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On Convergence of Sequences of Real Valued Functions

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Abstract. Using the notions of α-equal, α-uniform equal and α-strong uniform equal convergence of real valued functions, we study here the classes of functions which are α-equal limits, α-uniform equal limits and α-strong uniform equal limits of sequences of real valued functions belonging to certain class. We also show that exhaustiveness and different types of convergence are preserved under uniform conjugacy.

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1. **Introduction**

Besides uniform convergence and pointwise convergence, several other types of convergence of sequences of real valued functions have been defined and studied for example discrete, equal (quasinormal), uniform discrete, uniform equal, α-convergence, α-equal, α-uniform equal convergence and α- strong uniform equal convergence. [4, 5, 6, 7, 8, 9]. In [7, 9] authors have studied unification of different types of convergence. The notion of α-convergence (also known as continuous convergence in the literature [15]) and its variants turned out to be very useful. In [9], we have used α-uniform equal convergence to characterize compact metric spaces. In [11], authors have studied α-convergence in detail and have used this concept to obtain Ascoli type theorems dealing with exhaustiveness instead of equicontinuity. In section 2, we give the notations and terminologies used in subsequent sections. In [9], we had studied classes $\Phi^{u.d.}$ and $\Phi^{u.e.}$ consisting respectively of the real valued functions on a non empty set X which are uniform discrete limits and uniform equal limits of sequences of functions in a particular class Φ. In section 3, we study $\Phi^{\alpha-e}$, $\Phi^{\alpha-u.e.}$ and $\Phi^{\alpha-s.u.e.}$ consisting respectively of the real valued functions on a non-empty set X which are α-equal, α-uniform equal and α-strong uniform equal limits of sequences of functions in a particular class Φ and related results. Tian and Chen in [16] have defined the notion of uniform conjugacy for sequences of maps and have shown that chaoticity of maps is preserved under uniform conjugacy. In section 4, we show that exhaustiveness and different types of convergence are preserved under uniform conjugacy. In [1, 2, 3, 10, 14, 17] authors have studied transitivity and chaoticity of uniform limit maps. In [5] equal convergence is studied in relation to covering properties.

2. Notations and Terminologies

By $\mathbb{N}$, we mean the set of natural numbers and by $\mathbb{R}$, we mean the set of real numbers. If Γ is a set, then $|\Gamma|$ denotes the cardinality of Γ. If $x \in \mathbb{R}$, then $[x]$ denotes the integer part of x.

Let X be a non-empty set. By a function on X, we mean a real valued function on X. Let Φ be an arbitrary class of functions defined on X. Then we have the following definitions.

Definition 2.1. [13] A sequence of functions $\{f_n\}$ in Φ is said to converge *uniformly equally* to a function f in Φ (written as $f_n \xrightarrow{u.e.} f$) if there exists a sequence $(\epsilon_n)_{n\in\mathbb{N}}$ of positive reals converging to zero and a natural number n_0 such that the cardinality of the set $\{n \in \mathbb{N} : |f_n(x) - f(x)|\} \geq \epsilon_n$ is at most n_0, for each $x \in X$.

Definition 2.2. [13] A sequence of functions $\{f_n\}$ in Φ is said to converge *uniformly discretely* to a function f in Φ (written as $f_n \xrightarrow{u.d.} f$) if there exists a natural number n_0 such that the cardinality of the set $\{n \in \mathbb{N} : |f_n(x) - f(x)| > 0\}$ is at most n_0, for each $x \in X$.

We denote by $\Phi^{u.e.}$, the set of all functions on X which are uniform equal limits of sequences of functions in Φ. Similarly $\Phi^{u.d.}$ denotes the set of all functions on X which are uniform discrete limits of sequences of functions in Φ.

Note 2.1. One can observe that if $f \in \Phi^{u.e.}$, then for any sequence $(\Lambda_n)_{n\in\mathbb{N}}$ of positive reals converging to zero, there exists a sequence of functions in Φ which converges uniformly equally to f with witnessing sequence $(\Lambda_n)_{n\in\mathbb{N}}$.

