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Approach Regions of Lebesgue Measurable, Locally Bounded, Quasi-Continuous Functions

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Abstract

Quasi-continuity (in the sense of Kempisty) generalizes directional continuity of complex-valued functions on open subsets of $\mathbb{R}^n$ or $\mathbb{C}^n$, and in particular provides certain approach regions at every point. We show that these can be used as a proof tool for proving several properties for Lebesgue measurable, locally bounded, quasi-continuous functions e.g. that for such a function f the polynomial ring $C(M,K)[f]$ (where $K = \mathbb{R}$ or $\mathbb{C}$) satisfies that the equivalence classes under identification a.e. have cardinality one and an asymptotic maximum principle.

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1 Introduction

The purpose of this article is to consider some properties of Lebesgue measurable, locally bounded quasi-continuous functions by pointing out the existence of approach regions, and show that these in turn can be used as a proof tool. In this introduction we present some problems which can be treated using approach regions.

Let M be a domain in $\mathbb{R}^n$ ($\mathbb{C}^n$). We shall denote by $L^1(M)$ the set of equivalence classes of integrable functions, where f, g are called equivalent, denoted $f \sim g$, if they agree a.e. with respect to Lebesgue measure on M. Clearly $f, g \in C(M,K)$ ($K = \mathbb{C}$ or $\mathbb{R}$) together with $[f] = [g]$ implies that $f \equiv g$

(pointwise) due to the fact that a continuous function which vanishes a.e. vanishes identically. In other words the equivalence classes in $C(M,K)/\sim$ each consist of precisely one element.

Question. Is $C(M,K)$ maximal with this property?

The answer is no and we provide in Proposition 3.14 proper superspaces of $C(M,K)$ with this property. These are in turn subspaces of a superspace of $C(M,K)$ whose elements satisfy a generalization of directional continuity (see Section 2), and it turns out that this larger class shares several properties with the set of continuous functions (see Section 3). The given generalization of directional continuity turns out in certain natural settings to be equivalent to one introduced by Kempisty [8] called *quasi-continuity*, however it is known that there exists a quasi-continuous function which is (the axiom of choice is needed) not Lebesgue measurable (see Marcus [9]), and also e.g. the function $f(x) := x^{-\frac{1}{2}}, x \in (0,1), f(x) = 0, x \in (-1,0]$ is Lebesgue integrable, quasi-continuous but not locally bounded at $x = 0$. We shall in this text restrict ourselves to the the case of Lebesgue measurable, locally bounded, quasi-continuous functions on domains in $\mathbb{R}^n$ ($\mathbb{C}^n$). For an introduction to quasi-continuous functions please see Neubrunn [10] and the references therein. We shall consider some properties inherited by functions which can be locally L^1-approximated by certain families of continuous functions.

Example 1.1. *A function on a domain M of $\mathbb{C}^n$ which can be uniformly approximated by elements in a class of functions (K-valued, $K = \mathbb{R}$ or $\mathbb{C}$) which have restrictions M that are continuous, is itself continuous but if the convergence is only pointwise one obtains a so called Baire 1 function. Under certain natural circumstances the boundary maximum modulus principle will be inherited for continuous functions which can be uniformly approximated by functions which satisfy this principle.*

A study of not necessarily continuous but locally bounded Lebesgue measurable functions on smooth manifolds cannot assume local uniform approximability by continuous functions. There are exceptional cases where local L^1-approximability by appropriate families of continuous functions imply local uniform approximation, e.g. if in Example 1.1 we have that the domains of the approximating elements and that of the restrictions have the same dimension and if it concerns approximation by holomorphic functions then local L^1-approximation implies that the approximation is in fact locally uniform (see Hörmander [6], Theorem 1.2.4).

Question. Can we find an analogue of Example 1.1 for Lebesgue measurable locally bounded quasi-continuous functions?

For the first part of Example 1.1 we refer the reader to Kantorovich [7] for an analysis of the so called Baire order of a family of functions properly containing the continuous functions (see also Rodriguez-Salinas [11]). We mention that it is already known (see Natanson [12]) that a (real-valued) Lebesgue measurable function (in one real variable) is equivalent to a function of Baire class ≤ 2. We prove an analogue of the second part (see Proposition 3.24).
In a metric space continuity is equivalent to sequential continuity, and the generalization of directional continuity which quasi-continuity involves that there for every point exists an open set with the point in its closure, such that sequential continuity holds for any sequence chosen within the special open set (see Proposition 3.15), this is in one dimension precisely continuity to one side. In higher dimension it is a stronger property.

> The main results of this text are a uniqueness property for polynomial rings $C(M,K)[f]$ where f is approach compatible (see Proposition 3.14) and an asymptotic local maximum modulus principle (see Proposition 3.24).

2 Approach regions and Lebesgue measurable locally bounded quasi-continuous functions

We shall formulate the definition of Lebesgue measurable locally bounded quasi-continuous functions in terms of so called approach regions. But first we give the general definition of quasi-continuity (which will reduce to the definition involving approach regions for Lebesgue measurable, locally bounded functions on domain of $\mathbb{R}^n$ or $\mathbb{C}^n$).

Definition 2.1 (Quasi-continuous, see Kempisty [8]). *A function* $f : X \to Y$ *from a topological space* X *to a metric space* (Y,d) *is called* quasi-continuous *at* $x_0 \in X$ *if* $\forall \epsilon > 0$ *and each neighborhood* U_{x_0} *of* x_0 $\exists$ *a nonempty open set* $W_{x_0} \subset \mathrm{U}_{x_0}$ *such that* $d(f(x), f(x_0)) < \epsilon, \forall x \in W_{x_0}$.

We have for any domain M of $\mathbb{R}^n$ (or $\mathbb{C}^n$) $C(M,K) \subsetneqq \{f : M \to K;\ f$ is Lebesgue measurable, locally bounded, quasi-continuous$\} \subsetneqq \{f : M \to K; f$

is locally integrable} . Quasi-continuous functions are a generalization of functions $U \to \mathbb{R}, U \subset \mathbb{R}$ an open connected set subset, which at every point of U is left or right continuous. Of course if the function is both left and right continuous at a point then it is continuous at that point, however in higher dimension there are infinitely many directions of approach to every point.

Definition 2.2 (ϵ-approach region). *Let f be a measurable complex-valued function on an open subset M of some $\mathbb{R}^n$ or $\mathbb{C}^n$. Given a point $p_0 \in M$ and an $\epsilon > 0$, we define an* ϵ-approach region with respect to f at p_0, $\omega^f_{p_0,\epsilon}$ *to be the set,*

$$\omega^f_{p_0,\epsilon} := \{z \in M : |f(z) - f(p_0)| < \epsilon\}^\circ,$$

where we use the notation X° for the interior of a set X. Note that we always have $p_0 \in \{z \in M : |f(z) - f(p_0)| < \epsilon\}$. Note also that $\omega^f_{p_0,\epsilon}$ is always open but can be empty.

Definition 2.3 (Approach compatible function). *Let M be an open subset of $\mathbb{R}^n$ or $\mathbb{C}^n$. We call a locally bounded Lebesgue measurable function f,* approach compatible *at $p_0 \in M$, if for any open ambient neighborhood* U_{p_0} *of p_0, $\omega^f_{p_0,\epsilon} \cap \mathrm{U}_{p_0} \neq \emptyset, \forall \epsilon > 0$. If f is approach compatible at every point of M we simply say that f is approach compatible.*

Remark 2.4. *Local boundedness together with Lebesgue measurability implies that any approach compatible function is locally integrable. The requirement of local boundedness is important, indeed we wish that for a given $\epsilon > 0$, $\int_{\omega^f_{p_0,\epsilon}} |f|$ is not finite since we are interested in situations when f is locally integrable.*

Observation 2.5. *Let M be an open subset of some $\mathbb{R}^n$ (or $\mathbb{C}^n$) and let $K = \mathbb{C}$ or $\mathbb{R}$. Then a function $f : M \to K$ is approach compatible iff it is Lebesgue measurable, locally bounded and quasi-continuous on M.*

Proof. Let $p_0 \in M$ and let $\mathrm{U}_{p_0} \subset M$ be an open neighborhood of p_0. If f is quasi-continuous at p_0 then $|f(z) - f(p_0)| < \epsilon, \forall z \in W_{p_0}$ for some open $W_{p_0} \subset \mathrm{U}_{p_0}$. Now $W_{p_0} = \{z \in W_{p_0} : |f(z) - f(p_0)| < \epsilon\} \subseteq \mathrm{U}_{p_0} \cap \{z \in M : |f(z) - f(p_0)| < \epsilon\}$ and we have that W_{p_0} is open so $\mathrm{U}_{p_0} \cap \{z \in M : |f(z) - f(p_0)| < \epsilon\}$ must contain an interior point which in turn must lie in $\{z \in M : |f(z) - f(p_0)| < \epsilon\}^\circ \cap \mathrm{U}_{p_0} = \omega^f_{p_0} \cap \mathrm{U}_{p_0}$,
Conversely if f is approach compatible at p_0 then $\mathrm{U}_{p_0} \cap \omega^f_{p_0}$ is open and nonempty thus contains an open ball, and since this can be done for any $\epsilon > 0$, f must be quasi-continuous. $\square$

For simplicity we shall (in light of Observation 2.5 and because we shall only consider complex-valued functions on real (complex) open subsets of $\mathbb{R}^n$ ($\mathbb{C}^n$)) in the remaining part of this text use the term approach compatible instead of Lebesgue measurable locally bounded quasi-continuous.

