# A New Approach for Computing Multi-Fractal Dimension Based on Escape Time Method

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## Abstract

The multifractal decomposition behaves as expected for family of sets known as digraph recursive set. Fractal dimension is a measure of how fragmented a fractal object is. In fractal geometry, there are various approaches to compute the fractal dimension of an object. In this paper we propose an improved approach to compute the multi fractal dimension based on escape time method. This new approach depends on spreading points inside a specific window. The resulted values of some multifractals for the proposed method is coincided with the computer program values that was made with Delphi for the graph of self similarity with multi nodes (or Mauldin-Williams Graph) .

## 1- Introduction

Many objects in nature are very complex and irregular, therefore their description with traditional methods is insufficient. One possible approach is the fractals geometry. There are several concepts to classify a structure by dimension analysis. It is a tool to quantify structural  information of artificial and natural objects. There are a lot of methods of fractal dimension counting, but the most familiar is the box-counting method [3]. Fractal geometry is a useful way to describe and characterize complex shapes and surfaces. The idea of fractal

geometry was originally derived by Mandelbrot in 1967 to describe self-similar geometric figures such as the Von Koch curve. Fractal dimension $D$ is a key quantity in fractal geometry. The $D$ value can be a non-integer and can be used as an indicator of the complexity of the curves and surfaces. For a self-similar figure, it can be decomposed into $N$ small parts, where each part is a reduced copy of the original figure by a ratio $r$. The $D$ of a self-similar figure can be defined as $D=-\log(N)/\log(r)$. For curves and images that are not self-similar, there exist numerous empirical methods to compute $D$. Results from different estimators were found to differ from each other. This is partly due to the elusive definition of $D$, i.e., the Hausdorff–Besicovitch dimension [8].

A multifractal system is a generalization of a fractal system in which a single exponent (the fractal dimension) is not enough to describe its dynamics. Multifractal decomposition has been investigated by many authors, it has become a useful tool in the fractal analysis. Edgar and Mauldin studied the digraph multifractals [4], Falconer [10] randomized Cawley's results, and the latest results on random self-similar multifractals were done by Arbeiter and Patzschke under rather weak conditions [5]. Multifractal analysis was first introduced by physicists around 1980's, while studying the velocity and the energy dissipation of incompressible turbulence flow. It is consists in computing the Hausdorff dimension of the level sets of points collected according to the value of some point-wise Holder exponent [7]. A new multi fractal space is defined by Al-Shameri [16]. This space was generated by multiplying of a finite number of fractal spaces. Some generalization of the concepts in this space is considered by Al-Saidi [13].

Fractal geometry has various approaches to compute the fractal dimension of an object. These approaches can be classified as belonging to the Hausdorff-Besicovitch Dimension (like the Box-Counting and Dividers methods) or to the Bouligand-Minkowski Dimension (Minkowski fractal dimension method). Arkan [1] in her thesis was proposed a new method to find the dimension of some fractals based on escape time principle, using the method of spreading of the points inside a specific window. In this paper, this method is modified to be applied for multifractals, this proposed method is carried on for some graph directed IFS and compared to a computer program results for the well known theorem given in Edgar [5] for the graph of self similarity with multi nodes (or Mauldin-Williams graph). The program is made with Delphi and it can be used for analysis of different fractals.

The paper is organized as follows. In the next section, some background materials are included to assist readers less familiar with the detailed to consider. Section 3 deals with the definition of the multi fractal space which behaves as expected for family of sets known as digraph recursive fractal, in addition to the notion for graph directed IFS. In Section 4 the proposed method to compute the dimension of some multifractals are presented with the algorithm for computing fractal dimension based on Edgar theorem. Finally conclusions are drawn in Section 5.

## 2- Theoretical Background

This section presents an overview of the major concepts and results of fractals dimension and the Iterated Function System (IFS). A more detailed review of the topics in this section are as in [2,5,12]. Fractal is defined as the attractor of mutually recursive function called Iterated Function System (IFS). Such systems consist of sets of equations, which represent a rotation, translation, and scaling.

