infty} |b_{n,i}|z^{n+p+\alpha} \in \Sigma_{p,\alpha}, \quad (z \in E).$$

Since

$$\frac{zF''_{p,g}(\lambda, \alpha; z)}{F'_{p,g}(\lambda, \alpha; z)} = (p+\alpha-1) + \sum_{i=1}^{m} \beta_i \left[\frac{z(D^{k,p,\alpha}_{\lambda,g} f_i(z))'}{D^{k,p,\alpha}_{\lambda,g} f_i(z)} - (p+\alpha)\right] \qquad (1.8)$$

we find

$$
\begin{aligned}
&1+\frac{zF''_{p,g}(\lambda,\alpha;z)}{F'_{p,g}(\lambda,\alpha;z)}-(p+\alpha)\\
&=\sum_{i=1}^{m}\beta_i\left[\frac{(p+\alpha)z^{p+\alpha}-\sum_{n=2}^{\infty}\left(1+\frac{\lambda n}{p+\alpha}\right)^k(n+p+\alpha)|a_{n,i}||b_{n,i}|z^{n+p+\alpha}}{z^{p+\alpha}-\sum_{n=2}^{\infty}\left(1+\frac{\lambda n}{p+\alpha}\right)^k|a_{n,i}||b_{n,i}|z^{n+p+\alpha}}-(p+\alpha)\right]\\
&=\sum_{i=1}^{m}\beta_i\left[\frac{-\sum_{n=2}^{\infty}\left(1+\frac{\lambda n}{p+\alpha}\right)^k n|a_{n,i}||b_{n,i}|z^{n+p+\alpha}}{z^{p+\alpha}-\sum_{n=2}^{\infty}\left(1+\frac{\lambda n}{p+\alpha}\right)^k|a_{n,i}||b_{n,i}|z^{n+p+\alpha}}\right] \qquad (1.9)
\end{aligned}
$$

which is equivalent to

$$
1+\frac{zF''_{p,g}(\lambda,\alpha;z)}{F'_{p,g}(\lambda,\alpha;z)}-(p+\alpha)=-\sum_{i=1}^{m}\beta_i\left[\frac{-\sum_{n=2}^{\infty}n\left(1+\frac{\lambda n}{p+\alpha}\right)^k|a_{n,i}||b_{n,i}|z^n}{1-\sum_{n=2}^{\infty}\left(1+\frac{\lambda n}{p+\alpha}\right)^k|a_{n,i}||b_{n,i}|z^n}\right]
$$

□

Lemma 1.2. *For $f_i\in\Sigma_{p,\alpha}$, we have*

$$
\begin{aligned}
&1+\frac{zG''_{p,g}(\lambda,\alpha;z)}{G'_{p,g}(\lambda,\alpha;z)}-(p+\alpha)\\
&=-\sum_{i=1}^{m}\beta_i\left[\frac{\sum_{n=2}^{\infty}\left(1+\frac{\lambda n}{p+\alpha}\right)^k n(n+p+\alpha)|a_{n,i}||b_{n,i}|z^n}{(p+\alpha)-\sum_{n=2}^{\infty}\left(1+\frac{\lambda n}{p+\alpha}\right)^k(n+p+\alpha)|a_{n,i}||b_{n,i}|z^n}\right]
\end{aligned}
$$

where $G_{p,g}(\lambda,\alpha;z)$ is the integral operator given by (1.7).

Proof. Let

$$
f_i(z)=z^{p+\alpha}-\sum_{n=2}^{\infty}|a_{n,i}|z^{n+p+\alpha},\quad i\in\{1,2,\dots,m\}.
$$

From (1.7), it follows that

$$1+\frac{zG''_{p,g}(\lambda,\alpha;z)}{G'_{p,g}(\lambda,\alpha;z)}-(p+\alpha)=\sum_{i=1}^{m}\beta_i\left[\frac{z(D^{k,p,\alpha}_{\lambda,g}(f_i(z))''}{(D^{k,p,\alpha}_{\lambda,g}(f_i(z))'}-(p+\alpha-1)\right]$$

$$=\sum_{i=1}^{m}\beta_i\left[\frac{\left\{\begin{array}{l}(p+\alpha)(p+\alpha-1)z^{(p+\alpha-1)}\\ -\sum_{n=2}^{\infty}\left(1+\frac{\lambda n}{p+\alpha}\right)^k|a_{n,i}||b_{n,i}|(n+p+\alpha)(n+p+\alpha-1)z^{n+p+\alpha-1}\end{array}\right\}}{(p+\alpha)z^{(p+\alpha-1)}-\sum_{n=2}^{\infty}\left(1+\frac{\lambda n}{p+\alpha}\right)^k|a_{n,i}||b_{n,i}|(n+p+\alpha)z^{n+p+\alpha-1}}-(p+\alpha-1)\right]$$

$$=-\sum_{i=1}^{m}\beta_i\left[\frac{\sum_{n=2}^{\infty}\left(1+\frac{\lambda n}{p+\alpha}\right)^k n(n+p+\alpha)|a_{n,i}||b_{n,i}|z^{n+p+\alpha-1}}{(p+\alpha)z^{(p+\alpha-1)}-\sum_{n=2}^{\infty}\left(1+\frac{\lambda n}{p+\alpha}\right)^k|a_{n,i}||b_{n,i}|(n+p+\alpha)z^{n+p+\alpha-1}}\right]$$