3 Properties

For the purposes of the proofs of this text the quintessential properties of approach compatible functions are given Observation 3.10 and the combined properties which provide that the polynomial ring of an approach compatible function over the continuous functions belongs to the set of approach compatible functions. First of all we note the following consequence of approach compatibility.

Observation 3.1. *A function f approach compatible at a point p satisfies that $p \in \overline{\omega^f_{p,\epsilon}}$, recall the approach regions are open and nonempty. Also note that $\epsilon' > \epsilon \Rightarrow \omega^f_{p,\epsilon} \subseteq \omega^f_{p,\epsilon'}$ and since each $E_\epsilon := B_p(\delta) \cap \omega^f_{p,\epsilon}$ is by definition open for all $\delta > 0$ we have $dist(p, \omega^f_{p,\epsilon}) = 0$. A result of this is that if another function g satisfies that there exists for each $\epsilon > 0$ another $\epsilon' > 0$ such that,*

$$\omega^f_{p,\epsilon'} \cap B_p(\epsilon') \subseteq \omega^g_{p,\epsilon},$$

then g is approach compatible at p, namely any open neighborhood U_p of p has nonempty open intersection with $\omega^f_{p,\epsilon'} \cap B_p(\epsilon')$ (because there is a nonempty open subset of $\omega^f_{p,\epsilon'}$ containing p on the boundary) thus also has open intersection with $\omega^g_{p,\epsilon}$.

It is clear that approach compatible functions are not necessarily continuous and that continuous functions are approach compatible, and in one real variable functions continuous to one side at every point are approach compatible.

Example 3.2. *$F(x) := 1$ for $x \in [0, 1]$ and zero otherwise, is approach compatible since it is at every point left or right continuous. Note that this function is globally neither left continuous nor right continuous. Any neighborhood U_1 of the point $x = 1$ has open intersection with the open interval $(0, 1)$, and $0 = |F(x) - F(1)| < \epsilon, \forall \epsilon > 0, x \in [0, 1]$, in fact for any $\epsilon < 1$, $\omega^F_{0,\epsilon} := (0, 1)$. Similarly one checks the only other point of discontinuity $x = 0$.*

Example 3.3. *Let $M = (-1, 1)$ and let $f(x) = (1 - x/2) \cdot \sin\left(\frac{1}{x}\right), x \in (0, 1), f(x) = 0, x \in (-1, 0]$. This function is approach compatible since it* *is left or right continuous at every point of M. For any $x \in M$, there is a connected component of $\omega^f_{x,\epsilon}$ which contains x in its closure.*

The set of approach compatible functions is not closed under addition or multiplication.

Example 3.4. *Let*

$$F_1(x) := \begin{cases} \frac{1}{2} & ,x \geq 0 \\ 0 & ,else \end{cases} ; F_2(x) := \begin{cases} 0 & ,x > 0 \\ -\frac{1}{2} & ,else \end{cases}$$

where F_1 is right continuous and F_2 is left continuous everywhere (in particular both are approach compatible). Define also the approach compatible functions,

$$G_1(x) := \begin{cases} \frac{1}{2} & ,x \geq 0 \\ 1 & ,else \end{cases} ; G_2(x) := \begin{cases} 1 & ,x > 0 \\ -\frac{1}{2} & ,else \end{cases}$$

Then we have,

$$(F_1 + F_2)(x) = \begin{cases} \frac{1}{2} & ,x > 0 \\ 0 & ,x = 0 \\ -\frac{1}{2} & ,x < 0 \end{cases} ; (G_1 \cdot G_2)(x) = \begin{cases} \frac{1}{2} & ,x > 0 \\ -\frac{1}{4} & ,x = 0 \\ -\frac{1}{2} & ,x < 0 \end{cases}$$

neither of which is approach compatible at the origin.

Here is a complex-valued example.

Example 3.5. *Define the real smooth submanifold of $\mathbb{C}^2$,*

$$M := \{(z_1, z_2) \in \mathbb{C}^2 : |z| < 1\}.$$

Then the restriction to M, of the function,

$$f = z_1 e^{\lceil Imz_2 \rceil},$$

(where $\lceil \cdot \rceil$ denotes the least upper integer) is an approach compatible function since it satisfies on $\{z \in M : Imz_2 > 0\}$ that $f(z) = z_1 \cdot e$, (thus is a holomorphic function on the upper half-sphere, in particular any point has open nonempty ϵ-approach regions for all $\epsilon > 0$), and on the lower hemishpere $\{z \in M : Imz_2 \leq 0\}$ it satisfies $f(z) = z_1$ (a holomorphic function on the interior). If we take any point $p \in M \setminus \{Imz_2 \neq 0\}$ then p belongs to the interior of $\omega^f_{p,\epsilon}$. For $p \in M \setminus \{Imz_2 = 0\}$ let U be an open neighborhood of p in M. Then U has open intersection with the interior of each hemisphere. By the open mapping theorem for holomorphic functions (f being holomorphic in the interior of each hemisphere) f is approach compatible at p. This example is clearly Lebesgue measurable, locally bounded on M and non-continuous on $\{Imz_2 = 0, z_1 \neq 0\}$.

Another property of quasi-continuous functions which is not shared with the set of continuous functions is the fact that a bijective quasi-continuous function need not have quasi-continuous inverse (see Grande & Natkaniec [3]). For further examples of quasi-continuous functions see e.g. Neubrunn [10].

3.1 A uniqueness property for certain polynomial rings

After having seen several examples of properties different compared to the continuous subclass we now present some properties shared with the continuous functions.

Observation 3.6. *The set of approach compatible functions is closed under addition by the everywhere constant functions. Let $c \in \mathbb{C}$. If f is approach compatible on an open subset M of some $\mathbb{R}^n$ (or $\mathbb{C}^n$) and $p \in M$, then $\omega_{p,\epsilon}^{f+c} = \{z \in M : |(f(z)+c)-(f(p)+c)| = |f(z)-f(p)| < \epsilon\}^\circ = \omega_{p,\epsilon}^f$. It is also clear that $-f$ is approach compatible, since $\{|(-f(z))-(-f(p))| < \epsilon\} = \{|f(z)-f(p)| < \epsilon\}$, so $\omega_{p,\epsilon}^{-f} = \omega_{p,\epsilon}^f$.*

More can be said.

Proposition 3.7. *Let M be an open subset of some $\mathbb{R}^n$ (or $\mathbb{C}^n$). The set of approach compatible functions is closed under addition by continuous functions. Furthermore if f is approach compatible and $g \in C(M,K)$ (where $K = \mathbb{C}$ or $\mathbb{R}$), then for any $p \in M$ and $\epsilon > 0$, the ϵ-approach region of $f+g$ at p contains the (open) intersection with the ϵ'-approach region of f at p with $B_p(\epsilon')$ for some $\epsilon' > 0$.*

Proof. Let $p \in M$, let f be approach compatible on M and let $g \in C(M,K)$. For the ϵ-approach region at any $p \in M$ of g we know that it is an open neighborhood of p which implies that $\omega_{p,\epsilon}^f \cap \omega_{p,\epsilon}^g$ is open and nonempty. For the ϵ-approach regions with respect to $f+g$ we have,

$$\omega_{p,\epsilon}^{f+g} = \{z \in M : |(f(z)+g(z)) - (f(p)+g(p))| < \epsilon\}^\circ = \\ \{z \in M : \underbrace{|(f(z)-f(p)) + (g(z)-g(p))|}_{\leq |f(z)-f(p)|+|g(z)-g(p)|} < \epsilon\}^\circ.$$