Given a metric space $(X; d)$, we can construct a new metric space $(H(X); h)$, where $H(X)$ is the collection of nonempty compact set of $X$, and $h$ is the Hausdorff metric on $H(X)$ defined by:

$$h(A,B) = \max\{\inf\{\varepsilon > 0; B \subset N_\varepsilon(A)\}, \inf\{\varepsilon > 0; A \subset N_\varepsilon(B)\}\}$$

**Definition 1** [10]. For any two metric spaces $(X, d_X)$ and $(Y, d_Y)$, a transformation $w: X \rightarrow Y$ is said to be a contraction if and only if there exists a real number $s$, $0 < s < 1$, such that $d_Y(w(x_i), w(x_j)) \leq s d_X(x_i, x_j)$, for any $x_i, x_j \in X$, where $s$ is the *contractivity factor* for $w$.

**Theorem 1** *contraction mapping theorem* [5]. Let $w: X \rightarrow X$ be a contraction on a complete metric space $(X, d)$. Then, there exists a unique point $x_f \in X$ such that $w(x_f) = x_f$. Furthermore, for any $x \in X$, we have $\lim\limits_{n \to \infty} w^{\circ n}(x) = x_f$, where $w^{\circ n}$ denotes the n-fold composition of $w$.

**Definition 2** [12]. A hyperbolic IFS $\{X; w_1, w_2, \ldots, w_n\}$ consists of a complete metric space $(X, d)$ and a finite set of contractive transformations $w_n: X \rightarrow X$ with contractivity factors $s_n$, for $n = 1, \ldots, N$. The contractivity factor for the IFS is the maximum $s$ among $\{s_1, \ldots, s_N\}$. The *attractor* of the IFS is the unique fixed point in $H(X)$ of the transformation $W: H(X) \rightarrow H(X)$, defined by $A = W(A) = \bigcup_{i=1}^{N} w_i(B)$ for any $B \in H(X)$.

**Theorem 2** *(The Box-Counting Theorem)* [1]. Let $A \in H(R^m)$, where the Euclidean metric is used. Cover $R^m$ by closed square boxes of side length $(1/2^n)$. Let $N_n(A)$ denote the number of boxes of side length $(1/2^n)$ which intersect the attractor. If

$$D_B(A) = \lim_{n \to \infty}\left\{\frac{\ln(N_n(A))}{\ln(2^n)}\right\} \text{ then } A \text{ has Box dimension } D, \text{ where } D(A) = D_B(A).$$

**Definition 3** [12]. A *dynamical system* is a transformation $f: X \longrightarrow X$ on a metric space $(X, d)$. It is denoted by $\{X; f\}$. The *orbit* of a point $x \in X$ is the sequence $\{f^{\circ n}(x)\}_{n=0}^{\infty}$.

**Definition 4** [12]. Let $\{X; f_n, n = 1, 2, \ldots, N\}$ be totally disconnected IFS with attractor $A$. The associated shift transformation on $A$ is the transformation $S: A$

$\longrightarrow$ $A$, defined by, $S(a) = f_n^{-1}(a)$ for $a \in f_n(A)$, where $f_n$ is viewed as transformation on $A$. The dynamical system $\{A; S\}$ is called the *shift dynamical* system associated with the IFS.

## 3- Multifractals and the Graph Directed IFS

The goal of this section is to correlate the theory of classical fractal geometry to the theory of graph directed constructions. The use of fractal and multifractal geometry today spans from physics, through economy, biology, medicine, to computer science; among many other areas [15]. Fractals are considered as self-similar objects. This feature is a key issue in fractals. It implies that the object looks similar to its zoomed part. The theory of self-similar sets was formalized by John Hutchinson [9] in the 1981 paper, but popularized by Michael Barnsley in his 1993 book [12]. The iterated function system approach used to create self-similar sets was extended to graph-directed constructs by Mauldin and Williams in a 1988 paper [14]. So that the self similar sets is a special case of a much larger class of sets-graph directed constructions. To construct the multifractal space, the next subsection consider some important concepts, and for more details we can see [13,16].

### 3.1 Multi Fractal Space

Let $N$ be a set, assume that for each $i \in N$ we have a metric space $(X_i, d_i)$. The product space $X = \prod_{i \in N} X_i$ is the space that consists of all $N$-tuples $(X_i)_{i \in N}$ with $x_i \in X_i$ define $d: X \times X \to R$ as follows: If $X = (x_1, x_2, \ldots x_N)$ and, $Y = (y_1, y_2, \ldots y_N)$ are elements of $\chi$, then $d(X, Y) = max_{i=1,\ldots,N} d_i(x_i, y_i)$. It is easy to verify that $d$ satisfies the requirements to be a distance function, if $d$ represents the product metric in $\chi$ then $(\chi, d)$ is a metric space.