Definition 2.3. [8] A sequence of functions $\{f_n\}$ in Φ is said to *converge equally* to a function f in Φ (written as $f_n \xrightarrow{e} f$) if there exists a sequence $(\epsilon_n)_{n\in\mathbb{N}}$ of positive reals converging to zero such that, for each $x \in X$, there exists a natural number $n(x)$ satisfying $|f_n(x) - f(x)| < \epsilon_n$, for each $n \geq n(x)$. Also, $\{f_n\}$ is said to *converge discretely* to a function f in Φ (written as $f_n \xrightarrow{d} f$) if, for every $x \in X$, there exists $n(x) \in \mathbb{N}$ such that $f(x) = f_n(x)$ for all $n \geq n(x)$.

Note 2.2. For a sequence of functions in Φ, it is clear that we have the implications: uniform convergence implies uniform equal convergence and uniform equal convergence implies equal convergence. On the other hand uniform discrete convergence implies both discrete and uniform equal convergence.

Remark 2.4. There are examples showing that converse of all the above four implications may fail.[13]

Definition 2.5. [8] (a) Φ is called a *lattice* if Φ contains all constants and $f, g \in \Phi$ implies $max(f, g)$ and $min(f, g) \in \Phi$.
(b) Φ is called a *translation lattice* if it is a lattice and $f \in \Phi, c \in \mathbb{R}$ implies $f + c \in \Phi$.
(c) Φ is called a *congruence lattice* if it is a translation lattice and $f \in \Phi$ implies $-f \in \Phi$.
(d) Φ is called a *weakly affine lattice* if it is a congruence lattice and there is a set $C \subset (0, \infty)$ such that C is not bounded and $f \in \Phi, c \in C$ implies $cf \in \Phi$.
(e) Φ is called an *affine lattice* if it is a congruence lattice and $f \in \Phi, c \in \mathbb{R}$ implies $cf \in \Phi$.
(f) Φ is called a *subtractive lattice* if it is a congruence lattice and $f, g \in \Phi$ implies $(f - g) \in \Phi$.
(g) Φ is called an *ordinary class* if it is a subtractive lattice, $f, g \in \Phi$ implies $f.g \in \Phi$ and $f \in \Phi, f(x) \neq 0$, for all $x \in X$ implies $1/f \in \Phi$.

Let X be a metric space and let f, f_n, be real valued functions defined on X. Then we have the following definitions.

Definition 2.6. We say that a sequence of functions $\{f_n\}$ *α-converges* to f (written as $f_n \underset{\rightarrow}{\alpha} f$) if for any $x \in X$ and for any sequence (x_n) of points of X converging to x, $(f_n(x_n))$ converges to $f(x)$.

Remark 2.7. In the literature, α-convergence is known as continuous convergence [15]. It is clear from the definition that this convergence is stronger than the pointwise convergence but if the limit function is continuous then this convergence is weaker than uniform convergence.

Definition 2.8. [9] A sequence $\{f_n\}$ is said to *converge α-equally* to f in Φ (written as $f_n \underset{\longrightarrow}{\alpha - e} f$) if there exists a sequence $(\epsilon_n)_{n\in\mathbb{N}}$ of positive reals converging to zero such that, for each $x \in X$, and for any sequence (x_n) of points of X converging to x, there exists a natural number $n_0 \equiv n_0(x, (x_n))$ satisfying $|f_n(x_n) - f(x)| < \epsilon_n$, for all $n \geq n_0$.

Remark 2.9. It follows from the definition that α-equal convergence implies equal convergence defined by Cs̄aszar and Lazkovich [8]. The equal convergence is defined by Bukovoska with the name quasi-normal convergence in [4]. In [9], example 5.3 shows that equal convergence need not imply α-equal convergence.

Definition 2.10. [9] The sequence $\{f_n\}$ is said to converge *α-uniformly equally* to f (written as $f_n \xrightarrow{\alpha - u.e.} f$) if there exists a sequence $(\epsilon_n)_{n \in \mathbb{N}}$ of positive reals converging to zero and an $n_0 \in \mathbb{N}$ such that $|\{n \in \mathbb{N} : |f_n(x_n) - f(x)| \geq \epsilon_n\}| \leq n_0$, for each $x \in X$ and $x_n \to x$.