Hence the (due to the definition of approach regions necessarily nonempty open) intersection $\omega_{p,\epsilon/2}^f \cap \omega_{p,\epsilon/2}^g = \{z \in M : |f(z)-f(p)| < \epsilon/2\}^\circ \cap \{z \in M : |g(z)-g(p)| < \epsilon/2\}^\circ$, belongs to $\omega_{p,\epsilon}^{f+g}$, and this can be done for any $\epsilon > 0$. Now g being continuous means that $\omega_{p,\epsilon/2}^g$ contains an open ball say $B_p(\epsilon')$. This means by definition of approach region of f at p that,

$$\omega_{p,\epsilon}^{f+g} \supset \omega_{p,\epsilon/2}^f \cap \omega_{p,\epsilon/2}^g \cap B_p(\epsilon') = B_p(\epsilon') \cap \omega_{p,\epsilon/2}^f \neq \emptyset.$$

Finally, we recall that $\epsilon_1 \geq \epsilon_2 \Rightarrow \omega^f_{p,\epsilon_2} \subseteq \omega^f_{p,\epsilon_1}$ thus making sure that $\epsilon' < \epsilon/2$ implies that

$$\emptyset \neq B_p(\epsilon') \cap \omega^f_{p,\epsilon'} \subset \omega^{f+g}_{p,\epsilon}.$$

By Observation 3.1 this implies that $f + g$ is approach compatible. □

Proposition 3.8. *If f is approach compatible on an open subset M of some $\mathbb{R}^n$ (or $\mathbb{C}^n$) then so is f^k and $k \cdot f$ for all finite $k \in \mathbb{N}$. Furthermore for any $p \in M$ and sufficiently small $\epsilon > 0$ the ϵ-approach region of f^k and $k \cdot f$ at p each contain the (open) intersection with the ϵ'-approach region of f at p with $B_p(\epsilon')$ for some $\epsilon' > 0$.*

Proof. We have,

$$\omega^{kf}_{p,\epsilon} = \{z \in M : |kf(z) - kf(p)| < \epsilon\}^\circ = \{z \in M : k\,|f(z) - f(p)| < \epsilon\}^\circ = \{z \in M : |f(z) - f(p)| < \epsilon/k\}^\circ = \omega^f_{p,\frac{\epsilon}{k}}.$$

so any $\omega^{kf}_{p,\epsilon}$ contains $\omega^f_{p,\frac{\epsilon}{k}}$ whose intersection with any open neighborhood of p is nonempty open, so kf is approach compatible. As for f^k we first look at the case $k = 2$. We have because f is locally bounded that there exists (when restricting attention to a subset of a fix approach region) a $0 < C < \infty$ such that $|f(z) + f(p)| < C$ on (the open) $\omega^f_{p,\epsilon} \cap B_p(\epsilon)$. Then,

$$\omega^{f^2}_{p,\epsilon} \cap B_p(\epsilon) = \{z \in M : \left|f^2(z) - f^2(p)\right| < \epsilon\}^\circ \cap B_p(\epsilon) = \{z \in M : \underbrace{|(f(z) - f(p))(f(z) + f(p))|}_{\leq C|f(z)-f(p)|} < \epsilon\}^\circ \cap B_p(\epsilon).$$

Thus,

$$\omega^f_{p,\frac{\epsilon}{C}} \cap B_p(\epsilon) \subset \omega^{f^2}_{p,\epsilon},$$

so by Observation 3.1 f^2 is approach compatible at p, and we can set $\epsilon' := \epsilon/C$ in the statement of the proposition. Now we use induction in k, namely let $k > 2$ and assume f^j is approach compatible for $1 \leq j \leq k-1$, and that for any $\epsilon, \exists\epsilon'$ such that $\omega^f_{p,\epsilon'} \cap B_p(\epsilon') \subset \omega^{fj}_{p,\epsilon}$. We have,

$$\left|f^k(z) - f^k(p)\right| \leq \left|(f(z) + f(p))(f^{k-1}(z) - f^{k-1}(p)) + f(z)f^{k-1}(p) - f(p)f^{k-1}(z)\right| \leq |(f(z) + f(p))|\left|f^{k-1}(z) - f^{k-1}(p)\right| + |f(z)f(p)|\left|f^{k-2}(p) - f^{k-2}(z)\right|.$$

Again since f is locally bounded we have when restricting attention to a subset of a fix approach region that there is a C such that $\max\{|(f(z) + f(p))|, |f(z)f(p)|\} \leq C$. Thus,

$$\left|f^k(z) - f^k(p)\right| \leq C\left|f^{k-2}(z) - f^{k-2}(p)\right| + C\left|f^{k-1}(z) - f^{k-1}(p)\right|. \quad (1)$$

By the induction hypothesis there exists $\epsilon' > 0, \epsilon'' > 0$ such that, $\omega^f_{p,\epsilon'} \cap B_p(\epsilon') \subset \omega^{f^{k-1}}_{p,\epsilon}$, and $\omega^f_{p,\epsilon''} \cap B_p(\epsilon'') \subset \omega^{f^{k-2}}_{p,\epsilon}$. Set $\epsilon''' := \min\{\epsilon', \epsilon''\}$. Then,

$$\omega^f_{p,\epsilon'''} \cap B_p(\epsilon''') \subset \omega^{f^{k-1}}_{p,\epsilon} \cap \omega^{f^{k-2}}_{p,\epsilon}.$$

Now by Eqn.(1),

$$\omega^{f^k}_{p,2C\epsilon} \cap B_p(\epsilon) = \{z \in M : \left|f^k(z) - f^k(p)\right| < 2C\epsilon\}^\circ \cap B_p(\epsilon) \supset \omega^{f^{k-2}}_{p,\epsilon} \cap \omega^{f^{k-1}}_{p,\epsilon} \cap B_p(\epsilon).$$

Set $\delta := C\epsilon$. Then for each $\delta > 0, \exists \epsilon'''$ such that,

$$\omega^{f^k}_{p,\delta} \cap B_p(\epsilon''') = \{z \in M : \left|f^k(z) - f^k(p)\right| < \delta\}^\circ \cap B_p(\epsilon''') \supset \omega^f_{p,\epsilon'''} \cap B_p(\epsilon'''). \tag{2}$$

Thus by Observation 3.1 f^k is approach compatible at p, and Eqn.(2) clearly shows the last part of the statement of the proposition for f^k, so by induction we are done. □

Proposition 3.9. *Let M be an open subset of some $\mathbb{R}^n$ (or $\mathbb{C}^n$). The set of approach compatible functions is closed under multiplication by continuous functions i.e. for any $p \in M$ and $g \in C(M, K)$ ($K = \mathbb{R}$ or $\mathbb{C}$) it holds that $g \cdot f$ is approach compatible at p, and for $\epsilon > 0$ the ϵ-approach region of $f \cdot g$ at p each contain $B_p(\epsilon') \cap \omega^f_{p,\epsilon'}$ for some $\epsilon' > 0$.*

Proof. Let f be approach compatible and $g \in C(M, K)$. We use that

$$|f(z)g(z) - f(p)g(p)| =$$
$$\left|(f(z)+g(z))^2 - (f(p)+g(p))^2 - (f^2(z) - f^2(p)) - (g^2(z) - g^2(p))\right| \leq$$
$$\left|(f(z)+g(z))^2 - (f(p)+g(p))^2\right| + \left|f^2(z) - f^2(p)\right| + \left|g^2(z) - g^2(p)\right|.$$

In particular,

$$\omega^{fg}_{p,\epsilon} \supseteq \omega^{(f+g)^2}_{p,\frac{\epsilon}{3}} \cap \omega^{f^2}_{p,\frac{\epsilon}{3}} \cap \omega^{g^2}_{p,\frac{\epsilon}{3}}. \tag{3}$$

So (by Observation 3.1) we are done if we can show that the right hand side contains for some ϵ', the set $B_p(\epsilon') \cap \omega^f_{p,\epsilon'}$. By Proposition 3.8 applied to f^2 there exists ϵ_1 such that,

$$B_p(\epsilon_1) \cap \omega^f_{p,\epsilon_1} \subseteq \omega^{f^2}_{p,\frac{\epsilon}{3}}. \tag{4}$$

Since g is continuous its ϵ-approach regions are open so clearly there exists ϵ_2 such that,

$$B_p(\epsilon_2) \cap \omega^f_{p,\epsilon_2} \subseteq \omega^{g^2}_{p,\frac{\epsilon}{3}}. \tag{5}$$