Let $MH(X)$ denotes the multifractal space or multi Hausdorff space that is defined as $MH(X) = H(X_1) \times H(X_2) \times \ldots \times H(X_N) = \prod_{i=1}^{N} H(X_i)$ then $MH(X)$ is a Hausdorff space (*since the product of Hausdorff spaces is a Hausdorff space*). If $A = (A_1, A_2, \ldots, A_N)$ and $B = (B_1, B_2, \ldots, B_N)$ are element of $MH(X)$. Then $D(A, B)$ that is defined as $D(A, B) = max_{1 \leq i \leq N} d(A_i, B_i)$, satisfies the requirements to be a distance function where $A_i$ *and* $B_i$ are an n-tuple of elements of $H(X_i)$, which follows that $(MH(X), D)$ is a metric space, and moreover it is a complete metric space [13].

***Example*** *(Golden Rectangle Fractal)*

The concept of multifractal space given in [13,16] can be illustrated by the following example with two complete metric spaces represented by a square *S* and a rectangle *R*. Its structure is such that there is a transformation which maps the large square and rectangle into each other and into themselves, as shown in the Figure 1.

Since the two spaces *S* and *R* are complete metric space then there is a contractive transformation $F:S \to S$, where $F = \bigcup_{i=1}^{n} f_i$, and each $f_i$ is a contractive transformation defined on the space *S* with contractivity factor $s_i$, also there is a transformation $G:R \to R$ such that $G = \bigcup_{i=1}^{n} g_i$, where $g_i$ is contractive transformation on the space *R* with contractivity factor $r_i$.

Using the two spaces *S* and *R*, a new space $S \times R$ is constructed, and a new transformation $\mathcal{W}: S \times R \to S \times R$ is defined such that; $\mathcal{W} = (W_S, W_R)$, where

$W_S = f_1(S) \cup g_1(R) \cup g_2(R) \cup g_3(R)$, and $W_R = f_2(S) \cup g_4(R)$. Hence,

$\mathcal{W} = (W_S, W_R) = (f_1(S) \cup g_1(R) \cup g_2(R) \cup g_3(R), f_2(S) \cup g_4(R))$

By using iteration on these functions, we have:

$W_s^n = f_1(S^{n-1}) \cup g_1(R^{n-1}) \cup g_2(R^{n-1}) \cup g_3(R^{n-1})$.

$W_s^n = f_2(S^{n-1}) \cup g_4(R^{n-1})$.

After the second iteration Figure 1(b) is obtained, while Figure 1(c) is obtained after the third iteration.

Finally the sequence $\{S_n\}$ converges in the Hausdorff metric space to a non-empty compact set *S*, and the sequence $\{R_n\}$ converges to a non-empty compact set *R*. This pair is the required point, which means that from the contractivity of each map $f_i$ and $g_i$ we can prove that $\mathcal{W}$ is a contractive operator and by contraction mapping theorem, $\mathcal{W}$ has a unique fixed point $A = (S, R)$ in $S \times R$ satisfying the condition $A = \mathcal{W}(A)$, where $A = (A_S, A_R)$, and $\mathcal{W} = (W_S(A), W_R(A))$. This fixed point can be obtained after *n*-iteration as illustrated in Figure 1 (e).
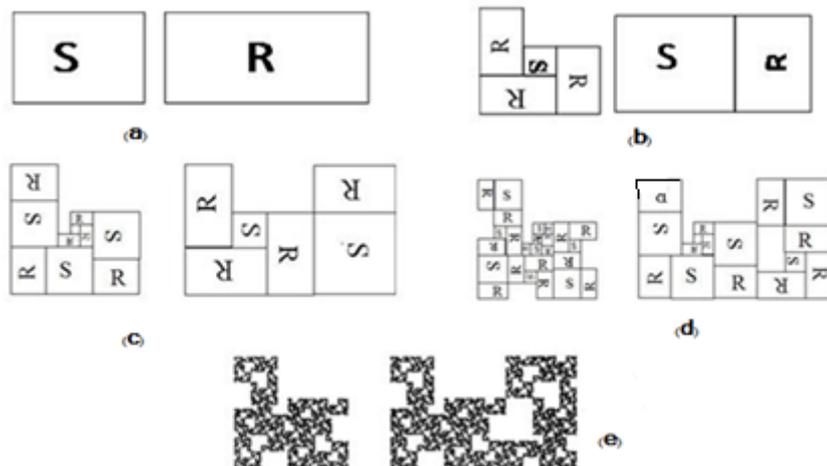


Figure 1. Golden rectangle fractal

**3.2 Graph Directed Iterated Function Systems**

The self-similar sets that have been described earlier is actually just a special case of a much broader class of sets, the graph-directed constructions. The graph-directed constructions that are described in this section are created by a recurrent scheme. It is create fractal sets by composition of different shaped sets. There is a generalization to self-similar sets that allowed to study the dimension of a much larger class with a finite set of vertices and directed edges. By allowing more than one edge between vertices, self-similar sets can be described. Each node corresponds to a subset, and the weight of the edge corresponds to a similarity ratio .

