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Growth of Polynomials Having Zeros Inside a Circle

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Abstract

In this paper we study the growth of polynomial $p(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$, $1 \leq \mu \leq n$, having all its zeros in $|z| \leq k$, $k \leq 1$. Using the notation $M(p,t) = \max_{|z|=t} |p(z)|$, we measure the growth of p by estimating $\frac{M(p,t)}{M(p,1)}$ from above for any $t \leq 1$.

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1 Introduction and Statement of Results

For an arbitrary entire function $f(z)$, let $M(f,r) = \max_{|z|=r} |f(z)|$ and $m(f,k) = \min_{|z|=k} |f(z)|$. Then for a polynomial $p(z)$ of degree n, it was shown by Zarantonello and Varga [7]

$$\max_{|z|=r} |p(z)| \geq r^n \max_{|z|=1} |p(z)| \quad \text{for} \quad r \leq 1. \tag{1.1}$$

The result is best possible and extremal polynomial is $p(z) = \lambda z^n$, $|\lambda| = 1$.

For polynomials not vanishing in $|z| < 1$, Rivlin [6] obtained stronger inequality and proved that if $p(z)$ is a polynomial of degree n having no zero in $|z| < 1$, then

$$\max_{|z|=r} |p(z)| \geq \left(\frac{1+r}{2}\right)^n \max_{|z|=1} |p(z)| \quad \text{for} \quad r \leq 1. \tag{1.2}$$

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The result is best possible and equality in (1.2) holds for $p(z) = \left(\frac{1+z}{2}\right)^n$.

Jain [2] obtained the following result for polynomials having all its zeros in $|z| \leq k$, $k > 1$.

Theorem A. *If $p(z)$ be a polynomial of degree n having all its zeros in $|z| \leq k$, $k > 1$, then for $k < R < k^2$*

$$\max_{|z|=R} |p(z)| \geq R^s \left(\frac{R+k}{1+k}\right) \max_{|z|=1} |p(z)|. \tag{1.3}$$

where s $(< n)$ is the order of a possible zero of $p(z)$ at origin.

Mir [3] proved the following theorem analogous to Theorem A for polynomials having all its zeros in $|z| \leq k \leq 1$.

Theorem B. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k \leq 1$ with s-fold zeros at the origin, then for $r \leq k \leq 1$*

$$\max_{|z|=r} |p(z)| \leq r^s \left(\frac{r+k}{1+k}\right) \max_{|z|=1} |p(z)|. \tag{1.4}$$

The result is best possible for $s = n-1$ and equality holds for $p(z) = z^{n-1}(z+k)$.

By involving the coefficients of the polynomial $p(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$, $1 \leq \mu \leq n$, having all its zeros in $|z| \leq k \leq 1$ with s-fold zeros at the origin, we have been able to obtain the precise estimate of its maximum modulus on $|z| = 1$, where $r \leq k \leq 1$. In this direction, we have been able to prove the following result.

Theorem 1. *Let $p(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$, $1 \leq \mu \leq n$, be a polynomial of degree n having all its zeros in $|z| \leq k$, $k \leq 1$. Then for $r \leq k \leq 1$*

$$\max_{|z|=r} |p(z)| \leq r^s \left\{ \frac{\begin{pmatrix} (k^{2\mu} + r^{n-s} k^{\mu-1})(n-s)|a_n| \\ +\mu |a_{n-\mu}| (k^{\mu-1} + r^{n-s}) \end{pmatrix}}{\begin{pmatrix} (k^{2\mu} + k^{\mu-1})(n-s)|a_n| \\ +\mu |a_{n-\mu}| (k^{\mu-1} + 1) \end{pmatrix}} \right\} \max_{|z|=1} |p(z)|, \tag{1.5}$$

where s is the order of a possible zero of $p(z)$ at origin with $s \leq n - \mu$.

The following corollary immediately follows by choosing $\mu = 1$ in Theorem 1.

Corollary 1. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n having all its zeros in $|z| \leq k$, $k \leq 1$. Then for $r \leq k \leq 1$*

$$\max_{|z|=r} |p(z)| \leq r^s \left\{ \frac{(k^2 + r^{n-s})(n-s)|a_n| + |a_{n-1}|(1 + r^{n-s})}{(k^2+1)(n-s)|a_n| + 2|a_{n-1}|} \right\} \max_{|z|=1} |p(z)|, \tag{1.6}$$

where s is the order of a possible zero of $p(z)$ at origin with $s \leq n - 1$.