Hence

$$1+\frac{zG''_{p,g}(\lambda,\alpha;z)}{G'_{p,g}(\lambda,\alpha;z)}-(p+\alpha)=-\sum_{i=1}^{m}\beta_i\left[\frac{\sum_{n=2}^{\infty}\left(1+\frac{\lambda n}{p+\alpha}\right)^k n(n+p+\alpha)|a_{n,i}||b_{n,i}|z^{n}}{(p+\alpha)-\sum_{n=2}^{\infty}\left(1+\frac{\lambda n}{p+\alpha}\right)^k|a_{n,i}||b_{n,i}|z^{n}}\right]$$

□

Lemma 1.3. *[6] Let $q(z)$ be univalent in E and let $\phi(z)$ be analytic in a domain containing $q(E)$. If $zq'(z)|\phi(q(z))$ is starlike, then*

$$z\psi'(z)\phi(\psi(z))\prec zq'(z)\phi(q(z))\quad(z\in E)$$

then $\psi(z)\prec q(z)$ and $q(z)$ is the best dominant.

Lemma 1.4. *[12] If $p(z)$ and $q(z)$ are analytic in E, $q(z)$ is convex univalent, α,β and γ are complex and $\gamma\neq 0$. Further assume that*

$$\Re\left\{\frac{\alpha}{\gamma}+\frac{2\beta}{\gamma}q(z)+\left(1+\frac{zq''(z)}{q(z)}\right)\right\}>0.$$

If $p(z)=1+cz+\dots$ is analytic in E and satisfies

$$\alpha p(z)+\beta p^2(z)+\gamma zp'(z)\prec\alpha q(z)+\beta q^2(z)+\gamma zq'(z),$$

then $p(z)\prec q(z)$ and $q(z)$ is the best dominant.

Theorem 1.1. *[11] Let $q(z)$ be convex univalent and $0 < \beta \leq 1$,*

$$\Re\left\{\frac{1-\beta}{\beta} + 2q(z) + \left(1 + \frac{zq''(z)}{q(z)}\right)\right\} > 0.$$

If $f \in \mathcal{A}$ satisfies

$$\frac{zf'(z)}{f(z)} + \beta z^2 \frac{f''(z)}{f'(z)} \prec (1-\beta)q(z) + \beta q^2(z) + \beta z q'(z)$$

then $\frac{zf'(z)}{f(z)} \prec q(z)$ and $q(z)$ is the best dominant.

Theorem 1.2. *[11] Let $q(z)$ be analytic in E, $q(0) = 1$ and $h(z) = zq'(z)/q(z)$ be starlike univalent in E. If $f \in \mathcal{A}$ satisfies $\frac{(zf(z))''}{f'(z)} - 2\frac{zf'(z)}{f(z)} \prec h(z)$ then $\frac{z^2 f'(z)}{f^2(z)} \prec q(z)$ and $q(z)$ is the best dominant.*

2 Sufficient Conditions for Functions in the Classes $KDF_{p,g}(\lambda, \alpha, \gamma, \mu, \beta_1, \ldots, \beta_m)$ and $KDG_{p,g}(\lambda, \alpha, \gamma, \mu, \beta_1, \ldots, \beta_m)$

In this section, we obtain sufficient conditions for a family of functions f_i belonging to the classes $KDF_{p,g}(\lambda, \alpha, \gamma, \mu, \beta_1, \ldots, \beta_m)$ and $KDG_{p,g}(\lambda, \alpha, \gamma, \mu, \beta_1, \ldots, \beta_m)$.

Theorem 2.1. *Let the function f_i belongs to the class $T_{p,\alpha}$ for $i \in \{1, 2, \ldots, m\}$. Then $f_i \in KDF_{p,g}(\lambda, \alpha, \gamma, \mu, \beta_1, \ldots, \beta_m)$ for $i \in \{1, 2, \ldots, m\}$ if and only if*

$$\sum_{i=1}^{m}\left[\sum_{n=2}^{\infty} \beta_i(\gamma+1)n\left(1+\frac{\lambda n}{p+\alpha}\right)^k |a_{n,i}||b_{n,i}| \Big/ 1 - \sum_{n=2}^{\infty}\left(1+\frac{\lambda n}{p+\alpha}\right)^k |a_{n,i}||b_{n,i}|\right] \leq p + \alpha - \mu \tag{2.1}$$

where $-(p+\alpha) \leq \mu < (p+\alpha)$ and $k \in \mathbb{N}_0$.