Also by Proposition 3.7 to $(f+g)$ followed directly by an application of Proposition 3.8 to the second power $(f+g)^2$ there exists $\epsilon_3 > 0$ such that,

$$B_p(\epsilon_3) \cap \omega^f_{p,\epsilon_3} \subseteq \omega^{(f+g)^2}_{p,\frac{\epsilon}{3}}. \tag{6}$$

Setting $\epsilon' := \min\{\epsilon_1, \epsilon_2, \epsilon_3\}$ we have that Eqn.(4), Eqn.(5) together with Eqn.(6) yield,

$$B_p(\epsilon') \cap \omega^f_{p,\epsilon'} \subseteq \omega^{(f+g)^2}_{p,\frac{\epsilon}{3}} \cap \omega^{f^2}_{p,\frac{\epsilon}{3}} \cap \omega^{g^2}_{p,\frac{\epsilon}{3}}.$$

We have by Eqn.(3) that $\omega^{fg}_{p,\epsilon}$ contains $B_p(\epsilon') \cap \omega^f_{p,\epsilon'}$ which is open nonempty (since f is approach compatible). Again by approach compatibility of f at p we have that for any open neighborhood U of p, $\mathrm{U} \cap B_p(\epsilon') \cap \omega^f_{p,\epsilon'} \subseteq \mathrm{U} \cap \cap \omega^f_{p,\epsilon'}$ so the right hand side must be open nonempty. Thus fg is approach compatible at p. furthermore it contains for any $\epsilon > 0$, $B_p(\epsilon') \cap \omega^f_{p,\epsilon'}$ for some $\epsilon' > 0$ so we are done. □

The following is another property which the approach compatible functions share with the continuous functions.

Observation 3.10. *If f is an approach compatible function on an open subset U (of $\mathbb{R}^n$ or $\mathbb{C}^n$) such that f vanishes a.e. then f must vanish pointwise.*

Proof. If $f(p) \neq 0$, then there exists an $\epsilon > 0$ such that $|f| > 0$ on the ϵ-approach region $\omega^f_{p,\epsilon}$. By definition of approach compatibility these approach regions have nonempty interior thus nonzero Lebesgue measure so it cannot hold that $f = 0$ a.e. on U, thus we must have $p \in f^{-1}(0), \forall p \in U$. □

Let M be an open subset of some $\mathbb{R}^n$ or $\mathbb{C}^n$. Let $\sim$ denote the equivalence relation on a given subset of the set of integrable functions, given by $f \sim g$, if they agree a.e. with respect to Lebesgue measure on M. We shall show a uniqueness property for certain superspaces of the continuous functions.

Definition 3.11. *Let M be an open subset of some $\mathbb{R}^n$ or $\mathbb{C}^n$, and let f be approach compatible on M. Denote by $C(M,K)[f]$ the set of polynomials in f with coefficients in $C(M,K)$, i.e. functions of the form $F(z) = \sum_{j=1}^N c_j(z) f^j(z), c_j(z) \in C(M,K)$, for some finite positive integer N.*

Proposition 3.12. *Let M be an open subset of some $\mathbb{R}^n$ or $\mathbb{C}^n$, and let f be an approach compatible K-valued ($K = \mathbb{C}$ or $\mathbb{R}$) function on M. Then $C(M,K)[f]$ belongs to the set of approach compatible functions.*

Proof. We shall need the following lemma.

Lemma 3.13. *If g, h are two functions approach compatible at p such that for any $\epsilon > 0$, $\omega^g_{p,\epsilon}$ contains $B_p(\epsilon') \cap \omega^f_{p,\epsilon'}$ and $\omega^h_{p,\epsilon}$ contains $B_p(\epsilon'') \cap \omega^f_{p,\epsilon''}$ for some $\epsilon' > 0, \epsilon'' > 0$, then $g + h$ is approach compatible.*

Proof. Setting $\epsilon''' := \min\{\epsilon', \epsilon''\}$ we have, $B_p(\epsilon''') \cap \omega^f_{p,\epsilon'''} \subseteq \omega^g_{p,\epsilon} \cap \omega^h_{p,\epsilon}$. Also as in the proof of Proposition 3.7,

$$\omega^{g+h}_{p,\epsilon} = \{z \in M : |(g(z) + h(z)) - (g(p) + h(p))| < \epsilon\}^\circ = \{z \in M : \underbrace{|(g(z) - g(p)) + (h(z) - h(p))|}_{\leq |f(z)-f(p)|+|g(z)-g(p)|} < \epsilon\}^\circ.$$

so $\omega^g_{p,\epsilon} \cap \omega^h_{p,\epsilon} \subseteq \omega^{g+h}_{p,2\epsilon}$, where the left hand side contains $B_p(\epsilon''') \cap \omega^f_{p,\epsilon'''}$, which by approach compatibility of f will have nonempty open intersection with any open neighborhood U of p in M (because $\mathrm{U} \cap (B_p(\epsilon''') \cap \omega^f_{p,\epsilon'''}) = (\mathrm{U} \cap (B_p(\epsilon''')) \cap \omega^f_{p,\epsilon'''}))$. This completes the proof. □

The proposition can now be realized as a consequence of Lemma 3.13 together with the accumulated result of Proposition 3.7 (closedness under addition by continuous functions), Proposition 3.8 (closedness under taking finite powers and sums, f^k, kf) and Proposition 3.9 (closedness under multiplication by continuous functions). Namely each proposition yields an approach compatible function which for any $\epsilon > 0$ contains $B_p(\epsilon') \cap \omega^f_{p,\epsilon'}$ for some $\epsilon' > 0$, so by Lemma 3.13 a finite sum of such functions is approach compatible at p and we are done. □

Proposition 3.14. *Let M be a an open subset of $\mathbb{R}^n$ ($\mathbb{C}^n$) and let f be an approach compatible K-valued ($K = \mathbb{C}$ or $\mathbb{R}$) function on M. Then,*

$$g \in C(M, K)[f] \Rightarrow \textit{the equivalence class of } g \textit{ in } C(M, K)[f]/\sim \textit{ has cardinality one.}$$

Proof. Let $g, h \in C(M, K)[f], g = h$ a.e. with respect to Lebesgue measure on M. By Proposition 3.9, Proposition 3.12 this means that $g+(-h)$ is also an approach compatible function. But by Observation 3.10 any approach compatible function which vanishes a.e. (which clearly $(g - h)$ must) vanishes identically. Thus $g \equiv h$ on M. □

3.2 A note on the perspective of generalizing directional continuity

In this section we point out that approach compatibility has a sequential formulation and we show that as a consequence certain operations e.g. taking the absolute value preserves approach compatibility. But first we look at a formulation generalizing sequential continuity.

Proposition 3.15. *Let M be an open subset of some $\mathbb{R}^n$ or $\mathbb{C}^n$, $p_0 \in M$, and let f be a measurable locally bounded function $M \to K$ (where $K = \mathbb{R}$ or $\mathbb{C}$). Then the following assertions are equivalent:*

(i) f is approach compatible at $p_0 \in M$.

(ii) There exists a open set $U_{p_0} \subset M$ with $p_0 \in \overline{U}_{p_0}$, such that for any sequence $\{z_j\}_{j\in\mathbb{N}}$ in U_{p_0} with $z_j \xrightarrow{j} p_0$, it holds that $f(z_j) \xrightarrow{j} f(p_0)$.

Proof. (i)⇒(ii).
By definition there exists for each $\epsilon > 0$ an (open) nonempty approach region at p, denoted $\omega^f_{p,\epsilon}$. Let $B_p(\epsilon)$ denote the ball in M of radius $\epsilon > 0$ centered at p, and set (see Figure 1),

$$E_\epsilon := \omega^f_{p_0,\epsilon} \cap B_{p_0}(\epsilon).$$

Let $1 > \epsilon_1 > 0$. If $\overline{E_{\epsilon_1}}$ intersects $\partial B_{p_0}(\epsilon_1)$ set $r_1 := \epsilon_1$. If instead $\overline{E_{\epsilon_1}}$ does not intersect $\partial B_{p_0}(\epsilon_1)$ (in which case we know that $\omega^f_{p_0,\epsilon_1} \cap B_{p_0}(\epsilon_1)$ is a nonempty open proper subset of $B_{p_0}(\epsilon_1)$ and contains p_0 in its closure) set $r_1 := \max_{z\in\partial E_{\epsilon_1}} |p_0 - z|$. Define the set,

$$F_1 := \left(B_{p_0}(r_1) \setminus \overline{B_{p_0}\left(\frac{r_1}{2}\right)}\right) \cap E_{\epsilon_1}, \tag{7}$$

is open and nonempty. Denoting $B_1 := B_{p_0}(\frac{r_1}{2})$, we see that $F_1 \cap B_1 = \emptyset$, but that F_1 is open and nonempty and $\operatorname{dist}(F_1, p_0) \leq r_1$. Now define $F_k, k \geq 2$, inductively as follows. If $\overline{E_{\frac{r_{k-1}}{2}}}$ intersects $\partial B_{p_0}\left(\frac{r_{k-1}}{2}\right)$ set $r_k := \frac{r_{k-1}}{2}$. If instead $\overline{E_{\frac{r_{k-1}}{2}}}$ does not intersect $\partial B_{p_0}\left(\frac{r_{k-1}}{2}\right)$ set $r_k := \max_{z\in\partial E_{\frac{r_{k-1}}{2}}} |p_0 - z|$. Set $B_k = B_{p_0}(\frac{r_k}{2})$ and set,

$$F_k := \left(B_{p_0}(r_k) \setminus \overline{B_{p_0}\left(\frac{r_k}{2}\right)}\right) \cap E_{\frac{r_{k-1}}{2}}. \tag{8}$$