If we involve $m = \min\limits_{|z|=k} |p(z)|$, then we are able to improve upon the bound in Theorem 1. More precisely, we prove

Theorem 2. *Let $p(z) = a_n z^n + \sum\limits_{j=\mu}^{n} a_{n-j} z^{n-j}$, $1 \le \mu \le n$, be a polynomial of degree n having all its zeros in $|z| \le k$, $k \le 1$. Then for $r \le k \le 1$*

$$\begin{aligned}\max_{|z|=r} |p(z)| \;\le\; & r^s \frac{(k^{2\mu} + r^{n-s} k^{\mu-1})(n-s)|a_n| + \mu |a_{n-\mu}|(k^{\mu-1} + r^{n-s})}{(k^{2\mu} + k^{\mu-1})(n-s)|a_n| + \mu |a_{n-\mu}|(k^{\mu-1} + 1)} \max_{|z|=1} |p(z)| \\ & - \frac{r^s}{k^s} \frac{(1 - r^{n-s})\{(n-s)|a_n| k^{\mu-1} + \mu |a_{n-\mu}|\}}{(k^{2\mu} + k^{\mu-1})(n-s)|a_n| + \mu |a_{n-\mu}|(k^{\mu-1} + 1)} \min_{|z|=k} |p(z)| \end{aligned} \tag{1.7}$$

where s is the order of a possible zero of $p(z)$ at origin with $s \le n - \mu$.

If we take $\mu = 1$ in Theorem 2, we get the following result.

Corollary 2. *Let $p(z) = \sum\limits_{j=0}^{n} a_j z^j$ be a polynomial of degree n having all its zeros in having all its zeros in $|z| \le k$, $k \le 1$. Then for $r \le k \le 1$*

$$\begin{aligned}\max_{|z|=r} |p(z)| \;\le\; & r^s \frac{(k^2 + r^{n-s})(n-s)|a_n| + |a_{n-1}|(1 + r^{n-s})}{(k^2 + 1)(n-s)|a_n| + 2|a_{n-1}|} \max_{|z|=1} |p(z)| \\ & - \frac{r^s}{k^s} \frac{(1 - r^{n-s})\{(n-s)|a_n| + |a_{n-1}|\}}{(k^2 + 1)(n-s)|a_n| + 2|a_{n-1}|} \min_{|z|=k} |p(z)|, \end{aligned}$$

where s is the order of a possible zero of $p(z)$ at origin with $s \le n - 1$.

2 Lemmas

For the proof of these theorems we need the following lemmas. The first lemma is due to Qazi [5].

Lemma 1. *If $p(z) = a_0 + \sum\limits_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, is a polynomial of degree n not vanishing in $|z| < k$, $k \ge 1$, then*

$$\max_{|z|=1} |p'(z)| \le n \frac{1 + \frac{\mu}{n} \left| \frac{a_\mu}{a_0} \right| k^{\mu+1}}{1 + k^{\mu+1} + \frac{\mu}{n} \left| \frac{a_\mu}{a_0} \right| (k^{\mu+1} + k^{2\mu})} \max_{|z|=1} |p(z)|. \tag{2.1}$$

Lemma 2. *If* $p(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \le \mu \le n$, *is a polynomial of degree* n *having all its zeros in* $|z| \ge k$, $k \ge 1$, *then*

$$\begin{aligned}\max_{|z|=1} |p'(z)| \quad &\le \quad n\left(\frac{n|a_0| + \mu|a_\mu|k^{\mu+1}}{n|a_0|(1+k^{\mu+1}) + \mu|a_\mu|(k^{\mu+1}+k^{2\mu})}\right)\max_{|z|=1}|p(z)| \\ &\quad -\frac{n}{k^n}\left(\frac{n|a_0|k^{\mu+1} + \mu|a_\mu|k^{2\mu}}{n|a_0|(1+k^{\mu+1}) + \mu|a_\mu|(k^{\mu+1}+k^{2\mu})}\right)\min_{|z|=k}|p(z)|. \quad (2.2)\end{aligned}$$

The above lemma is due to Dewan, Singh and Yadav [1].