Proof. Suppose that (2.1) holds. It suffices to show that

$$\gamma\left|1 + \frac{zF''_{p,g}(\lambda,\alpha;z)}{F'_{p,g}(\lambda,\alpha;z)} - (p+\alpha)\right| - Re\left(1 + \frac{zF''_{p,g}(\lambda,\alpha;z)}{F'_{p,g}(\lambda,\alpha;z)} - (p+\alpha)\right) \leq p + \alpha - \mu$$

where $-(p+\alpha) \leq \mu < (p+\alpha)$.
We have

$$\begin{aligned}\gamma&\left|1 + \frac{zF''_{p,g}(\lambda,\alpha;z)}{F'_{p,g}(\lambda,\alpha;z)} - (p+\alpha)\right| - Re\left(1 + \frac{zF''_{p,g}(\lambda,\alpha;z)}{F'_{p,g}(\lambda,\alpha;z)} - (p+\alpha)\right)\\ &\leq (\gamma+1)\left|1 + \frac{zF''_{p,g}(\lambda,\alpha;z)}{F'_{p,g}(\lambda,\alpha;z)} - (p+\alpha)\right|\end{aligned}$$

Applying Lemma 1.1, we obtain

$$
\begin{aligned}
&(\gamma+1)\left|1+\frac{zF''_{p,g}(\lambda,\alpha;z)}{F'_{p,g}(\lambda,\alpha;z)}-(p+\alpha)\right| \\
&=(\gamma+1)\left|\sum_{i=1}^{m}\beta_i\left[\frac{\sum_{n=2}^{\infty}n\left(1+\frac{\lambda n}{p+\alpha}\right)^k|a_{n,i}||b_{n,i}|z^n}{1-\sum_{n=2}^{\infty}\left(1+\frac{\lambda n}{p+\alpha}\right)^k|a_{n,i}||b_{n,i}|z^n}\right]\right| \\
&\le(\gamma+1)\sum_{i=1}^{m}\beta_i\left[\frac{\sum_{n=2}^{\infty}n\left(1+\frac{\lambda n}{p+\alpha}\right)^k|a_{n,i}||b_{n,i}||z|^n}{1-\sum_{n=2}^{\infty}\left(1+\frac{\lambda n}{p+\alpha}\right)^k|a_{n,i}||b_{n,i}||z|^n}\right] \\
&\le(\gamma+1)\sum_{i=1}^{m}\beta_i\left[\frac{\sum_{n=2}^{\infty}n\left(1+\frac{\lambda n}{p+\alpha}\right)^k|a_{n,i}||b_{n,i}|}{1-\sum_{n=2}^{\infty}\left(1+\frac{\lambda n}{p+\alpha}\right)^k|a_{n,i}||b_{n,i}|}\right]
\end{aligned}
$$

The last expression is bounded above by $(p+\alpha)-\mu$ and hence we have

$$\gamma\left|1+\frac{zF''_{p,g}(\lambda,\alpha;z)}{F'_{p,g}(\lambda,\alpha;z)}-(p+\alpha)\right|-Re\left(1+\frac{zF''_{p,g}(\lambda,\alpha;z)}{F'_{p,g}(\lambda,\alpha;z)}-(p+\alpha)\right)\le p+\alpha-\mu$$

or equivalently

$$Re\left(1+\frac{zF''_{p,g}(\lambda,\alpha;z)}{F'_{p,g}(\lambda,\alpha;z)}\right)\ge\gamma\left|1+\frac{zF''_{p,g}(\lambda,\alpha;z)}{F'_{p,g}(\lambda,\alpha;z)}-(p+\alpha)\right|+\mu.$$

Hence $f_i\in KDF_{p,g}(\lambda,\alpha,\gamma,\mu,\beta_1,\ldots,\beta_m)$.

Conversely, suppose $f_i\in KDF_{p,g}(\lambda,\alpha,\gamma,\mu,\beta_1,\ldots,\beta_m)$ for $i\in\{1,2,\ldots,m\}$.

From (2.1) and Lemma 1.1, we obtain that

$$
\begin{aligned}
&(p+\alpha)-\sum_{i=1}^{m}\beta_i\left[\sum_{n=2}^{\infty}n\left(1+\frac{\lambda n}{p+\alpha}\right)^k|a_{n,i}||b_{n,i}|z^n/1-\sum_{n=2}^{\infty}\left(1+\frac{\lambda n}{p+\alpha}\right)^k|a_{n,i}||b_{n,i}|z^n\right] \\
&\quad\ge\gamma\left|\sum_{i=1}^{m}\beta_i\left|\sum_{n=2}^{\infty}n\left(1+\frac{\lambda n}{p+\alpha}\right)^k|a_{n,i}||b_{n,i}|z^n/1-\sum_{n=2}^{\infty}\left(1+\frac{\lambda n}{p+\alpha}\right)^k|a_{n,i}||b_{n,i}|z^n\right|\right|+\mu \\
&\quad\ge\gamma\sum_{i=1}^{m}\beta_i\left[\sum_{n=2}^{\infty}n\left(1+\frac{\lambda n}{p+\alpha}\right)^k|a_{n,i}||b_{n,i}|z^n/1-\sum_{n=2}^{\infty}\left(1+\frac{\lambda n}{p+\alpha}\right)^k|a_{n,i}||b_{n,i}|z^n\right]+\mu
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
&\sum_{i=1}^{m}\beta_i\left[\sum_{n=2}^{\infty}n\gamma\left(1+\frac{\lambda n}{p+\alpha}\right)^k|a_{n,i}||b_{n,i}|z^n/1-\sum_{n=2}^{\infty}\left(1+\frac{\lambda n}{p+\alpha}\right)^k|a_{n,i}||b_{n,i}|z^n\right] \\
&\qquad+\sum_{i=1}^{m}\beta_i\left[\sum_{n=2}^{\infty}n\left(1+\frac{\lambda n}{p+\alpha}\right)^k|a_{n,i}||b_{n,i}|z^n/1-\sum_{n=2}^{\infty}\left(1+\frac{\lambda n}{p+\alpha}\right)^k|a_{n,i}||b_{n,i}|z^n\right] \\
&\qquad\le p+\alpha-\mu
\end{aligned}
$$