Given such a graph $G=(V,E)$, where $V$ is the set of vertices, and $E$ is the set of edges. An IFS realizing the graph is a set made up as follows: each vertex, $v \in V$, corresponds to a compact metric space, $X_i$, and each edge, $e \in E$, corresponds to similarities $S_e$, with similarity ratio $w(e)$ such that $\mathcal{F}_e : R^n \to R^n$. The invariant set for such IFS is $\{X_i = \cup \{\mathcal{F}_{ij}(X_j): (i, j) \in E\}$, and the construction object is defined by $F = \bigcup_{i=1}^n X_i$.  We assume that all similarity mappings are disjoint. The set of similarities, $\{\mathcal{F}e : e \in E\}$, is called a *graph-directed iterated function system*, and the sets $\{X_1,...,X_n\}$ are called graph-directed sets [15].

There is another representation to the IFS and self similar set analyzed by Mauldin and Williams (MW), uses directed graph, is called *Mauldin-Williams graphs.*  For more details we can see [5,14].

A Mauldin–Williams Graph is a directed multi-graph $(V,E)$ together with a positive number $r_e$ for each edge $e \in E$. The system $(r_e)$ is strictly contracting if and only if $r_e < 1$ for all $e$ [6]. The directed graph for the previous  example is illustrated in  Figure 2, and its dimension is calculated using the methods proposed in the next section.
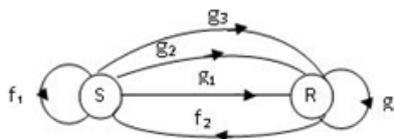


Figure 2. Directed graph for the golden rectangle fractal.

## 4-  Dimension for Multifractal as Graph directed IFS

From Euclid works, we know that the dimension of a point is considered 0, the dimension of a line is 1, the dimension of a square is 2, and that the dimension of a cube is 3. Roughly, we can say that the dimension of a set describes how much space the set fills. We will build the theory of fractal dimension from the basic Euclidean definition to the more mathematically exhaustive definitions of Hausdorff and Box-counting dimensions. One might ask why there are several different definitions of dimension. This is simply because a certain definition might be useful for one purpose, but not for another according to its properties,

and its performance and efficiency parameters. In many cases the definitions are equivalent, but when they are not, it is their particular properties that makes them more suitable for the task at hand [15].

In the next section the escape time algorithm for computing attractors of IFS given in [5], is modified to the escape time algorithm for computing multifractal attractors of the IFS.

### 4.1 The ETA for multifractal attractor constructed by IFS

The Escape Time Algorithm can be applied to any dynamical system of the form $\{R^2;f\}$ or $\{C;f\}$. We need only to specify a viewing window $W$ and a region $V$, in which orbits of points in $W$ might escape. The result will be a "picture" of $W$, wherein the pixel corresponding to the point $z$ is colored according to the smallest value of the positive integer $n$, such that $f^{\circ n}(z) \in V$. A special color, such as black, may be reserved to represent points whose orbits does not reaches $V$ before $(n+1)$ iterations.

The relationship between the dynamical system $\{R^2;f\}$ and the IFS=$\{R^2;$ $w_1, w_2, ..., w_n\}$ is the shift dynamical system $\{A,f\}$ associated with the IFS, where $A$ is the multifractal attractor constructed by this IFS.

The **Escape Time Algorithm (ETA)** given in [12] is illustrated as follows:

### The Algorithm.

1- Given $W \subseteq R^2$, s.t. $W=\{(x,y): a \leq x \leq c, b \leq y \leq d\}$, the array of points in $W$ is defined by, $x_{p,q}=(a+p(c-a)/M , b+q(d-b)/M)$, $p,q=1,2,...,M$, for any +ve integer $M$.