We have that $F_k \cap B_k = \emptyset$, but that F_k is open and nonempty and $\operatorname{dist}(F_k, p_0) \leq r_k$. For $z \in E_\epsilon$ it holds simultaneously that $|z - p_0| < \epsilon$ and $|f(z) - f(p_0)| < \epsilon$, so by construction $|f(z) - f(p_0)| < r_k$, as soon as $z \in F_k$. Furthermore, $r_k \to 0$ as $k \to \infty$. Finally setting,

$$U_{p_0} := \bigcup_{k=1}^{\infty} F_k, \tag{9}$$

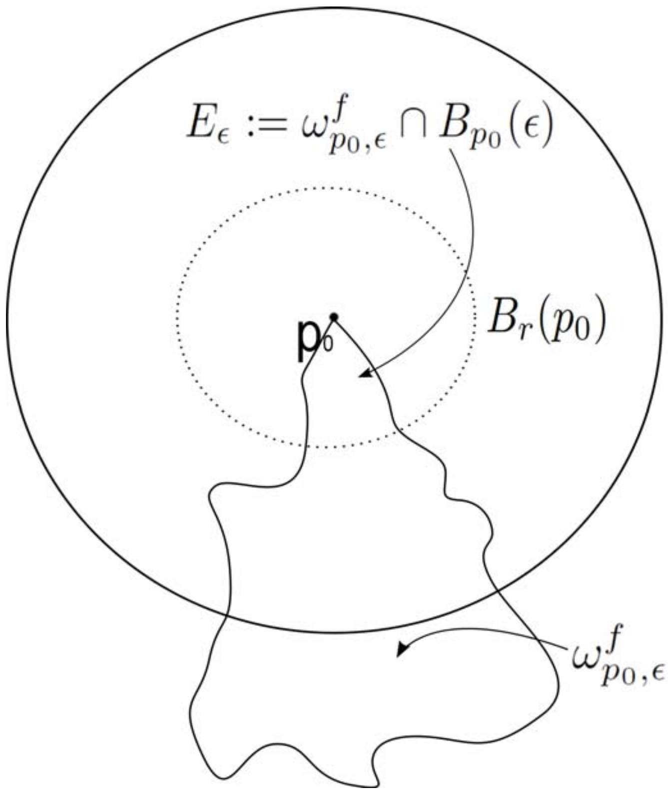

Figure 1: Part of the ϵ-approach region of f at p_0 belonging to $B_{p_0}(r)$ and the intersection with the depicted ball of radius ϵ shrinks towards p_0 as ϵ decreases. By definition of approach region each such intersection is open in M, and the figure depicts a case when there is only one connected component of the ϵ-approach region.

we obtain that for any sequence $\{z_j\}_{j\in\mathbb{N}}$ in U_{p_0} with $z_j \xrightarrow{j} p_0$, it holds that $f(z_j) \xrightarrow{j} f(p_0)$. This proves the implication (i)⇒(ii).

(ii)⇒(i).
By assumption there exists a open set $U_{p_0} \subset M$ with $p_0 \in \overline{U}_{p_0}$, satisfying that for any sequence $\{z_j\}_{j\in\mathbb{N}}$ in U_{p_0} with $z_j \xrightarrow{j} p_0$, it holds that $f(z_j) \xrightarrow{j} f(p_0)$. Let $\epsilon > 0$ be fix. If f is not approach compatible at p_0 then there exists a ball $B_{p_0}(r_0), 0 < r_0 < \epsilon$, such that $B_{p_0}(r_0) \cap \omega^f_{p_0,\epsilon} = \emptyset$. This implies that $\{|f(z) - f(p_0)| \geq \epsilon\} \cap B_{p_0}(r_0)$ is dense. But $B_{p_0}(r_0) \cap U_{p_0}$ is open and nonempty since $p_0 \in \overline{U}_{p_0}$, so we can find a sequence $\{z_j\}_{j\in\mathbb{N}}, z_j \in B_{p_0}(r_0) \cap U_{p_0}, z_j \to p_0$ such that $|f(z_j) - f(p_0)| \geq \epsilon, \forall j \in \mathbb{N}$, contradicting the assumed property of U_{p_0}. This completes the proof. □

Note that if in the proof it holds $p \in U_p$, that will imply that f is sequentially continuous at p and recall that for metric spaces continuity is equivalent to sequential continuity. Proposition 3.15 immediately yield the following corollary.

Corollary 3.16. *Let M be an open subset of some $\mathbb{R}^n$ (or $\mathbb{C}^n$) and let for any approach compatible function f on M, and $p_0 \in M$, U_{p_0} be as in Proposition 3.15 (i.e. U_{p_0} is associated to f). Let G be a self-map on the space of functions $M \to K$, which preserves convergence along a sequences within the approach region, in the following sense: if $f : M \to K$, is an approach compatible function and $\{z_j\}_{j\in\mathbb{N}}$ is a sequence in U_{p_0} converging to p_0, such that $\{f(z_j)\}_{j\in\mathbb{N}}$ converges (necessarily to $f(p_0)$), then $\{G(f)(z_j)\}_{j\in\mathbb{N}}$ converges to $G(f)(p_0)$. Then also $G(f)$ is approach compatible at p_0.*

Proof. For U_{p_0} as in Proposition 3.15, and $\{z_j\}_{j\in\mathbb{N}}$ a sequence in U_{p_0} with $z_j \overset{j}{\to} p_0$, we know that $f(z_j) \overset{j}{\to} f(p_0)$. This yields a sequence of numbers in K, $\{G(f)(z_j)\}_{j\in\mathbb{N}}$, where the particular preservation of convergence property for G gives that $G(f)(z_j) \to G(f)(p_0)$. Since this holds for any $\{z_j\}_{j\in\mathbb{N}}$ in U_{p_0} with $z_j \overset{j}{\to} p_0$, $G(f)$ satisfies the conditions of Proposition 3.15. $\square$

Observation 3.17. *Let $\{z_j\}_{j\in\mathbb{N}}$ be a sequence of complex numbers such that $z_j \to p_0$. Let f be a function such that $f(z_j) \to f(p_0)$. Let $\{\epsilon_j\}_{j\in\mathbb{N}}$ be a sequence of positive number $\epsilon_j \to 0$. For points z_j within $B_{z_j}(\epsilon_j) \cap \omega^f_{z_j,\epsilon_j}$ we have $|f(z_j) - f(p_0)| < \epsilon_j$. Thus because $z_j \to p_0$ we can always find $\{\epsilon_j\}_{j\in\mathbb{N}}$ such that, $|f(p_0)| - |f(z_j) - f(p_0)| \leq |f(z_j)| \leq |f(z_j) - f(p_0)| + |f(p_0)|$ implies $|f(z_j)| \to |f(p_0)|$. A consequence of this is that taking the absolute value ($f \mapsto |f|$) yields a map G which has the property of Corollary 3.16.*

3.3 An application in the presence of appropriate approximation

If M is an open subset of $\mathbb{R}^n$ (or $\mathbb{C}^n$) and $f \in C(M, K)$ can be locally represented on say $\omega \Subset M$ ($\Subset$ meaning relatively compact) represented as the pointwise limit of a sequence $\{P_j\}_{j\in\mathbb{N}}$, then if the moduli of the approximating elements are constant a.e. on ω, so is the modulus of the local limit. We consider the analogous statement for approach compatible f.