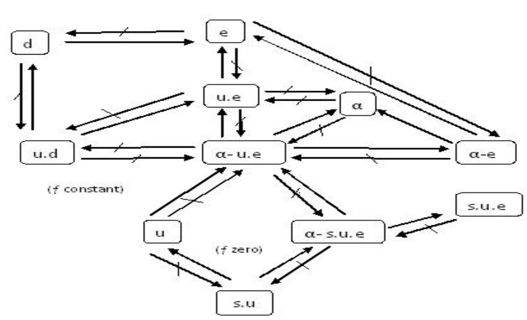
Remark 2.11. It is clear from the definition that $\alpha - u.e.$ convergence implies both α-convergence and $u.e.$ convergence defined by Papanastassiou in [13]. However example 4.4(i) in [9] shows that the converse of each of the above implications may fail.

Definition 2.12. We say that a sequence $\{f_n\}$ of real valued functions converges *strongly uniformly* to a function f (written as $f_n \xrightarrow{s.u.} f$) if there exists a convergent series $\sum_{n=0}^{\infty} \epsilon_n$ of positive reals and an $n_0 \in \mathbb{N}$ such that $|f_n(x_n) - f(x)| < \epsilon_n$ for all $n \geq n_0$ and for each $x \in X$.

Definition 2.13. We say that a sequence $\{f_n\}$ of real valued functions converges *α-strongly uniformly equally* to a function f (written as $f_n \xrightarrow{\alpha - s.u.e.} f$) if there exists a convergent series $\sum_{n=0}^{\infty} \epsilon_n$ of positive reals and an $n_0 \in \mathbb{N}$ such that $|\{n \in \mathbb{N} : |f_n(x_n) - f(x)| \geq \epsilon_n\}| \leq n_0$, for each $x \in X$ and $x_n \to x$.

Remark 2.14. From the definition it follows that the notion of $\alpha - s.u.e.$ convergence is stronger than $\alpha - u.e.$ convergence as well as strong uniform equal convergence. But example 4.16 in [9] justifies that converse of each of the above two implications may fail. Also by definition strong uniform convergence is stronger than uniform convergence and if limit function is zero function then strong uniform convergence implies α-strong uniform equal convergence.

Note 2.3. The symbol $f_n \not\twoheadrightarrow^{\beta} f$ means $\{f_n\}$ does not converge to f in the respective β-convergence. The following figure shows the relation between all the convergences discussed above:



3. On the classes of α-equal, α-uniform equal and α-strong uniform equal convergences

Proposition 3.1. *Let f_n be a sequence of real valued functions on X and let f be a real valued function on X. If (ϵ_n) and (λ_n) are two zero sequences of positive reals such that $0 < \epsilon_n \leq \lambda_n$ for every $n \in \mathbb{N}$ and (ϵ_n) witnesses the α-uniform equal convergence of f_n to f on X, then (λ_n) also witnesses the same.*

Proof. Since (ϵ_n) witnesses the α-uniform equal convergence of f_n to f therefore there exists an $n_0 \in \mathbb{N}$ such that

$$|\{n \in \mathbb{N} : |f_n(x_n) - f(x)| \geq \epsilon_n\}| \leq n_0,$$

for each $x \in X$ and $x_n \to x$. Since

$$\{n \in \mathbb{N} : |f_n(x_n) - f(x)| \geq \epsilon_n\} \supseteq \{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \lambda_n\}$$

therefore

$$|\{n \in \mathbb{N} : |f_n(x_n) - f(x)| \geq \lambda_n\}| \leq |\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \epsilon_n\}|$$

and hence

$$|\{n \in \mathbb{N} : |f_n(x_n) - f(x)| \geq \lambda_n\}| \leq n_0,$$

for each $x \in X$ and $x_n \to x$. □

Remark 3.2. By similar arguments the above result holds for α-equal and $\alpha-$uniform equal convergences also.