Lemma 3. *If* $p(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \le \mu \le n$, *is a polynomial of degree* n *having all its zeros in* $|z| \ge k$, $k \ge 1$, *then for* $r \le k \le R$,

$$\max_{|z|=r}|p(z)| \ge \frac{\begin{pmatrix} n|a_0|r^{n-1}(r^{\mu+1}+k^{\mu+1}) \\ +\mu|a_\mu|r^n k^{\mu+1}(r^{\mu-1}+k^{\mu-1}) \end{pmatrix}}{\begin{pmatrix} n|a_0|\{R^n + k^{\mu+1}r^{n-\mu-1}\}r^\mu \\ +\mu|a_\mu|r^{\mu-1}k^{\mu+1}\{R^n + k^{\mu-1}r^{n-\mu+1}\} \end{pmatrix}}\max_{|z|=R}|p(z)|. \quad (2.3)$$

Proof of Lemma 3. The proof of Lemma 3 follows on the same lines as that of Lemma 4, by using Lemma 1 instead of Lemma 2. We omit the details.

Lemma 4. *If* $p(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \le \mu \le n$, *is a polynomial of degree* n *having all its zeros in* $|z| \ge k$, $k \ge 1$, *then for* $r \le k \le R$,

$$\begin{aligned}\max_{|z|=R}|p(z)| \quad &\le \quad \frac{\begin{pmatrix} n|a_0|\{R^n + k^{\mu+1}r^{n-\mu-1}\}r^\mu \\ +\mu|a_\mu|r^{\mu-1}k^{\mu+1}\{R^n + k^{\mu-1}r^{n-\mu+1}\} \end{pmatrix}}{\begin{pmatrix} n|a_0|(r^{\mu+1}+k^{\mu+1})r^{n-1} \\ +\mu|a_\mu|r^n k^{\mu+1}(r^{\mu-1}+k^{\mu-1}) \end{pmatrix}}\max_{|z|=r}|p(z)| \\ &\quad -\frac{1}{k^n}\frac{r^{n-1}(R^n - r^n)(n|a_0|k^{\mu+1} + \mu|a_\mu|k^{2\mu}r)}{\begin{pmatrix} n|a_0|(r^{\mu+1}+k^{\mu+1})r^{n-1} \\ +\mu|a_\mu|r^n k^{\mu+1}(r^{\mu-1}+k^{\mu-1}) \end{pmatrix}}\min_{|z|=k}|p(z)|. \quad (2.4)\end{aligned}$$

Proof of Lemma 4. Let $0 \le r \le k$. Since $p(z)$ has all its zeros in $|z| \ge k$, $k \ge 1$, the polynomial $T(z) = p(rz)$ has all its zeros in $|z| \ge \frac{k}{r}$, $\frac{k}{r} \ge 1$, therefore applying

Lemma 2 to $T(z)$, we get

$$\begin{aligned}\max_{|z|=1}|T'(z)| &\leq n\left(\frac{n|a_0|+\mu|r^\mu a_\mu|\frac{k^{\mu+1}}{r^{\mu+1}}}{n|a_0|\left(1+\frac{k^{\mu+1}}{r^{\mu+1}}\right)+\mu|r^\mu a_\mu|\left(\frac{k^{\mu+1}}{r^{\mu+1}}+\frac{k^{2\mu}}{r^{2\mu}}\right)}\right)\max_{|z|=1}|T(z)| \\ &\quad -\frac{nr^n}{k^n}\left(\frac{n|a_0|\frac{k^{\mu+1}}{r^{\mu+1}}+\mu|r^\mu a_\mu|\frac{k^{2\mu}}{r^{2\mu}}}{n|a_0|\left(1+\frac{k^{\mu+1}}{r^{\mu+1}}\right)+\mu|r^\mu a_\mu|\left(\frac{k^{\mu+1}}{r^{\mu+1}}+\frac{k^{2\mu}}{r^{2\mu}}\right)}\right)\min_{|z|=\frac{k}{r}}|T(z)|.\end{aligned}$$

Replacing $T(z)$ by $p(rz)$, we get

$$\begin{aligned}\max_{|z|=r}|p'(z)| \leq n&\left(\frac{n|a_0|r^\mu+\mu|a_\mu|r^{\mu-1}k^{\mu+1}}{n|a_0|(r^{\mu+1}+k^{\mu+1})+\mu|a_\mu|(k^{\mu+1}r^\mu+k^{2\mu}r)}\right)\max_{|z|=r}|p(z)| \\ &-\frac{nr^{n-1}}{k^n}\left(\frac{n|a_0|k^{\mu+1}+\mu|a_\mu|rk^{2\mu}}{n|a_0|(r^{\mu+1}+k^{\mu+1})+\mu|a_\mu|(k^{\mu+1}r^\mu+k^{2\mu}r)}\right)\min_{|z|=k}|p(z)|. \quad (2.5)\end{aligned}$$