2- Let $\mathcal{C}$ be a circle centered at the origin, the set $\mathcal{V}$ is defined such that,
$$\mathcal{V} = \{(x, y) \in R^2: x^2 + y^2 > r\},$$ where $r$ is sufficiently large number.

3- Let $f$ be a complex function with the orbit of point $\{f^{\circ n}(x_{p,q}\}_{n=0}^{\infty}$, where $x_{p,q} \in W$.

4- Repeat, $\forall\ x_{p,q} \in W$.
IF $\{f^{\circ n}(x_{p,q}\}_{n=0}^{\infty} \subseteq \mathcal{V}$, then $x_{p,q}$ is colored with the color indexed by $n$.
Else it is colored Black.
End IF.

5- Change all that is not black color to the white color, such that:
$$x_{p,q=}\begin{cases} White & if\ f^{\,n}(x_{p,q}) \in V\ for\ some\ n \leq N \\ Black & if\ f^{\,n}(x_{p,q}) \notin V\ for\ all\ n \leq N \end{cases}$$

6- Then the set of escape time point $A$ in $W$ is defined as follows,
$A= \{x_{p,q} \in W: f^{\partial n}(x_{p,q}) \notin \mathcal{V}$, for all $n \leq N\}$
$= \{x_{p,q} \in W$, such that $x_{p,q}$ is black point$\}$.

**Output:** The set $A$ which represent the multifractal attractor constructed by ETA.

### 4.2 Escape Time Dimension for multifractal attractors

New approach for computing multifractal dimension based on escape time algorithm is proposed. In this method, a separate scanning to a point inside the window $\mathcal{W}$ is used, starting from the center of the window and separate this point to the whole region $\mathcal{W}$. Let $f : R^2 \rightarrow R^2$ be a multifractal attractor constructed by ETA. Now to find the dimension $D$ for multifractals attractor, the following algorithm which is known as **Escape Time Dimension (ETD)** algorithm is applied.

### The Algorithm

**For** $n$=1 to $N$ do.

  $\varepsilon_n$=1/2$^n$, choose $\mathcal{W}$={(x,y): -2$\leq x \leq$2, -2$\leq y \leq$2}.

- Using double binary tree method in the Euclidian plane to find the coordinates of the points in the window $\mathcal{W}$, such that each binary tree will form a coordinate in one dimension by taking $\varepsilon_n$ =1/2$^n$, $n$ = 0, 1, … $N$, as given in Figure 3.

- The points (x,y) inside the windows $\mathcal{W}$ is computed as follows,

  M=2n, K=2*M-1

  **For** I=-K to K step 2 **Do**.

    **For** J=-K to K step 2 **Do**.

      x=I/M, y=J/M

  **Next** J,I.

  These points forms new Window $\mathcal{W}_0 \subseteq \mathcal{W}$. We have to find the points in this windows with black color that form a new set $B$, where

  $B$= {$x_{p,q} \in \mathcal{W}_0$: $f^{[n]}(x_{p,q}) \notin \mathcal{V}$, for all $n \leq N$}, such that $B \subseteq A$ , $\mathcal{W}_0 \subseteq \mathcal{W}$.

- Let $N_n(A)$ be the number of points inside $\mathcal{W}$ depend on ε, where $A \subseteq MH(X)$.

- The number of black points $N_n(B)$, $\forall$ $B \subseteq \mathcal{W}_0$, and $x_{p,q} \in \mathcal{W}_0$, is computed as follows,

    **For** I=1,…,$n$ Do.

      **IF** $x_{p,q} \notin \mathcal{V}$, **Then** $N_n(B)$= $N_n(B)$+1

      **End IF**.

    **Next** I.

    Calculate $D_{E_n} = \dfrac{\ln(\mathcal{N}_n(B))}{\ln(2^n)}$ ,

  **Next** n.

This approximate value depends on chosen some tolerance (tol) (i.e if $|D_{E_n}(B) - D_{E_{n-1}}(B)|$ <tol, then $D(B) = D_{E_n}(B)$.

Hence $D$=lim$_{n \to \infty} D_{E_n}(A)$.