Proposition 3.3. *If $\sum_{n=0}^{\infty} \epsilon_n$ and $\sum_{n=0}^{\infty} \lambda_n$ are two convergent series of positive reals such that $0 < \epsilon_n \leq \lambda_n$ for every $n \in \mathbb{N}$ and $\sum_{n=0}^{\infty} \epsilon_n$ witnesses the α-strong uniform equal convergence then $\sum_{n=0}^{\infty} \lambda_n$ also witnesses the α-strong uniform equal convergence.*

Proof. It is similar to the previous proof. □

Proposition 3.4. *If $\{f_n\} \xrightarrow{\alpha - u.e.} f$ and $\{f_{n_k}\}$ is a subsequence of $\{f_n\}$ then $\{f_{n_k}\} \xrightarrow{\alpha - u.e.} f$.*

Proof. There exists a sequence $(\epsilon_n)_{n\in\mathbb{N}}$ of positive reals converging to zero and an $n_0 \in \mathbb{N}$ such that $|\{n \in \mathbb{N} : |f_n(x_n) - f(x)| \geq \epsilon_n\}| \leq n_0$, for each $x \in X$ and $x_n \to x$. Consider a subsequence $\{f_{n_k}\}$ of $\{f_n\}$. The subsequence $\{\epsilon_{n_k}\}$ of $\{\epsilon_n\}$ also converges to zero and

$$\begin{aligned}|\{n_k \in \mathbb{N} : |f_{n_k}(x_{n_k}) - f(x)| \geq \epsilon_{n_k}\}| &\leq |\{n \in \mathbb{N} : |f_n(x_n) - f(x)| \geq \epsilon_n\}| \\ &\leq n_0,\end{aligned}$$

for each $x \in X$ and $x_{n_k} \to x$. □

Note 3.1. (i) If $\{f_n\} \xrightarrow{\alpha - e} f$ and $\{f_{n_k}\}$ is a subsequence of the sequence $\{f_n\}$ then $\{f_{n_k}\} \xrightarrow{\alpha - e} f$.
(ii) If $\{f_n\} \xrightarrow{\alpha - s.u.e.} f$ and $\{f_{n_k}\}$ is a subsequence of the sequence $\{f_n\}$ then $\{f_{n_k}\} \xrightarrow{\alpha - s.u.e.} f$.

Proposition 3.5. *Let Φ be a class of functions on X. If Φ is a lattice, a translation lattice, a convergence lattice, a weakly affine lattice, an affine lattice or a subtractive lattice then so is $\Phi^{\alpha-u.e.}$.*

Proof. Suppose Φ is a lattice then Φ contains all constant functions. If f is a constant function in Φ, then by taking $f_n = f$ for all $n \in \mathbb{N}$, for all sequences x_n converging to x and for all positive sequences ϵ_n converging to 0, $\{n \in \mathbb{N} : |f_n(x_n) - f(x)| \geq \epsilon_n\}$ is empty and hence $f \in \Phi^{\alpha-u.e.}$. Also, if $f_n \xrightarrow{\alpha - u.e.} f$ then $|f_n| \xrightarrow{\alpha - u.e.} |f|$. Next, if $f_n \xrightarrow{\alpha - u.e.} f$, $g_n \xrightarrow{\alpha - u.e.} g$ and $a, b \in \mathbb{R}$ then there exist positive sequences ϵ_n and λ_n converging to 0 and $n_0, n_1 \in \mathbb{N}$ satisfying

$$|\{n \in \mathbb{N} : |af_n(x_n) - af(x)| \geq |a|\epsilon_n\}| \leq n_0,$$

and

$$|\{n \in \mathbb{N} : |bg_n(x_n) - bg(x)| \geq |b|\lambda_n\}| \leq n_1,$$

for all $x \in X$ and $x_n \to x$. Observe that

$$\{n \in \mathbb{N} : |(af_n(x_n) - (af(x))| \geq |a|\epsilon_n\} \cup \{n \in \mathbb{N} : |(bg_n(x_n) - (bg(x))| \geq |b|\lambda_n\}$$

contains

$$\{n \in \mathbb{N} : |(af_n + bg_n)(x_n) - (af + bg)(x))| \geq |a|\epsilon_n + |b|\lambda_n\}$$

and therefore

$$|\{n \in \mathbb{N} : |(af_n + bg_n)(x_n) - (af + bg)(x))| \geq |a|\epsilon_n + |b|\lambda_n\}| \leq n_0 + n_1.$$