which reduces to

$$\sum_{i=1}^{m}\beta_i\left[\frac{\sum_{n=2}^{\infty}n(\gamma+1)\left(1+\frac{\lambda n}{p+\alpha}\right)^k|a_{n,i}||b_{n,i}|z^n}{1-\sum_{n=2}^{\infty}\left(1+\frac{\lambda n}{p+\alpha}\right)^k|a_{n,i}||b_{n,i}|z^n}\right]\leq p+\alpha-\mu.$$

when $z\to 1^-$ along the real axis, we obtain the required inequality:

$$\sum_{i=1}^{m}\beta_i\left[\frac{\sum_{n=2}^{\infty}n(\gamma+1)\left(1+\frac{\lambda n}{p+\alpha}\right)^k|a_{n,i}||b_{n,i}|}{1-\sum_{n=2}^{\infty}\left(1+\frac{\lambda n}{p+\alpha}\right)^k|a_{n,i}||b_{n,i}|}\right]\leq p+\alpha-\mu.$$

□

Employing the same procedure as in proof of Theorem 2.1 and applying Lemma 1.2, we have

Theorem 2.2. *Let f_i belongs to the class $T_{p,\alpha}$ for $i\in\{1,2,\ldots,m\}$. Then $f_i\in KDG_{p,g}(\lambda,\alpha,\gamma,\mu,\beta_1,\ldots,\beta_m)$ for $i\in\{1,2,\ldots,m\}$ if and only if*

$$\sum_{i=1}^{m}\left[\sum_{n=2}^{\infty}\beta_i(\gamma+1)\left(1+\frac{\lambda n}{p+\alpha}\right)^k n(n+p+\alpha)|a_{n,i}||b_{n,i}|\Big/\right.$$
$$\left.(p+\alpha)-\sum_{n=2}^{\infty}\left(1+\frac{\lambda n}{p+\alpha}\right)^k(n+p+\alpha)|a_{n,i}||b_{n,i}|\right]\leq p+\alpha-\mu \quad (2.2)$$

where $-(p+\alpha)\leq\mu<(p+\alpha)$ and $k\in\mathbb{N}_0$.

3 Integral Operators $F_{p,g}(\lambda,\alpha;z)$ and $G_{p,g}(\lambda,\alpha;z)$ on the classes $M_{p,\alpha}(\delta)$, $N_{p,\alpha}(\delta)$, $MT_{p,\alpha}(\gamma,\mu)$ and $NT_{p,\alpha}(\gamma,\mu)$

In this section some of the properties of the integral operators $F_{p,g}(\lambda,\alpha;z)$ and $G_{p,g}(\lambda,\alpha;z)$ on the classes $M_{p,\alpha}(\delta)$, $N_{p,\alpha}(\delta)$, $MT_{p,\alpha}(\gamma,\mu)$ and $NT_{p,\alpha}(\gamma,\mu)$ are discussed.

Theorem 3.1. *Let $\beta_i\in\mathbb{R}$ with $\beta_i>0$, $i\in\{1,2,\ldots,m\}$, $f_i\in\Sigma_{p,\alpha}$ and $\left|\frac{(D_{\lambda,g}^{k,p,\alpha}f_i(z))'}{D_{\lambda,g}^{k,p,\alpha}f_i(z)}\right|\prec M_i$. If $f_i\in MT_{p,\alpha}(\gamma_i,\mu_i)$, $0\leq\alpha<1$ then $F_{p,g}(\lambda,\alpha;z)\in N(\delta')$ where $\delta'=p+\alpha+\sum_{i=1}^{m}\beta_i\mu_i(\gamma_iM_i+p+\alpha)$.*

Proof. It is clear from (1.7) that $F_{p,g}(\lambda,\alpha;z) \in \Sigma_{p,\alpha}$. On differentiating $F_{p,g}(\lambda,\alpha;z)$, given by (1.7), we get

$$F'_{p,g}(\lambda,\alpha;z) = (p+\alpha)z^{p+\alpha-1}\prod_{i=1}^{m}\left(\frac{D^{k,p,\alpha}_{\lambda,g}f_i(z)}{z^{p+\alpha}}\right)^{\beta_i} \tag{3.1}$$