Proposition 3.18. *Let U be an open subset of $\mathbb{R}^n$ (or $\mathbb{C}^n$) and let f be an approach compatible function on U such that $|f|$ is bounded on U. Assume the nonnegative real-valued function $|f|$ is the L^1-limit of the nonnegative real-valued functions on $\mathbb{R}^n$ (or $\mathbb{C}^n$) $\{\varphi_j\}_{j\in\mathbb{N}}$, where each φ_j satisfies*

$\varphi_j(z) \equiv c_j, \forall z \in U$ *(c_j a constant $c_j \in \mathbb{R}_{\geq 0}, \forall j \in \mathbb{N}$). Then $|f| \equiv$constant pointwise on U.*

Proof. Set $F := |f|$. As soon as U has nonzero measure we have that $c := \lim_j c_j$ exists and is finite because F is locally integrable as a consequence of f being locally integrable (by definition). Furthermore $(F - c) = 0$ a.e. on U. Since f is bounded on U, Observation 3.17 yields that F is approach compatible on U (since f is). The set of approach compatible functions is closed under addition by constant functions, see Observation 3.6, thus $(F - c)$ is approach compatible so if $(F(p) - c) > 0$ then $(F - c) > 0$ on some approach region $\omega^F_{p,\epsilon}$, with ϵ sufficiently small, which by definition has nonzero Lebesgue measure so that would contradict vanishing a.e., hence we must have $p \in F^{-1}(c), \forall p \in U$. p being arbitrary in U implies $|f| \equiv c$ pointwise on U. □

Remark 3.19. *In our application of Proposition 3.18 we shall consider non-negative real-valued functions of the form $\{|P_j|\}_{j\in\mathbb{N}}$ where each P_j has constant modulus $|P_j(z)| \equiv c_j, \forall z \in U$ and where the P_j locally L^1-approximate f near some reference point.*

Proposition 3.18 can be combined with the following lemma. As before we let the field K be $\mathbb{R}$ or $\mathbb{C}$.

Lemma 3.20 (Ambient critical point). *Let* A *be a subspace of the C^2-functions on a possibly higher dimensional open superset $\tilde{M}$ (of some $\mathbb{R}^N$ or $\mathbb{C}^N$) of an open subset M (of some $\mathbb{R}^n$ or $\mathbb{C}^n$) whose restrictions to M obey the weak maximum principle*[1]*. Let $q \in M$, and let $P \in$* A *satisfy $d(ReP)_q = d(ImP)_q = 0$, where $q \in M$. Assume there exists a connected open $M-$neighborhood, V, of q such that $|P(z)| \leq |P(q)|, \forall z \in V$. Then either $|P|_M| \equiv$constant or there exists a point $q' \in V$ with $|P(q')| = |P(q)|, dP_{q'} \neq 0$, or $\exists q'' \in M$ with $|P(q'')| = |P(q)|, dP_{q''} = 0$ such that both Hess$(ReP)_{q''}$, Hess$(ImP)_{q''}$ are singular.*

Proof. If there is no Cauchy sequence $\{z_j\}_{j\geq 1}, z_j \in V$, such that $z_j \to q$, and such that $|P(z_j)| = |P(q)|, \forall j \geq 1$, then we can replace V by a smaller $M-$open subset containing q such that $|P(q)| > |P(z)|, \forall z \in V$. But the restriction of an element of A, cannot attain strict local maximum at any point of M. Thus we can assume that there is a Cauchy sequence $\{z_j\}_{j\geq 1}, z_j \in V$, such that $z_j \to q$, and such that $|P(z_j)| = |P(q)|$, i.e. q is an accumulation point for a sequence of points, all rendering weak local maximum for $|P|$ on V. We now

[1] By the weak maximum principle we mean that no element attains strict local maximum at any point.

use that a nondegenerate critical point is isolated from other critical points[2] (see e.g. Guillemin & Pollack [4], p.43). If $d(\mathrm{Re}P)_{z_j} = d(\mathrm{Re}P)_{z_j} = 0, \forall j \in \mathbb{N}$, q is not an isolated critical point for neither $\mathrm{Re}(P)$ nor $\mathrm{Im}(P)$. But if q cannot be a nondegenerate (since it is not isolated $z_j \to q$) critical point for $\mathrm{Re}(P)$ or $\mathrm{Im}(P)$ we have that that the real Hessians of $\mathrm{Re}(P)$ and $\mathrm{Im}(P)$, at q are singular, and we already know that $d(\mathrm{Re}P)_q = d(\mathrm{Re}P)_q = 0$ and q renders a weak local maximum. This completes the proof of the lemma. □

Note that if in Observation 3.20 A consists of holomorphic functions then for $P \in \mathrm{A}$, $dP_q = 0 \Rightarrow d(\mathrm{Re}P)_q = d(\mathrm{Re}P)_q = 0$, (where dP_q denotes the complex differential).

Corollary 3.21. *Let M be an open subset of $\mathbb{R}^n$ or $\mathbb{C}^n$ and* A *a subspace of the C^2-functions (on a possibly higher dimensional $\tilde{M}$), such that the restrictions to M obey the weak maximum principle. Assume ($P \in \mathrm{A}, P|_M \not\equiv$constant attains weak local maximum within M at $p \in M$)$\Rightarrow$ ($d(ReP)_q = d(ReP)_q = 0$ and at least one of Hess(ReP), Hess(ImP) is nonsingular at p). Then any approach compatible function, f, L^1-approximable on some $\omega \Subset M$, by restrictions of elements in* A, *satisfies that $|f|_\omega| \equiv$constant or it is locally L^1-approximable by elements of* A *which never attain weak maximum on M.*

Proof. If infinitely many elements in a local (on ω) approximating sequence $\{P_j\}_{j\in\mathbb{N}}$ attain weak local maximum on ω then by Lemma 3.20 applied to each they each have constant modulus. But they also constitute a subsequence again L^1-converging to f on ω thus by Proposition 3.18 $|f|_\omega| \equiv$constant. If only finitely many elements of $\{P_j\}_{j\in\mathbb{N}}$ attain weak local maximum we can start from a sufficiently large index so that the remaining elements never attain weak local maximum. □

Remark 3.22. *With regards to Corollary 3.21, note that already in the case when* A *consists of functions holomorphic near M it is certainly not true that restrictions of holomorphic functions in general obey the weak maximum principle, for example let $M \subset \mathbb{C}^3$ be a graph $M = \{(z,w) \in \mathbb{C} \times \mathbb{C}^2 : y = |w|^2\}, z := x + iy$. Then the restriction of the function (holomorphic near M) $\frac{1}{P}$, where, $P(z) = i + z = i(1+y) + x$, (note it is a holomorphic polynomial) attains strict local maximum at the origin since $|P(z,w)| = \left|i(1+|w|^2) + x\right| \geq 1 + (x^2 + 2|w|^2 + |w|^4)$.*

[2]This can be seen by setting $G = (\frac{\partial}{\partial x_1} g, \ldots, \frac{\partial}{\partial x_k} g)$, then $dg_x = 0 \Leftrightarrow G = 0$, and dG_x is nonsingular since it is the (real) Hessian of g thus G is a diffeomorphism to a neighborhood of the origin, i.e. $G = 0$ only at x for a small enough neighborhood of x.

Before we move on we make the following observation on the size of approach regions for differentiable functions. We shall need the following observation.

Observation 3.23. *If M is a domain in $\mathbb{R}^n$ (or $\mathbb{C}^n$), then for a given ball $B_0(R) \subset M, R > 0$, and fix $\epsilon > 0$, any differentiable function on M satisfies that the Lebesgue measure of the $\omega^f_{\epsilon,p} \cap B_p(\epsilon), p \in B_0(R)$, is bounded from below (as p varies) by a positive number. Namely on a relatively compact open set ω, both f and the modulus of its Jacobian are bounded by say $C < \infty$, where C is independent of p. Thus given p we have (by the mean value theorem) a Lipschitz property, $|f(p) - f(z)| \leq C\,|p - z|\,,\ z \in \omega$. Hence the inverse image $f^{-1}\left(y \in \mathbb{C} : \{|f(p) - y| < \epsilon\}\right)$ contains the ball $\{|p - z| < \epsilon/C\}$ (note that we are not implying the function is open, since $\{|f(p) - y| < \epsilon\}$ is not necessarily open) and $\{|p - z| < \epsilon/C\} \subseteq B_p(\epsilon)$, as soon as $C \geq 1$. For $p \in B_0(R)$ the intersection of the balls $\{|p - z| < \epsilon/C\} \cap B_p(\epsilon) \cap B_0(R)$ is nonempty and eventhough the measure can be smaller than the measure of each of the balls (e.g. when $B_p(\epsilon)$ intersects the complement of $B_0(R)$), C is independent of p so the Lebesgue measure of $\omega^f_{p,\epsilon} \cap B_p(\epsilon)$, $p \in B_0(R)$ is, for fix ϵ, bounded from below by a positive number (this is not trivial, in fact for the general case it is necessary that $\omega^f_{p,\epsilon}$ contains an open neighborhood of p, to ensure that a sufficiently large part does not leave $B_0(R)$ and even then without a Lipschitz type of condition we cannot guarantee that the size those parts is bounded from below). A consequence is the following: Let $\Omega \subset M$ be a domain and let $U \Subset \Omega$ be a relatively compact domain with smooth boundary and let $\{P_j\}_{j\in\mathbb{N}}$ be a sequence of C^1-functions on M such that the Jacobians $\{JacP_j\}_{j\in\mathbb{N}}$ is a uniformly bounded family on Ω. Then for any fix $\epsilon > 0$ the Lebesgue measure of $\omega^{P_j}_{p,\epsilon} \cap B_p(\epsilon)$, $p \in U$, is bounded from below by a positive number η_ϵ (independent of j). Note that it is vital that the boundary be at least C^1, indeed for $p \in U$ the ball $B_p(\epsilon)$ can in general intersect the complement of U in M.*