Hence $(af_n + bg_n) \xrightarrow{\alpha - u.e.} (af + bg)$ which implies

$$\frac{f_n + g_n}{2} + \frac{|f_n - g_n|}{2} \xrightarrow{\alpha - u.e.} \frac{f + g}{2} + \frac{|f - g|}{2} = max(f, g)$$

This proves $max(f, g) \in \Phi^{\alpha-u.e.}$. Similarly $min(f, g) \in \Phi^{\alpha-u.e.}$. Thus $\Phi^{\alpha-u.e.}$ is a lattice. Remaining part of the proposition can be proved using similar arguments. □

Remark 3.6. By similar techniques the above result holds for $\Phi^{\alpha-e}$ and $\Phi^{\alpha-s.u.e.}$

Proposition 3.7. *Let $f_n : X \to \mathbb{R}, n \in \mathbb{N}$. If $f_n \xrightarrow{\alpha - u.e.} 0$ then $f_n^2 \xrightarrow{\alpha - u.e.} 0$.*

Proof. $f_n \xrightarrow{\alpha - u.e.} 0$ implies there exist positive sequence ϵ_n converging to 0 and $n_0 \in \mathbb{N}$ satisfying $|\{n \in \mathbb{N} : |f_n(x_n)| \geq \epsilon_n\}| \leq n_0$ for all $x \in X$ and $x_n \to x$. Since

$$\{n \in \mathbb{N} : |f_n(x_n)| \leq \epsilon_n\} \subseteq \{n \in \mathbb{N} : |f_n(x_n)|^2 \leq \epsilon_n^2\} = \{n \in \mathbb{N} : |f_n^2(x_n)| \leq \epsilon_n^2\}$$

we have

$$|\{n \in \mathbb{N} : |f_n^2(x_n)| \geq \epsilon_n^2\}| \leq |\{n \in \mathbb{N} : |f_n(x_n)| \geq \epsilon_n\}| \leq n_0$$

for each $x \in X$ and each $x_n \to x$. □

Proposition 3.8. *Let $f_n : X \to \mathbb{R}, n \in \mathbb{N}$. If f is a non-zero constant function and $f_n \xrightarrow{\alpha - u.e.} f$ then $f_n.f \xrightarrow{\alpha - u.e.} f^2$.*

Proof. Let $f(x) = c$ for each $x \in X$, where $c \neq 0$ is a constant. Note that $f_n \xrightarrow{\alpha - u.e.} f$ implies there exist positive sequence ϵ_n converging to 0 and $n_0 \in \mathbb{N}$ satisfying $|\{n \in \mathbb{N} : |f_n(x_n) - |f(x)| \geq \epsilon_n\}| \leq n_0$ for each $x \in X$ and $x_n \to x$. Since

$$\{n \in \mathbb{N} : |f_n(x_n) - f(x)| \leq \epsilon_n\} \subseteq \{n \in \mathbb{N} : |(f_n.f)(x_n) - (f.f)(x)| \leq |c|\epsilon_n\}\}$$

therefore

$$|\{n \in \mathbb{N} : |(f_n.f)(x_n) - (f.f)(x)| \geq |c|\epsilon_n\}| \leq |\{n \in \mathbb{N} : |f_n(x_n) - f(x)| \geq \epsilon_n\}|$$
$$\leq n_0$$

for all $x \in X$ and $x_n \to x$. This proves $(f_n.f) \xrightarrow{\alpha - u.e.} (f^2)$. □

Note 3.2. If $f_n \xrightarrow{\alpha - u.e.} f$ then $f_n^2 \xrightarrow{\alpha - u.e.} f^2$.