Differentiating (3.1) logarithmically and multiplying by z, we obtain

$$\frac{zF''_{p,g}(\lambda,\alpha;z)}{F'_{p,g}(\lambda,\alpha;z)} = (p+\alpha-1) + \sum_{i=1}^{m}\beta_i\left[\frac{z(D^{k,p,\alpha}_{\lambda,g}f_i(z))'}{D^{k,p,\alpha}_{\lambda,g}f_i(z)} - (p+\alpha)\right]$$

equivalently

$$1+\frac{zF''_{p,g}(\lambda,\alpha;z)}{F'_{p,g}(\lambda,\alpha;z)} = (p+\alpha) + \sum_{i=1}^{m}\beta_i\left[\frac{z(D^{k,p,\alpha}_{\lambda,g}f_i(z))'}{D^{k,p,\alpha}_{\lambda,g}f_i(z)} - (p+\alpha)\right] \tag{3.2}$$

Taking real part of both sides of (3.2) we get

$$\begin{aligned} Re\left[1+\frac{zF''_{p,g}(\lambda,\alpha;z)}{F'_{p,g}(\lambda,\alpha;z)}\right] &= (p+\alpha) + \sum_{i=1}^{m}\beta_i\left[Re\left(\frac{z(D^{k,p,\alpha}_{\lambda,g}f_i(z))'}{D^{k,p,\alpha}_{\lambda,g}f_i(z)} - (p+\alpha)\right)\right] \\ &\leq (p+\alpha) + \sum_{i=1}^{m}\beta_i\left|\frac{z(D^{k,p,\alpha}_{\lambda,g}f_i(z))'}{D^{k,p,\alpha}_{\lambda,g}f_i(z)} - (p+\alpha)\right| \end{aligned}$$

Since $f_i \in MT_{p,\alpha}(\gamma_i,\mu_i)$ for $i \in \{1,2,\ldots,m\}$ we have

$$\begin{aligned} Re\left[1+\frac{zF''_{p,g}(\lambda,\alpha;z)}{F'_{p,g}(\lambda,\alpha;z)}\right] &< (p+\alpha) + \sum_{i=1}^{m}\beta_i\mu_i\left|\gamma_i\frac{z(D^{k,p,\alpha}_{\lambda,g}f_i(z))'}{D^{k,p,\alpha}_{\lambda,g}f_i(z)} + (p+\alpha)\right| \\ &< (p+\alpha) + \sum_{i=1}^{m}\beta_i\mu_i\gamma_i\left|\frac{(D^{k,p,\alpha}_{\lambda,g}f_i(z))'}{D^{k,p,\alpha}_{\lambda,g}f_i(z)}\right| + \sum_{i=1}^{m}\beta_i\mu_i(p+\alpha) \\ &< (p+\alpha) + \sum_{i=1}^{m}\beta_i\mu_i(\gamma_i M_i + p + \alpha). \end{aligned}$$

As $\alpha + \sum_{i=1}^{m}\beta_i\mu_i(\gamma_i M_i + p+\alpha) > 0$, $F_{p,g}(\lambda,\alpha;z) \in N_p(\delta')$ where

$\delta' = p+\alpha+\sum_{i=1}^{m}\beta_i\mu_i(\gamma_i M_i + p+\alpha).$ □

By substituting $p = m = 1$, $\beta_1 = \beta$, $M_1 = M$, $f_1 = f$, $k = 0$ and $b_n = 1$ $\forall$ $n \geq 2$ in Theorem 3.1, we obtain

Corollary 3.1. *Suppose $\beta \in R$ with $\beta > 0$, $f \in \Sigma_{1,\alpha}$ and $\left|\frac{f'(z)}{f(z)}\right| < M$, M is fixed. If $f \in MT_\alpha(\gamma, \mu)$ then*

$$\int_0^z (1+\alpha)t^\alpha \left(\frac{f(t)}{t^{1+\alpha}}\right)^\beta dt \in N(\beta\mu(\gamma M + 1 + \alpha) + (1+\alpha))$$

Theorem 3.2. *Let $\beta_i > 0$ and $f_i \in M_{p,\alpha}(\delta_i)$ for $i \in \{1, 2, \ldots, m\}$, $0 \leq \alpha < 1$ with $\delta_i > p + \alpha$. Then $F_{p,g}(\lambda, \alpha; z) \in N_p(\delta')$ where*
$\delta' = p + \alpha + \sum_{i=1}^{m} \beta_i(\delta_i - (p+\alpha))$, $i \in \{1, 2, \ldots, m\}$.