3.4 An asymptotic maximum principle

In this section we show an analogue of the second part of Example 1.1.

Proposition 3.24. *Let M be an open subset of $\mathbb{R}^n$ (or $\mathbb{C}^n$), let $\Omega \subset M$ be a domain and let f be approach compatible on M, which can be locally L^1-approximated by C^1-functions which obey the strong maximum principle on Ω and whose Jacobians are a uniformly bounded sequence on Ω . Then $\forall p_0 \in M, \exists$ a domain $p_0 \in U \subseteq \Omega$ such that for any domain $\omega \Subset U$ (where by $\Subset$ we mean relatively compact) with smooth boundary and $p_0 \in \omega$:*

$$(|f(p_0)| > |f(z)|, \forall z \in \omega \setminus \{p_0\}) \Rightarrow (dist(\{|f(p_0)| - \epsilon < |f(z)|\}, \partial\omega) < \epsilon, \forall \epsilon > 0)$$

Proof. Approach compatible functions are in particular locally bounded, thus there is a domain $U' \subset \Omega$ containing p_0 on which f is bounded. Also there is a domain $U'' \subset \Omega$ together with a sequence of C^1-functions $\{P_k(z)\}_{k\in\mathbb{N}}$ on M, and whose restrictions obey the strong maximum principle which L^1-converge on U'' to the function f and whose Jacobians are uniformly bounded family on Ω. If infinitely many of the P_k are constant then f can be L^1-approximated by constant functions, thus we can assume that only finitely many of the P_k are constant and after passing over to a subsequence (which we again denote $\{P_j\}_{j\in\mathbb{N}}$ we can assume that $P_j \not\equiv$constant on Ω, for any $j \in \mathbb{N}$. Set $U := U' \cap U''$, let $\omega \Subset U$ and let $p_0 \in \omega$.
First we prove the result for the case when ω is a ball centered at p_0. The proof is then easily adapted to the case of a relatively compact domain $\omega \Subset U$.
Let $\epsilon > 0$ be fix and let η_0 be half of the Lebesgue measure of the open set $\omega^f_{p_0,\frac{\epsilon}{3}} \cap \omega$. L^1-convergence implies that there is a subsequence $\{P_{j_k}\}_{k\in\mathbb{N}}$ with a.e. pointwise convergence to f on U (see e.g. Rudin [12], p.68). Let $r_\epsilon > 0$ be such that $B_{p_0}(r_\epsilon) \subset \omega$ and $\operatorname{dist}(B_{p_0}(r_\epsilon), \partial\omega) < \epsilon$. By Egoroff's theorem there exists a set A_{η_0} of Lebesgue measure $< \eta_0$ such that $P_{j_k} \to f$ uniformly on $U \setminus A_{\eta_0}$. In particular we have uniform convergence on the nonempty set $\omega^f_{p_0,\frac{\epsilon}{3}} \cap U \setminus A_{\eta_0}$. Let $z^\epsilon \in \left(\omega^f_{p_0,\frac{\epsilon}{3}} \cap \omega\right) \setminus A_{\eta_0}$, and let $\tilde{U}^k$ denote the connected component of $\{|P_{j_k}| > |P_{j_k}(z^\epsilon)|\}$ containing z^ϵ on the boundary. Since the restrictions of the P_{j_k} obey the strong maximum principle it must hold that each $\tilde{U}^k$ is open and furthermore that $\tilde{U}^k \cap \partial\omega \neq \emptyset$. Setting $U^k := \tilde{U}^k \cap \left(\omega \setminus \overline{B}_{p_0}(r_\epsilon)\right)$ implies that U^k is open connected nonempty (for finite k) and all points of U^k have distance $< \epsilon$ to $\partial\omega$. Let $N \in \mathbb{N}, N < \infty$ be such that (using uniform convergence on $U \setminus A_{\eta_0}$),

$$|P_{j_N}(z) - f(z)| \leq \frac{\epsilon}{3}, \forall z \in U \setminus A_{\eta_0}. \tag{10}$$

Now assume that there is a sufficiently large choice of integer N such that $U^N \setminus A_{\eta_0}$ contains at least one point w_N (see Figure 3.4), by construction $\operatorname{dist}(w_N, \partial\omega) < \epsilon$. Furthermore we have (because $w_N \in U^N$),

$$|P_{j_N}(w_N)| \geq |P_{j_N}(z^\epsilon)| \geq |f(p_0)| - \underbrace{|P_{j_N}(z^\epsilon) - f(p_0)|}_{\leq \left|P_{j_N}(z^\epsilon) - f(z^\epsilon)\right| + |f(z^\epsilon) - f(p_0)|} > |f(p_0)| - 2\frac{\epsilon}{3} \tag{11}$$

On the other hand we have noted that the inequality of Eqn.(10) holds on $U \setminus A_{\eta_0}$ which we are assuming contains w_N thus,

$$|P_{j_N}(w_N)| \leq |P_{j_N}(w_N) - f(w_N)| + |f(w_N)| < \frac{\epsilon}{3} + |f(w_N)|. \tag{12}$$

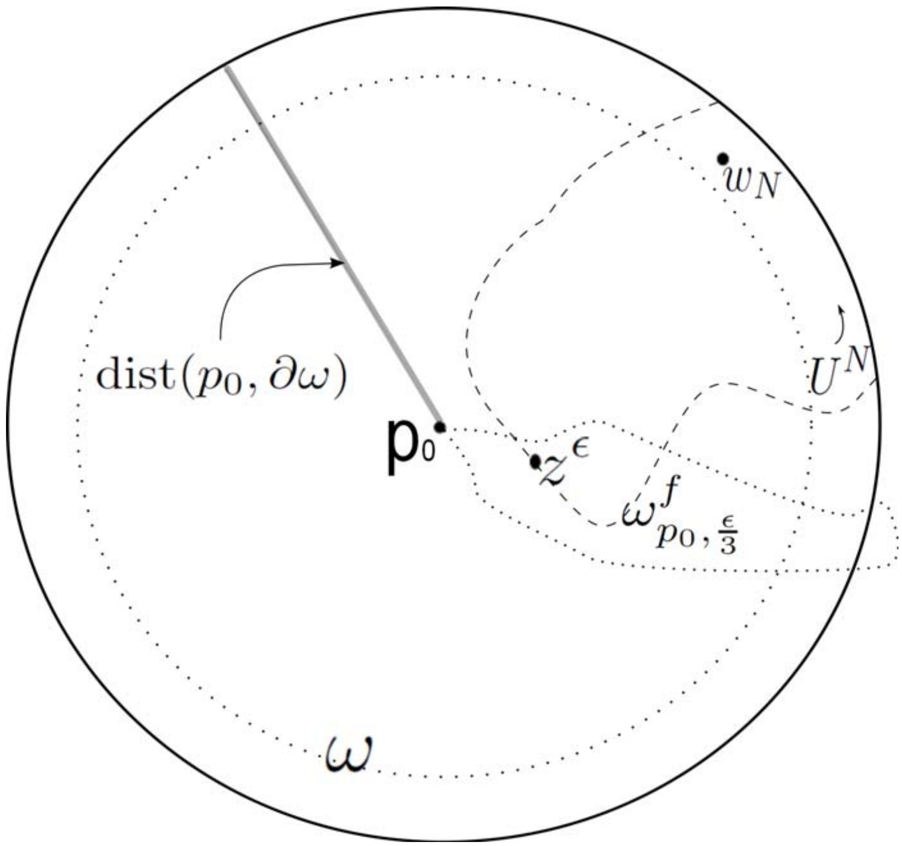

Figure 2: The choice of w_N in the proof of Proposition 3.24 has distance $< \epsilon$ to $\partial\omega$ and simultaneously lies in U^N. For simplicity the figure depicts a case when there is one and only one connected component of the approach region at p_0.