4. **Properties preserved under uniform conjugacy**

We recall the following notion of uniform conjugacy defined by Tian and Chen in [16].

Definition 4.1. *Let (X, d) and (Y, ρ) be metric spaces, $F = \{f_k\}_{k=1}^{\infty}$ and $G = \{g_k\}_{k=1}^{\infty}$ be two sequences of maps in X and Y respectively and $h : X \to Y$ be a homeomorphism. If for any $k \in \{1, 2, ...\}$, $g_k(h(x)) = h(f_k(x))$ for every $x \in X$, then F and G are said to be h-conjugate. In particular, if h is a uniform homeomorphism then F and G are said to be uniformly h-conjugate.*

They have also defined a sequence of maps on X to be chaotic in iterative and in successive way and proved that chaoticity in iterative way as well as in successive way is preserved under uniform conjugacy (Theorem 3.1 in [16]). Next, we recall the notion of exhaustiveness for a sequence of maps defined in [11].

Definition 4.2. *Let (X, d) and (Y, ρ) be metric spaces, $f_n : X \to Y, n \in \mathbb{N}$. The sequence $\{f_n\}_{n=1}^{\infty}$ is called exhaustive at $x \in X$ if for every $\epsilon > 0$ there exists a $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for all $a \in S(x, \delta)$ and for all $n \geq n_0$, we have $\rho(f_n(a), f_n(x)) < \epsilon$, where $S(x, \delta) = \{a \in X | d(x, a) < \delta\}$. The sequence $\{f_n\}_{n=1}^{\infty}$ is called exhaustive if it is exhaustive at each $x \in X$.*

Authors have used this notion which is closed to equicontinuity to describe relation between pointwise convergence and continuous convergence for sequence of functions. They have also used this notion to obtain some Ascoli type theorem dealing with exhaustiveness instead of equicontinuity. In the following we prove that exhaustiveness is preserved under uniform conjugacy.

Theorem 4.3. *Let (X, d) and (Y, ρ) be metric spaces, $F = \{f_k\}_{k=1}^{\infty}$ and $G = \{g_k\}_{k=1}^{\infty}$ be two sequences of maps in X and Y respectively. If there exists a homeomorphism $h : X \to Y$ such that h and h^{-1} both are uniformly continuous and F and G are h-conjugate, then F is exhaustive if and only if G is.*

Proof. Let $F = \{f_k\}_{k=1}^{\infty}$ be exhaustive on X. Then $\{f_k\}_{k=1}^{\infty}$ is exhaustive at each $x \in X$. Let $y \in Y$. We have to show that $\{g_k\}_{k=1}^{\infty}$ is exhaustive at y. Let $\epsilon > 0$ be given. Since h is uniformly continuous therefore there exists $\alpha > 0$ such that

$$d(x_1, x_2) < \alpha \Rightarrow \rho(h(x_1), h(x_2)) < \varepsilon. \tag{1}$$

Now h being onto, there exists $x \in X$ such that $h(x) = y$. Since $\{f_k\}_{k=1}^{\infty}$ is exhaustive at x, there exists $\eta > 0$ and $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} & a \in S(x, \eta) \Rightarrow d(f_n(a), f_n(x)) < \alpha \\ or \quad & d(a, x) < \eta \Rightarrow d(f_n(a), f_n(x)) < \alpha \end{aligned} \tag{2}$$

for all $n \geq n_0$.