Proof. From (3.2), we have

$$\begin{aligned} Re\left(1 + \frac{zF''_{p,g}(\lambda,\alpha;z)}{F'_{p,g}(\lambda,\alpha;z)}\right) &= \sum_{i=1}^{m} \beta_i Re\left(\frac{z(D^{k,p,\alpha}_{\lambda,g} f_i(z))'}{D^{k,p,\alpha}_{\lambda,g} f_i(z)}\right) - (p+\alpha)\sum_{i=1}^{m}\beta_i + (p+\alpha) \\ &< \sum_{i=1}^{m}\beta_i\delta_i - (p+\alpha)\sum_{i=1}^{m}\beta_i + (p+\alpha) \\ &= (p+\alpha) + \sum_{i=1}^{m}\beta_i(\delta_i - (p+\alpha)). \end{aligned}$$

Since $\delta_i > p + \alpha$, it is clear that $\alpha + \sum_{i=1}^{m} \beta_i(\delta_i - (p+\alpha)) > 0$ and hence $F_{p,g}(\lambda, \alpha; z) \in N_p(\delta')$ where $\delta' = p + \alpha + \sum_{i=1}^{m} \beta_i(\delta_i - (p+\alpha))$. □

Letting $p = 1 = m$, $\beta_1 = \beta$, $\delta_1 = \delta$, $K = 0$, $b_n = 1 \ \forall \ n \geq 2$ and $f_1 = f$ in Theorem 3.2 we obtain

Corollary 3.2. *Suppose $\beta > 0$, $f \in M_\alpha(\delta)$ with $\delta > 1 + \alpha$, $0 \leq \alpha < 1$. Then*

$$\int_0^z (1+\alpha)t^\alpha \left(\frac{f(t)}{t^{1+\alpha}}\right)^\beta dt \in N((1+\alpha) + \beta[\delta - (1+\alpha)]).$$

Theorem 3.3. *Let $\beta_i > 0$ and $f_i \in N_{p,\alpha}(\delta_i)$ for $i \in \{1, 2, \ldots, m\}$, $0 \leq \alpha < 1$ with $\delta_i > p + \alpha$. Then $G_{p,g}(\lambda, \alpha; z) \in N_p(\delta')$ with*
$\delta' = p + \alpha + \sum_{i=1}^{m} \beta_i(\delta_i - (p+\alpha))$, $i \in \{1, 2, \ldots, m\}$.

Proof. From the definition of $G_{p,g}(\lambda,\alpha;z)$, given by (1.7) we have

$$\begin{aligned} Re\left(1+\frac{G''_{p,g}(\lambda,\alpha;z)}{G'_{p,g}(\lambda,\alpha;z)}\right) &= \sum_{i=1}^{m}\beta_i\left[Re\left\{1+\left(\frac{z(D^{k,p,\alpha}_{\lambda,g}f_i(z))''}{(D^{k,p,\alpha}_{\lambda,g}f_i(z))'}\right)\right\}\right]-(p+\alpha)\sum_{i=1}^{m}\beta_i+(p+\alpha)\\ &< (p+\alpha)+\sum_{i=1}^{m}\beta_i(\delta_i-(p+\alpha)) \end{aligned}$$

As $\delta_i>(p+\alpha)$, it is clear that $\alpha+\sum_{i=1}^{m}\beta_i(\delta_i-(p+\alpha))>0$ and hence we conclude that $G_{p,g}(\lambda,\alpha;z)\in N_p(\delta')$ with $\delta'=p+\alpha+\sum_{i=1}^{m}\beta_i(\delta_i-(p+\alpha))$. □

By substituting $p=1=m$, $\beta_1=\beta$, $\delta_1=\delta$, $k=0$, $b_n=1\ \ \forall\ \ n\geq 2$ and $f_1=f$ in Theorem 3.3, we obtain the following corollary:

Corollary 3.3. *Let $\beta>0$ and $f\in N_\alpha(\delta)$ with $\delta>1+\alpha$. Then*

$$\int_0^z (1+\alpha)t^\alpha\left(\frac{f'(t)}{(1+\alpha)t^\alpha}\right)^\beta dt\in N((1+\alpha)+\beta(\delta-(1+\alpha))).$$

Theorem 3.4. *Let $f_i\in NT_{p,\alpha}(\gamma_i,\mu_i)$, $\beta_i\in\mathbb{R}$ with $\beta_i>0$ and $\left|\frac{(D^{k,p,\alpha}_{\lambda,g}f_i(z))''}{(D^{k,p,\alpha}_{\lambda,g}f_i(z))'}\right|<M_i$, $i\in\{1,2,\ldots,m\}$. Then $G_{p,g}(\lambda,\alpha;z)\in N_p(\delta')$ where $\delta'=p+\alpha+\sum_{i=1}^{m}\beta_i\mu_i\{\gamma_i(1+M_i)+(p+\alpha)\}$.*

Proof. From the definition of $G_{p,g}(\lambda,\alpha;z)$, given by (1.7), we have

$$\begin{aligned} Re\left(1+\frac{G''_{p,g}(\lambda,\alpha;z)}{G'_{p,g}(\lambda,\alpha;z)}\right) &\leq (p+\alpha)\sum_{i=1}^{m}\beta_i\left|\frac{z(D^{k,p,\alpha}_{\lambda,g}f_i(z))''}{(D^{k,p,\alpha}_{\lambda,g}f_i(z))'}-(p+\alpha-1)\right|\\ &< (p+\alpha)+\sum_{i=1}^{m}\beta_i\mu_i\left|\gamma_i\left(1+\frac{z(D^{k,p,\alpha}_{\lambda,g}f_i(z))''}{(D^{k,p,\alpha}_{\lambda,g}f_i(z))'}\right)+(p+\alpha)\right|\\ &< p+\alpha+\sum_{i=1}^{m}\beta_i\mu_i\gamma_i\left(1+\left|\frac{z(D^{k,p,\alpha}_{\lambda,g}f_i(z))''}{(D^{k,p,\alpha}_{\lambda,g}f_i(z))'}\right|\right)+\sum_{i=1}^{m}\beta_i\mu_i(p+\alpha)\\ &< p+\alpha+\sum_{i=1}^{m}\beta_i\mu_i\{\gamma_i(1+M_i)+(p+\alpha)\}. \end{aligned}$$