When ($N \longrightarrow \infty$), $\bar{B} = A$ ($B$ is dense in $A$). Therefore dimension of $B \cong$ dimension of $A$, and by box counting theorem,

$D(A) = \lim_{n \to \infty} \dfrac{\ln(\mathcal{N}_n(A))}{\ln(2^n)}$ . Hence: $D_{E_n}(B) = \dfrac{\ln(\mathcal{N}_n(B))}{\ln(2^n)}$ ,
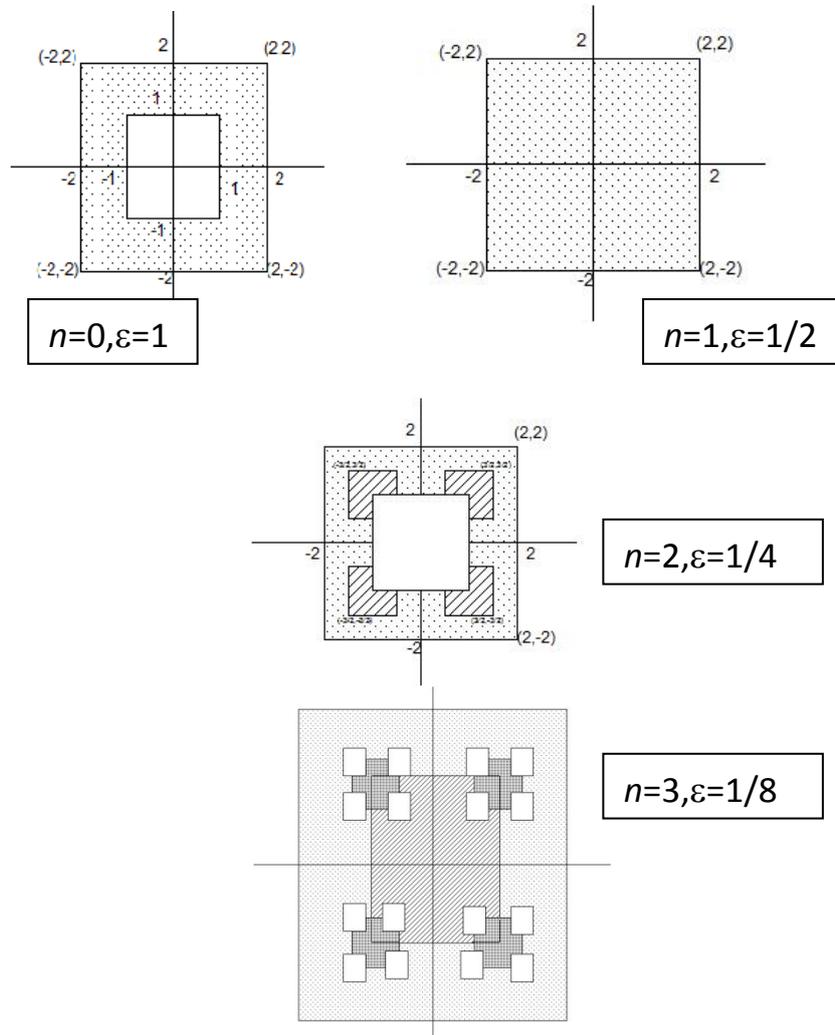
**Output:** $D(B)$=$D(A)$.

Figure 3. The coordinate of the points in the window $w$

In the following example, this algorithm have been applied to the golden rectangle fractal as a multifractal contracted by IFS.

**Example 2:** Consider the IFS = $\{R^2; \omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6 \}$, where

$$\omega_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \varphi^{-3}\cos(\frac{\Pi}{2}) & -\varphi^{-3}\sin(\frac{\Pi}{2}) \\ \varphi^{-3}\sin(\frac{\Pi}{2}) & \varphi^{-3}\cos(\frac{\Pi}{2}) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \varphi^{-1} \\ \varphi^{-2} \end{pmatrix}$$

$$\omega_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \varphi^{-2}\cos(\frac{\Pi}{2}) & -\varphi^{-2}\sin(\frac{\Pi}{2}) \\ \varphi^{-2}\sin(\frac{\Pi}{2}) & \varphi^{-2}\cos(\frac{\Pi}{2}) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \varphi^{-2} \\ \varphi^{-2} \end{pmatrix}$$

$$\omega_3\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\varphi^{-2}\cos(-\Pi) & -\varphi^{-2}\sin(-\Pi)\\ \varphi^{-2}\sin(-\Pi) & \varphi^{-2}\cos(-\Pi)\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}\varphi^{-1}\\ \varphi^{-2}\end{pmatrix}$$

$$\omega_4\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\varphi^{-2}\cos(-\frac{\Pi}{2}) & -\varphi^{-2}\sin(-\frac{\Pi}{2})\\ \varphi^{-2}\sin(-\frac{\Pi}{2}) & \varphi^{-2}\cos(-\frac{\Pi}{2})\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}\varphi^{-1}\\ \varphi^{-1}\end{pmatrix}$$

$$\omega_5\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}0\\ 0\end{pmatrix}, \qquad \omega_6\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\varphi^{-1}\cos(\frac{\Pi}{2}) & -\varphi^{-1}\sin(\frac{\Pi}{2})\\ \varphi^{-1}\sin(\frac{\Pi}{2}) & \varphi^{-1}\cos(\frac{\Pi}{2})\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}\varphi\\ 0\end{pmatrix}$$