Since h^{-1} is uniformly continuous therefore there exists $\theta > 0$ such that

$$\rho(y_1, y_2) < \theta \Rightarrow d(h^{-1}(y_1), h^{-1}(y_2)) < \eta \tag{3}$$

Let $b \in S(y, \theta)$ then $\rho(b, y) < \theta$ which by (3) gives $d(h^{-1}(b), h^{-1}(y)) < \eta$. Using (2), we get $d(f_n(a), f_n(x)) < \alpha$ for all $n \geq n_0$, where $a = h^{-1}(b)$. Finally using (1), we get $\rho(h(f_n(a)), h(f_n(x))) < \epsilon$ for all $n \geq n_0$.
Since $F = \{f_k\}_{k=1}^{\infty}$and $G = \{g_k\}_{k=1}^{\infty}$are h-conjugate, we get

$$\begin{aligned} & \rho(g_n(h(a)), g_n(h(x))) < \epsilon, \quad for \ \ all \ \ n \geq n_0 \\ or \quad & \rho(g_n(b), g_n(y)) < \epsilon, \quad for \ \ all \ \ n \geq n_0 \end{aligned}$$

Hence $G = \{g_k\}_{k=1}^{\infty}$is exhaustive at y. Since $y \in Y$ is chosen arbitrarily, we get that $G = \{g_k\}_{k=1}^{\infty}$is exhaustive on Y.
Conversely, let $G = \{g_k\}_{k=1}^{\infty}$be exhaustive on Y. Since $F = \{f_k\}_{k=1}^{\infty}$and $G = \{g_k\}_{k=1}^{\infty}$are h-conjugate, $G = \{g_k\}_{k=1}^{\infty}$and $F = \{f_k\}_{k=1}^{\infty}$are h^{-1}-conjugate. Since $G = \{g_k\}_{k=1}^{\infty}$is exhaustive on Y, by above observations we get $F = \{f_k\}_{k=1}^{\infty}$is exhaustive on X. □

The next result shows that uniform convergence is preserved under uniform conjugacy.

Theorem 4.4. *Let (X, d) and (Y, ρ) be two metric spaces,, $h\colon X \to Y$ be a uniformly continuous homeomorphism such that $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ on X and Y respectively are h-conjugate. If $\{f_k\}_{k=1}^{\infty}$ is uniformly convergent then $\{g_k\}_{k=1}^{\infty}$ is also uniformly convergent.*

Proof. Let $\{f_k\}_{k=1}^{\infty} \to f$ uniformly, where $f\colon X \to X$. Let $\epsilon > 0$ be given. Since h is uniformly continuous, there exists $\alpha > o$ such that

$$d(x_1, x_2) < \alpha \Rightarrow \rho(h(x_1), h(x_2)) < \epsilon \tag{4}$$

Since $\{f_k\}_{k=1}^{\infty}$ converges uniformly to f, therefore there exists $n_0 \in \mathbb{N}$ such that $d(f_k(x), f(x)) < \alpha$ for all $k \geq n_0$ and for every $x \in X$. By (4) we get

$$\rho(h(f_k(x)), h(f(x))) < \epsilon \tag{5}$$

for all $k \geq n_0$ and for every $x \in X$.

Since $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are uniformly conjugate, $h \circ f_k = g_k \circ h$ for every $k \in \mathbb{N}$ and hence (5) gives $\rho(g_k(h(x)), g(h(x))) < \epsilon$ for all $k \geq n_0$ and for every $x \in X$.
Since h is bijective, we get $\rho(g_k(y), g(y)) < \epsilon$ for every $k \geq n_0$ and $y \in Y$. □

Remark 4.5. If $h^{-1}\colon Y \to X$ is uniformly continuous, then $\{g_k\}_{k=1}^{\infty}$ is uniformly convergent implies $\{f_k\}_{k=1}^{\infty}$ is uniformly convergent.

Theorem 4.6. *Let (X, d) and (Y, ρ) be two metric spaces,, $h\colon X \to Y$ be a uniformly continuous homeomorphism such that $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ on X and Y respectively are h-conjugate. If $\{f_k\}_{k=1}^{\infty}$ α-converges to $f\colon X \to X$ then $\{g_k\}_{k=1}^{\infty}$ α-converges to $g\colon Y \to Y$ where $g = h \circ f \circ h^{-1}$.*