As $\alpha+\sum_{i=1}^{m}\beta_i\mu_i\{\gamma_i(1+M_i)+(p+\alpha)\}>0$, we observe that $G_{p,g}(\lambda,\alpha;z)\in N_p(\delta')$ where $\delta'=p+\alpha+\sum_{i=1}^{m}\beta_i\mu_i\{\gamma_i(1+M_i)+(p+\alpha)\}$. □

By taking $p = 1 = m$, $\beta_1 = \beta$, $M_1 = 1$, $f_1 = f$, $k = 0$, $b_n = 1$ $\forall$ $n \geq 2$ in Theorem 3.4, we obtain the following corolary:

Corollary 3.4. *Let $\beta \in \mathbb{R}$ with $\beta > 0$, $f \in NT_\alpha(\gamma, \mu)$ and $\left|\frac{f''(z)}{f'(z)}\right| < M$, M is fixed. Then $\int_0^z (1+\alpha)t^\alpha \left(\frac{f'(t)}{(1+\alpha)t^\alpha}\right)^\beta dt \in N(1+\alpha+\beta\mu[\gamma(1+M)+(1+\alpha)])$.*

4 Subordination Results for $f \in \Sigma_{p,\alpha}$ Associated with the Operator $D_{\lambda,g}^{k,p,\alpha}$

Motivated by the work of the authors in [12], in this section we extend Theorem 1.1 and Theorem 1.2 for functions $f \in \Sigma_{p,\alpha}$ defined through the operator given by (1.5).

Theorem 4.1. *Let $q(z)$ be convex univalent, $\beta \neq 0$ and*

$$\Re\left\{\frac{(1-\beta)(p+\alpha)}{\lambda\beta} + \frac{2(p+\alpha)}{\lambda}q(z) + \left(1 + \frac{zq''(z)}{q'(z)}\right)\right\} > 0$$

If $f(z) \in \Sigma_{p,\alpha}$ satisfies

$$\frac{D_{\lambda,g}^{k+1,p,\alpha}f(z)}{D_{\lambda,g}^{k,p,\alpha}f(z)}\left\{1-\beta+\beta\frac{D_{\lambda,g}^{k+2,p,\alpha}f(z)}{D_{\lambda,g}^{k+1,p,\alpha}f(z)}\right\} \prec (1-\beta)q(z)+\beta q^2(z)+\frac{\beta\lambda}{p+\alpha}zq'(z) \tag{4.1}$$

then

$$\frac{D_{\lambda,g}^{k+1,p,\alpha}f(z)}{D_{\lambda,g}^{k,p,\alpha}f(z)} \prec q(z) \tag{4.2}$$

and $q(z)$ is the best dominant.

Proof. Define the function

$$\psi(z) = \frac{D_{\lambda,g}^{k+1,p,\alpha}f(z)}{D_{\lambda,g}^{k,p,\alpha}f(z)} \tag{4.3}$$

Logarithmic differential of (4.3) yields

$$\frac{\psi'(z)}{\psi(z)} = \frac{(D_{\lambda,g}^{k+1,p,\alpha}f(z))'}{D_{\lambda,g}^{k+1,p,\alpha}f(z)} - \frac{(D_{\lambda,g}^{k,p,\alpha}f(z))'}{D_{\lambda,g}^{k,p,\alpha}f(z)}$$

which is equivalent to

$$\frac{z\psi'(z)}{\psi(z)} = \frac{z(D_{\lambda,g}^{k+1,p,\alpha} f(z))'}{D_{\lambda,g}^{k+1,p,\alpha} f(z)} - \frac{z(D_{\lambda,g}^{k,p,\alpha} f(z))'}{D_{\lambda,g}^{k,p,\alpha} f(z)} \tag{4.4}$$

By using (1.6) in equation (4.4), we obtain

$$\frac{z\psi'(z)}{\psi(z)} = \frac{p+\alpha}{\lambda}\left[\frac{D_{\lambda,g}^{k+2,p,\alpha} f(z)}{D_{\lambda,g}^{k+1,p,\alpha} f(z)} - \frac{D_{\lambda,g}^{k+1,p,\alpha} f(z)}{D_{\lambda,g}^{k,p,\alpha} f(z)}\right] \tag{4.5}$$

Using (4.3) in (4.5), we get

$$\frac{D_{\lambda,g}^{k+2,p,\alpha} f(z)}{D_{\lambda,g}^{k+1,p,\alpha} f(z)} = \frac{\lambda}{p+\alpha}\left[z\frac{\psi'(z)}{\psi(z)} + \frac{p+\alpha}{\lambda}\psi(z)\right] \tag{4.6}$$