Eqn.(11) in combination with Eqn.(12) yield,

$$|f(w_N)| + \frac{\epsilon}{3} > |f(p_0)| - 2\frac{\epsilon}{3},$$

i.e. we have found a point $w_N \in \omega$ satisfying,

$$|f(w_N)| > |f(p_0)| - \epsilon, \mathrm{dist}(w_N, \partial\omega) < \epsilon. \quad (13)$$

This proves the the result for the case when there exists a sufficiently large N such that $U^N \setminus A_{\eta_0}$ contained at least one point w_N (and ω a ball centered at p_0).
Assume now that for the fixed ϵ, $U^N \subset A_{\eta_0}$ for all sufficiently large N. Let l_0 be half of the Lebesgue measure of the open set $\omega^f_{p_0,\frac{\epsilon}{4}} \cap \omega$. Since the $\{P_{j_k}\}_{k\in\mathbb{N}}$ had Jacobians which formed a uniformly bounded sequence, we have by Observation 3.23 that the Lebesgue measure of the $\omega^{P_{j_k}}_{z,\frac{\epsilon}{4}} \cap B_z(\frac{\epsilon}{4})$, $z \in \omega$, are

bounded from below by a positive number, say $\kappa_{\frac{\epsilon}{4}}$, independent of k. Set η_1 =min$\{l_0, \kappa_{\frac{\epsilon}{4}}\}$. We know that the Lebesgue measure of A_{η_1} is $< \eta_1 \leq l_0$. We can choose an integer N' such that,

$$\left|P_{j_{N'}}(z) - f(z)\right| < \frac{\epsilon}{4}, \forall z \in U \setminus A_{\eta_1}. \tag{14}$$

Let $\zeta^\epsilon \in \omega^f_{p_0,\frac{\epsilon}{4}} \setminus A_{\eta_1}$, and let $\tilde{V}^k$ denote the connected component of $\{|P_{j_k}| > |P_{j_k}(\zeta^\epsilon)|\}$ with ζ^ϵ on the boundary and $V^k := \tilde{V}^k \cap \left(\omega \setminus \overline{B}_{p_0}(r_\epsilon)\right)$. By the choice of η_1, there is a point $w_{N'} \in V^{N'} \cap A_{\eta_1} \cap \left\{\frac{7\epsilon}{16} < \mathrm{dist}(z, \partial\omega) < \frac{9\epsilon}{16}\right\}$, such that the set $\omega^{P_{j_{N'}}}_{w_{N'},\frac{\epsilon}{4}} \cap B_{w_{N'}}\left(\frac{\epsilon}{4}\right) \setminus \overline{B_{p_0}(\epsilon)}$ has Lebesgue measure $\geq \kappa_{\frac{\epsilon}{4}}$, so has nonempty intersection with $A^c_{\eta_1}$ (^{c} denoting complement). We can thus find $v_{N'} \in \omega^{P_{j_{N'}}}_{w_{N'},\frac{\epsilon}{4}} \setminus A_{\eta_1}$ with $\mathrm{dist}(v_{N'}, \partial\omega) < \epsilon$. This gives,

$$\left|P_{j_{N'}}(v_{N'})\right| \geq \left|P_{j_{N'}}(w_{N'})\right| - \left|P_{j_{N'}}(w_{N'}) - P_{j_{N'}}(v_{N'})\right| > \left|P_{j_{N'}}(w_{N'})\right| - \frac{\epsilon}{4}. \tag{15}$$

We have by Eqn.(14) that $\left|P_{j_{N'}}(v_{N'}) - f(v_{N'})\right| < \frac{\epsilon}{4}$ and $\left|P_{j_{N'}}(\zeta^\epsilon)\right| > |f(\zeta^\epsilon)| - \left|f(\zeta^\epsilon) - P_{j_{N'}}(\zeta^\epsilon)\right| > |f(\zeta^\epsilon)| - \frac{\epsilon}{4}$. By the choice of $V^{N'}$ and $v_{N'}$ we know that,

$$\left|P_{j_{N'}}(w_{N'})\right| > \left|P_{j_{N'}}(\zeta^\epsilon)\right| \text{ and } |f(v_{N'})| \geq \left|P_{j_{N'}}(v_{N'})\right| - \frac{\epsilon}{4}. \tag{16}$$

Also it is clear that $|f(\zeta^\epsilon)| \geq |f(p_0)| - |f(\zeta^\epsilon) - f(p_0)| > |f(p_0)| - \frac{\epsilon}{4}$. This in combination with Eqn.(16) and Eqn.(15) yields,

$$|f(v_{N'})| > |f(p_0)| - \epsilon. \tag{17}$$

Since $\mathrm{dist}(v_{N'}, \partial\omega) < \epsilon$, this proves the proposition for the case when ω is a ball centered at p_0. The proof can now be repeated for the case of a relatively compact domain $\omega \Subset U$ containing p_0 by replacing the sets $B_{p_0}(r_\epsilon) \subset \omega$ which satisfied $\mathrm{dist}(B_{p_0}(r_\epsilon), \partial\omega) < \epsilon$, with domains $\omega_\epsilon \subset \omega$ such that,

$$\max_{z \in \partial\omega_\epsilon} \mathrm{dist}(z, \partial\omega) < \epsilon. \tag{18}$$

Such ω_ϵ always exists, namely we can cover $\partial\omega$ by $\bigcup_{z\in\partial\omega} B_z(\frac{\epsilon}{2})$. Since $\partial\omega$ is compact we can find a finite subcover $\{B_{z_j}(\frac{\epsilon}{2})\}_{1\leq j<R}$, some $R < \infty$, of $\partial\omega$. Thus we can set $\omega_\epsilon := \omega \setminus \bigcup_{1\leq j<R} B_{z_j}(\frac{\epsilon}{2})$. This completes the proof. □

The restriction of a holomorphic function a smooth hypersurface $M \subset \mathbb{C}^n$ which is convex-concave at every point, satisfies the strong maximum principle (i.e. a weak local maximum implies reduction to a constant). By this is meant the following: Assume M has local defining function $\rho \in C^\infty(U, V)$, for some domains $U \subset \mathbb{C}^n$ and $V \subset \mathbb{R}$, $M \cap U = \{\rho = 0\}, d\rho_z \neq 0, z \in U$. If the matrix $\left[\frac{\partial^2\rho}{\partial z_j \partial \bar{z}_k}(p)\right]_{jk}$ has at least one positive and one negative eigenvalue then M is called convex-concave at p. This immediately gives the following corollary.

Corollary 3.25. *If $M \subset \mathbb{C}^n$ is a smooth hypersurface which is convex-concave at each point of an open $U \Subset M$. If f is an approach compatible function which is locally L^1-approximable by restrictions of entire functions, then f satisfies the asymptotic maximum principle of Proposition 3.24 on U.*

Finding examples of approach compatible functions locally L^1-approximable by entire functions can under some circumstances be easy indeed for real smooth embedded submanifolds it sufficient (see Hounie & Malagutti [5]) that the function be L^1_{loc} and annihilated by the so called tangential CR vector fields in the weak sense, and for hypersurfaces in $\mathbb{C}^n$ of the form $\{\text{Im} z_n = 0\}$, these vector fields simply reduce to $\frac{\partial}{\partial \bar{z}_j}, j = 1, \dots, n-1$.

Example 3.26. *Define the smooth hypersurface,* $M := \{(z_1, z_2) \in \mathbb{C}^2 : y_2 = 0\}, z_2 := x_2 + iy_2$. *Then the function,* $f = x_2 \cdot \chi(\{|x_2| < 1\}) \cdot e^{z_1}$, $f = 0$, *otherwise, where* χ *denotes the characteristic function, is an* L^1_{loc}*-function locally* L^1*-approximable by entire functions. This example is clearly approach compatible, noncontinuous on* $\{|x_2| = 1\}$, *and also not open, since for any ambient open neighborhood,* U_{p_0}, *of* $p_0 \in \{|x_2| = 1\}$, *the image of* f *will contain* 0, *as an isolated point.*

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