As $p'(z)$ is a polynomial of degree atmost $n-1$, by Maximum Modulus Principle [4, p. 158, Problem III, 269], we have

$$\frac{M(p',t)}{t^{n-1}} \leq \frac{M(p',r)}{r^{n-1}} \text{ for } \ t \geq r. \tag{2.6}$$

Inequality (2.6) in conjunction with inequality (2.5) yields

$$\begin{aligned}\max_{|z|=t}|p'(z)| &\leq \frac{nt^{n-1}}{r^{n-1}}\left(\frac{n|a_0|r^\mu+\mu|a_\mu|r^{\mu-1}k^{\mu+1}}{n|a_0|(r^{\mu+1}+k^{\mu+1})+\mu|a_\mu|(k^{\mu+1}r^\mu+k^{2\mu}r)}\max_{|z|=r}|p(z)|\right. \\ &\quad \left.-\frac{r^{n-1}}{k^n}\frac{n|a_0|k^{\mu+1}+\mu|a_\mu|rk^{2\mu}}{n|a_0|(r^{\mu+1}+k^{\mu+1})+\mu|a_\mu|(k^{\mu+1}r^\mu+k^{2\mu}r)}\min_{|z|=k}|p(z)|\right).\end{aligned}$$

Now, for $0 \leq \theta < 2\pi$, we have

$$\begin{aligned}&|p(Re^{i\theta})-p(re^{i\theta})| \\ &\quad\leq \int_r^R |p'(te^{i\theta})|dt \\ &\quad\leq \left(\frac{n|a_0|r^\mu+\mu|a_\mu|r^{\mu-1}k^{\mu+1}}{n|a_0|(r^{\mu+1}+k^{\mu+1})+\mu|a_\mu|(k^{\mu+1}r^\mu+k^{2\mu}r)}\max_{|z|=r}|p(z)|\right. \\ &\quad\left.-\frac{r^{n-1}}{k^n}\frac{n|a_0|k^{\mu+1}+\mu|a_\mu|rk^{2\mu}}{n|a_0|(r^{\mu+1}+k^{\mu+1})+\mu|a_\mu|(k^{\mu+1}r^\mu+k^{2\mu}r)}\min_{|z|=k}|p(z)|\right)\int_r^R\frac{nt^{n-1}}{r^{n-1}}dt \\ &= \frac{R^n-r^n}{r^{n-1}}\left(\frac{n|a_0|r^\mu+\mu|a_\mu|r^{\mu-1}k^{\mu+1}}{n|a_0|(r^{\mu+1}+k^{\mu+1})+\mu|a_\mu|(k^{\mu+1}r^\mu+k^{2\mu}r)}\max_{|z|=r}|p(z)|\right. \\ &\quad\left.-\frac{r^{n-1}}{k^n}\frac{n|a_0|k^{\mu+1}+\mu|a_\mu|rk^{2\mu}}{n|a_0|(r^{\mu+1}+k^{\mu+1})+\mu|a_\mu|(k^{\mu+1}r^\mu+k^{2\mu}r)}\min_{|z|=k}|p(z)|\right).\end{aligned}$$

This is equivalent to

$$M(p,R) \le 1+\left(\frac{R^n-r^n}{r^{n-1}}\right)\frac{n|a_0|r^{\mu}+\mu|a_\mu|r^{\mu-1}k^{\mu+1}}{n|a_0|(r^{\mu+1}+k^{\mu+1})+\mu|a_\mu|(k^{\mu+1}r^{\mu}+k^{2\mu}r)}\max_{|z|=r}|p(z)|$$
$$-\left(\frac{R^n-r^n}{k^n}\right)\frac{n|a_0|k^{\mu+1}+\mu|a_\mu|rk^{2\mu}}{n|a_0|(r^{\mu+1}+k^{\mu+1})+\mu|a_\mu|(k^{\mu+1}r^{\mu}+k^{2\mu}r)}\min_{|z|=k}|p(z)|, \quad (2.7)$$

from which we get the desired result.