Proof. Let $y \in Y$ and $\{y_n\} \to y$ in Y. Then h being onto, there exist $x_n \in X$, $n \in \mathbb{N}$ and $x \in X$ such that $h(x_n) = y_n$ and $h(x) = y$. Since $y_n \to y$ therefore $h(x_n) \to h(x)$. Now h^{-1} being continuous, we get $h^{-1}(h(x_n)) \to h^{-1}(h(x))$ i.e. $x_n \to x$. Since $\{f_k\}$ α-converges to f therefore $f_n(x_n) \to f(x)$. Again h being continuous, $h(f_n(x_n)) \to h(f(x))$. Since $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are h-conjugate therefore $(g_n \circ h)(x_n) \to g(h(x))$ i.e. $g_n(y_n) \to g(y)$. Hence $\{g_k\}$ α-converges to g. □

Remark 4.7. By similar arguments we can prove that if $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are h-conjugate and $\{f_k\}_{k=1}^{\infty}$ converges to f pointwise on X then $\{g_k\}_{k=1}^{\infty}$ converges to g pointwise on Y where $g = h \circ f \circ h^{-1}$.

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References

[1] Raghib Abu-Saris, Kifah Al-Hami, *Uniform convergence and chaotic behaviour*, Nonlinear Analysis **65**(2006), 933-937.

[2] Raghib M.Abu-Saris, Félix Martinez-Gimēnez, Alfredo Peris, *Erratum to Uniform Convergence and chaotic behaviour [NonlinearAnalysis **65(4)**(2006), 933-937.]*, Nonlinear Analysis **68**(2008), 1406-1407.

[3] Indranil Bhaumik and B.S.Choudhury, *Uniform convergence and sequence of maps on a compact metric space with some chaotic properties*, Anal.Theory Appl., **26(1)**(2010),53-58.

[4] Z. Bukovska, *Thin sets in trigonometrical series and quasinormal convergence*, Math. Slovaca, **40**(1990), 53-62.

[5] Lev Bukovsky and Jozef Hales, *QN-Spaces, WQN-spaces and covering properties*, Topology and its Applications, **154**(2007), 848-858.

[6] L.Bukovsky, I.Reclaw and M.Repicky, *Spaces not distinguishing pointwise and quasinormal convergence of real functions*, Topology and its Applications **41**(1991), 25-40.

[7] L.Bukovsky, I.Reclaw and M.Repicky, *Spaces not distinguishing convergences of real-valued functions*, Topology and its Applications, **112** (2001), 13-40.

[8] Ā, Csāszār and M.Laczkovich, *Discrete and equal convergence*, Studia Sci. Math. Hugar. **10** (1975),463-472.

[9] Ruchi Das and Nikolaos Papanastassiou, *Some types of convergence of sequences of real valued functions*, Real Analysis Exchange **29**, 2003/2004, 43-58.

[10] Alecsandro Fedeli and Attilio Le Donne, *A note on the uniform limit of transitive dynamical Systems*, Bull. Belg. Math. Soc. Simon Stevin **16**(2009), 59-66.

[11] V. Gregoriades and N. Papanastassiou, *The notion of exhaustiveness and Ascoli-type theorems*, Topology and its Applications **155**(2008), 1111-1128.

[12] E. Holā and T. Sālāt, *Graph convergence, uniform, quasi-uniform and continuous convergence and some characterizations of compactness*, Acta Math. Univ. Comenian,**54-55**(1988), 121-132.

[13] Nikolaos Papanastassiou, *On a new-type of convergence of sequences of functions*, Atti Sem. Mat. Fis. Univ. Modena, **50** (2002), 493-506.

[14] Heriberto Romān-Flores, *Uniform convergence and transitivity*,Chaos, Solitons and Fractals **38**(2008), 148-153.

[15] S.Stoilov, *Continuous convergence*, Rev. Math. Pares Appl. **4**(1959),341-344.

[16] Chuanjun Tian and Guanrong Chen, *Chaos of sequence of maps in a metric space*, Chaos, Solitons and Fractals **28**(2006), 1067-1075.

[17] Kesong Yan, Fanping Zeng and Gengrang Zhang, *Devaney's chaos on uniform limit maps*,Chaos, Solitons and Fractals **44**(2011), 522-525.

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