The multifractal attractor of the IFS is the golden rectangle fractal *G* given in Figure 1. The relationship between the dynamical system $\{R^2; f\}$, and multifractal attractor IFS $\{R^2; \omega_1,\omega_2,\omega_3,\omega_4,\omega_5,\omega_6\}$ is a shift dynamical system $\{G, f\}$ associated with the IFS.

Calculate the inverse transformations for the above system, which are as follows.

$$\text{Define } f\colon R^2 \longrightarrow R^2, \text{ by, } f\begin{pmatrix}x\\y\end{pmatrix}=\begin{cases}\omega_1^{-1}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0 & \varphi^3\\ -\varphi^3 & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}-\varphi\\ \varphi^2\end{pmatrix},\\[4pt] \omega_2^{-1}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0 & \varphi^2\\ -\varphi^2 & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}-1\\ 1\end{pmatrix},\\[4pt] \omega_3^{-1}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-\varphi^2 & 0\\ 0 & -\varphi^2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}\varphi\\ 1\end{pmatrix},\\[4pt] \omega_4^{-1}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0 & -\varphi^2\\ \varphi^2 & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}\varphi\\ -\varphi\end{pmatrix},\\[4pt] \omega_5^{-1}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}0\\ 0\end{pmatrix},\\[4pt] \omega_6^{-1}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0 & \varphi\\ -\varphi & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}0\\ \varphi^2\end{pmatrix}\end{cases}$$

Now apply the ETA for the two sets $\mathcal{V}$ and $\mathcal{W}$ chosen appropriately. The points whose numerical orbits require sufficiently much iteration before they reach $\mathcal{V}$, are plotted with white color, and the points whose orbit does not reach $\mathcal{V}$, are plotted with black color. This algorithm is used to find the attractor,

$G= \{x_{p,q} \in \mathcal{W}_0\colon (f^{\circ n} x_{p,q}) \notin \mathcal{V}, \text{ for all } n \le N\}$,

With the following calculation the modified escape time algorithm for computing multifractal attractor is used, we can easily deduce the escape time dimension $D_E$ for the MWG "*G*" of the golden rectangle fractal, with $n=1$, $\varepsilon=1/2$. The number of black point is=3, given tol=0.0001, *N* is large enough, then calculate: $D_{E_n}(G) = \ln N_n(G))/\ln(2^n)$, where $N_n(G)$ = the number of black points in $\mathcal{W}_0$.

The process is stop if: $|D_{E_n} - D_{E_{n-1}}| < \text{tol}$, then $D(G) = D_{E_n}(G)$.

Table 1. The resulted values for number of iteration

| $n$ | $D_{En}(G)= \ln N_n(G))/\ln(2^n)$ | $D_{En}(G)-D_{En-1}(G)$ |
|---|---|---|
| 1 | ln(3) / ln(2)= 1.5849625007211561 | 1.5849625007211561 |
| 2 | ln(11)/2ln(2)= 1.729715809318648 | 0.1447533085974921 |
| 3 | ln(41)/3ln(2)= 1.785850668206032 | 0.0561348588985999 |
| 4 | ln(144)/4ln(2)= 1.79248125036057866 | 0.00663058215425866 |
| 5 | ln(499)/5ln(2)= 1.79257920167455493 | 0.0000979508 |

Therefore the dimension of *G* is  *D(G)*=1.79257920167455493.

To validate the values given by the proposed method in Table 1, it was compared to the computer program results written in Delphi for the well known method given in Edgar [5] for the graph of self similarity with multi nodes (or MWG). The results of Edgar method shows that, the dimension is obtained after 6 iteration, while by applying the proposed method (Escape Time Dimension) as in Table 1, we found that, the dimension is obtained after 5 iteration. This verify that our method is faster than Edgar method.