Hence from (4.6)

$$\frac{D_{\lambda,g}^{k+1,p,\alpha} f(z)}{D_{\lambda,g}^{k,p,\alpha} f(z)}\left\{1-\beta+\beta\frac{D_{\lambda,g}^{k+2,p,\alpha} f(z)}{D_{\lambda,g}^{k+1,p,\alpha} f(z)}\right\} = (1-\beta)\psi(z) + \beta\psi^2(z) + \frac{\lambda\beta}{p+\alpha}z\psi'(z) \tag{4.7}$$

In view of (4.7), the subordination (4.1) becomes

$$(1-\beta)\psi(z) + \beta\psi^2(z) + \frac{\lambda\beta}{p+\alpha}z\psi'(z) \prec (1-\beta)q(z) + \beta q^2(z) + \frac{\lambda\beta}{p+\alpha}zq'(z)$$

Applying Lemma (1.4), we conclude that $\frac{D_{\lambda,g}^{k+1,p,\alpha} f(z)}{D_{\lambda,g}^{k,p,\alpha} f(z)} \prec q(z)$ and $q(z)$ is the best dominant. □

Theorem 4.2. *Let $q(z)$ be univalent in E, $\beta \neq 0$, $q(0) = 1$ and $zq'(z)/q(z)$ be starlike univalent in E. If $f(z) \in \Sigma_{p,\alpha}$ satisfies*

$$\frac{D_{\lambda,g}^{k+2,p,\alpha} f(z)}{D_{\lambda,g}^{k+1,p,\alpha} f(z)} - \beta\frac{D_{\lambda,g}^{k+1,p,\alpha} f(z)}{D_{\lambda,g}^{k,p,\alpha} f(z)} \prec \frac{\lambda}{p+\alpha}\frac{zq'(z)}{q(z)} + 1 - \beta \tag{4.8}$$

then

$$\frac{z^{(p+\alpha)(\beta-1)} D_{\lambda,g}^{k+1,p,\alpha} f(z)}{(D_{\lambda,g}^{k,p,\alpha} f(z))^{\beta}} \prec q(z) \tag{4.9}$$

and $q(z)$ is the best dominant.

Proof. Define the function $\psi(z)$ by

$$\psi(z) = \frac{z^{(p+\alpha)(\beta-1)} D_{\lambda,g}^{k+1,p,\alpha} f(z)}{(D_{\lambda,g}^{k,p,\alpha} f(z))^{\beta}} \tag{4.10}$$

Logarithmic differentiation of (4.10) yields

$$\frac{\psi'(z)}{\psi(z)} = \frac{(p+\alpha)(\beta-1)}{z} + \frac{(D_{\lambda,g}^{k+1,p,\alpha} f(z))'}{D_{\lambda,g}^{k+1,p,\alpha} f(z)} - \frac{\beta(D_{\lambda,g}^{k,p,\alpha} f(z))'}{D_{\lambda,g}^{k,p,\alpha} f(z)} \tag{4.11}$$

Multiplying the relation (4.11) by z, we get

$$\frac{z\psi'(z)}{\psi(z)} = (p+\alpha)(\beta-1) + \frac{z(D_{\lambda,g}^{k+1,p,\alpha} f(z))'}{D_{\lambda,g}^{k+1,p,\alpha} f(z)} - \frac{\beta z(D_{\lambda,g}^{k,p,\alpha} f(z))'}{D_{\lambda,g}^{k,p,\alpha} f(z)} \tag{4.12}$$

Using (1.6) in (4.12), we obtain

$$\frac{z\psi'(z)}{\psi(z)} = \frac{(p+\alpha)(\beta-1)}{\lambda} + \frac{p+\alpha}{\lambda}\frac{D_{\lambda,g}^{k+2,p,\alpha} f(z)}{D_{\lambda,g}^{k+1,p,\alpha} f(z)} - \beta\frac{p+\alpha}{\lambda}\frac{D_{\lambda,g}^{k+1,p,\alpha} f(z)}{D_{\lambda,g}^{k,p,\alpha} f(z)}$$

which is equivalent to

$$\frac{D_{\lambda,g}^{k+2,p,\alpha} f(z)}{D_{\lambda,g}^{k+1,p,\alpha} f(z)} - \beta\frac{D_{\lambda,g}^{k+1,p,\alpha} f(z)}{D_{\lambda,g}^{k,p,\alpha} f(z)} = \frac{\lambda}{p+\alpha}\frac{z\psi'(z)}{\psi(z)} + 1 - \beta \tag{4.13}$$

Since by hypothesis (4.8), we obtain $\frac{z\psi'(z)}{\psi(z)} \prec \frac{zq'(z)}{q(z)}$.
From Lemma (1.3), it is observed that

$$\frac{z^{(p+\alpha)(\beta-1)} D_{\lambda,g}^{k+1,p,\alpha} f(z)}{(D_{\lambda,g}^{k,p,\alpha} f(z))^{\beta}} \prec q(z)$$

and $q(z)$ is the best dominant. □

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