3 Proofs of the theorems

Proof of Theorem 1. The proof of Theorem 1 follows on the same lines as that of Theorem 2, but instead of using Lemma 4, we use Lemma 3. We omit the details.

Proof of Theorem 2. Since $p(z)$ has all its zeros in $|z| \le k \le 1$ with s-fold zeros at the origin, therefore $q(z)=z^n\overline{p(1/\overline{z})}$ is of degree $(n-s)$ and has all its zeros in $|z| \ge \frac{1}{k} \ge 1$. On applying Lemma 4 to $q(z)$ with $r=1$, we have

$$\max_{|z|=R}|q(z)| \le \frac{\left\{R^{n-s}+\frac{1}{k^{\mu+1}}\right\}(n-s)|a_n|+\mu|a_{n-\mu}|\frac{1}{k^{\mu+1}}\left\{R^{n-s}+\frac{1}{k^{\mu-1}}\right\}}{(1+\frac{1}{k^{\mu+1}})(n-s)|a_n|+\mu|a_{n-\mu}|\frac{1}{k^{\mu+1}}\left(1+\frac{1}{k^{\mu-1}}\right)}\max_{|z|=1}|q(z)|$$
$$-k^{n-s}\frac{(R^{n-s}-1)\left\{(n-s)|a_n|\frac{1}{k^{\mu+1}}+\mu|a_{n-\mu}|\frac{1}{k^{2\mu}}\right\}}{\left(1+\frac{1}{k^{\mu+1}}\right)(n-s)|a_n|+\mu|a_{n-\mu}|\frac{1}{k^{\mu+1}}\left(1+\frac{1}{k^{\mu-1}}\right)}\min_{|z|=\frac{1}{k}}|q(z)|.$$

which is equivalent to

$$R^n\max_{|z|=\frac{1}{R}}|p(z)| \le \frac{\left\{R^{n-s}+\frac{1}{k^{\mu+1}}\right\}(n-s)|a_n|+\mu|a_{n-\mu}|\left\{\frac{R^{n-s}}{k^{\mu+1}}+\frac{1}{k^{2\mu}}\right\}}{\left(1+\frac{1}{k^{\mu+1}}\right)(n-s)|a_n|+\mu|a_{n-\mu}|\left(\frac{1}{k^{\mu+1}}+\frac{1}{k^{2\mu}}\right)}\max_{|z|=1}|p(z)|$$
$$-\frac{k^{n-s}}{k^n}\frac{(R^{n-s}-1)\left\{(n-s)|a_n|\frac{1}{k^{\mu+1}}+\mu|a_{n-\mu}|\frac{1}{k^{2\mu}}\right\}}{\left(1+\frac{1}{k^{\mu+1}}\right)(n-s)|a_n|+\mu|a_{n-\mu}|\left(\frac{1}{k^{\mu+1}}+\frac{1}{k^{2\mu}}\right)}\min_{|z|=k}|p(z)|.$$

Now replacing R by $\frac{1}{r}$ in above inequality so that $\frac{1}{r} \geq \frac{1}{k} \geq 1$ or $r \leq k \leq 1$, we get

$$\begin{aligned}\max_{|z|=r}|p(z)| \quad \leq \quad & r^n \frac{\left\{\frac{1}{r^{n-s}}+\frac{1}{k^{\mu+1}}\right\}(n-s)|a_n|+\mu|a_{n-\mu}|\left\{\frac{1}{r^{n-s}k^{\mu+1}}+\frac{1}{k^{2\mu}}\right\}}{\left(1+\frac{1}{k^{\mu+1}}\right)(n-s)|a_n|+\mu|a_{n-\mu}|\left(\frac{1}{k^{\mu+1}}+\frac{1}{k^{2\mu}}\right)}\max_{|z|=1}|p(z)| \\ & -\frac{r^n}{k^s}\frac{\left(\frac{1}{r^{n-s}}-1\right)\left\{(n-s)|a_n|\frac{1}{k^{\mu+1}}+\mu|a_{n-\mu}|\frac{1}{k^{2\mu}}\right\}}{\left(1+\frac{1}{k^{\mu+1}}\right)(n-s)|a_n|+\mu|a_{n-\mu}|\left(\frac{1}{k^{\mu+1}}+\frac{1}{k^{2\mu}}\right)}\min_{|z|=k}|p(z)|,\end{aligned}$$

from which we get the desired result.

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