The  algorithm for Edgar method is as follows;

### The Algorithm.

**INPUT**:     **READ** $P_0$ , $P_1$

        tol         { Tolerance }

        $N_0$         { Number of Iterations }

    **FOR** i=1 **TO** 4

        **READ** $M_i$

        $A_i$=0

        **FOR** j=1 **TO** $M_i$

            **READ** $r_{ij}$

            $A_i = A_i + r_{ij}^{P0}$

        **ENDFOR** {j}

    **ENDFOR** {i}

**PROCESS**: $Q_0 = \dfrac{(A_1 + A_4) + \sqrt{(A_1 - A_4) + 4A_2 A_3}}{2} - 1$

    k=0

    **REPEAT**

      k=k+1

      **FOR** i=1 **TO** 4

        $A_i$=0

        **FOR** j=1 **TO** $M_i$   $A_i = A_i + r_{ij}^{P1}$

      **ENDFOR** {i}

      $Q_1 = \dfrac{(A_1 + A_4) + \sqrt{(A_1 - A_4) + 4A_2 A_3}}{2} - 1$

      $P = P_1 - \dfrac{Q_1(P_1 - P_0)}{(Q_1 - Q_0)}$

      $P_2 = P_1$ , $P_0 = P_1$ , $P_1 = P$ , $Q_0 = Q_1$

      **UNTIL** (k = $N_0$) **OR** (|P-$P_2$| < TOL)

**OUTPUT**: The Dimension is , P

## 5- Conclusions

Multifractal system is a generalization of fractal system and behaves as expected for family of sets known as digraph recursive set. There are many approaches to compute the fractal dimension of an object. In this work we modify a new approach to be applied for multifractal objects. This approach is based on the principle of escape time point, and on spreading points inside a specific windows. We conclude that this new method is a generalization of the well known method, "the Box counting dimension", while  it is considered as an efficient method comparing to the Box counting dimension in term of time.

## References

[1] A. Jasim, On dimensions of  some fractals, Ph.D. thesis, Al-Mustansiriyah University, 2005.

[2]  B.B. Mandelbrot, The Fractal Geometry of Nature. W.H. Freeman and Co., New York, NY 480. 1983.

[3] F. Talos, Using local fractal dimension in multifractals analysis. Proceedings of the Interdisciplinary approaches in fractal analysis, IAFA 2003, Bucharest, Romania, 2003.

[4] G. A. Edgar, R. D. Mauldin, Multifractal decompositions of digraph recursive fractals. Proc London Math Soc, 65(1992), 604-628.

[5] G.A. Edger, Measure, topology, and fractal geometry, New York: Springer-Verlag, 1990.

[6] G.A. Edgar, J.A. Golds, Fractal Dimension Estimate for a Graph-directed Iterated Function System of Non-similarities. Indiana University Mathematics Journal, 489(1999).

[7] G. Boffetta, A. Mazzino and A.Vulpiani, Twenty-five years of multifractals in fully developed turbulence : a tribute to Giovanni Paladin, J. Phys. A : Math. Theor., 41(2008), 1–25.

[8] G. Zhou, N.S.N. Lam, A comparison of fractal dimension estimators based on multiple surface generation, Computers & Geosciences, 31(2005), 260–1269.

[9] J. Hutchinson,  Fractals and self-similarity. Indiana Univ. J. Math. 30(1981), 713-747.

[10] K. Falconer, Fractal Geometry Mathematical Foundations and Application, John Wiley & Sons Ltd, 2nd edition, 2003.

[11] M. Arbeiter, N. Patzschke, Random self-similar multifractals, Math Nachr, 181(1996),  5-42.

[12]  M.F. Barnsley, Fractals everywhere, 2nd ed.  Academic Press Professional, Inc., San Diego, CA, USA, 1993.

[13]  N. Al-Saidi, M. Rushdan and A. Ahmed, About Fuzzy Fixed Point Theorem in the Generalized Fuzzy Fractal Space, Far East Journal of Mathematical Sciences, 53-1(2011), 53-64.

[14] R.D. Mauldin, S.C. Williams, Hausdorff dimension in graph directed constructions. Trans. of the American Mathematical Society, 309(1988), 881–829.

[15] T. Lofstedt, Fractal Geometry, Graph and Tree Constructions. Master's Thesis in Mathematics, Umea University 2008.

[16] W.F. Al-Shameri, On Nonlinear Iterated Function Systems, Ph.D. thesis, Al-Mustansiriyah University, 2001.