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Ergodic Theorem and Strong Law of Large Numbers for Unbounded and Nonconvex Random Sets

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Abstract. The main purpose of this paper is to prove, in the first part, some results of Birkhoff ergodic type theorem for random sets whose values are unbounded and nonconvex closed subsets of a Banach space E. Results are given with respect to the Mosco convergence and Wijsman-topology. The second part deals with the random sets whose values are convex in E. The convergence holds, this time, with respect the slice-topology. Consequently, the Multivalued strong laws of large numbers are obtained.

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1. Introduction

The study of the convergence of sequences of closed sets, not necessarily bounded in a Banach space E has been important in applications involving variational methods, such as in stochastic optimization, in mathematical economics and mechanics. The fact that the sets can be considered unbounded makes it possible to translate some similarities in terms of set-convergence of functional (see e.g. Attouch [3]).

In the literature, the strong law of large numbers (abbreviated SLLN) in the real case, known as Kolmogorov's theorem was proved in various ways, including from the ergodic theorem of Birkhoff (see e.g. Krengel [19]). In recent decades there has been considerable interest in the multivalued case. In this context, and in the case of finite dimension, let us mention the work of Artstein and Vitale [2] in the bounded case and of Artstein and Hart [1] in the unbounded case. The infinite dimensional case was treated in particular by Hess [13, 14, 15] and Hiai [16, 17], where the random sets are not bounded. In Ziat [29], the link between multivalued SLLN and reversed martingales has been established.

On the other hand, as is well-known, the Birkhoff ergodic theorem is one of the most important results of probability theory. Its generalization has been studied in various directions. In particular by Choirat, Hess and Seri for normal integrands and consequently for convex random sets in finite dimensional case (see [8], Theorem 2.6). Their result ([8], Theorem 2.6) was obtained without condition on the measure-preserving transformation θ, but including the condition of convexity on random sets in finite dimensional case and convergence used is that of Painleve-Kuratowski. Note also that several versions of ergodic theorems have already been proved for random sets. Let us mention the works of Schorger [23] for subadditive Superstationary families of convex compact sets in finite dimension and of Krupa [20] and Hansen and Hulse [11] in infinite dimension.

In Theorem 3.4 and Theorem 3.7 we aim to propose a new kind of Birkhoff ergodic theorem for random sets without the restriction of convexity or boundedness, where random sets take values of closed subsets in separable Banach space. More precisely, given a probability space $(\Omega, \mathcal{A}, P)$, a separable Banach space $(E, \|.\|)$ and a closed integrable random set F. Under an assumption of strong ergodicity of the measure-preserving transformation θ, we show that for almost all $\omega \in \Omega$

$$\lim_{n\to\infty} \frac{1}{n}\left(\overline{F(\theta\omega) + \cdots + F(\theta^n\omega)}\right) = \overline{\mathrm{co}}\,\mathbb{E}(F), \tag{1.1}$$

where $\mathbb{E}(F)$ denotes the multivalued integral (or expectation), originally introduced by Aumann [4] and the convergence will be taken from both Mosco sense, and that of Wijsman. Under the assumption of ergodicity of θ above which implies that the limit of ergodic average in (1.1) is almost certain, a convexification technique is highlighted. In theorem 3.9 we present a version of Birkhoff ergodic theorem for integrable convex random set F without restriction on the measure-preserving transformation θ, but assuming that there exists a sequence (x'_k) in the unit ball $B_{E'}$ of the dual space E' such that for almost all ω in Ω we have

$$\mathrm{D}(B, \mathbb{E}(F|\mathcal{T})(\omega)) = \sup_{k\geq 1}[-\delta^*(x'_k, \mathbb{E}(F|\mathcal{T})(\omega)) - \delta^*(-x'_k, B)]$$

for all closed bounded convex subset B of E. Under these conditions we show that

$$\tau_{\mathrm{S}}\text{-}\lim \frac{1}{n}\left(F(\theta\omega)\dot{+}\cdots\dot{+}F(\theta^n\omega)\right) = \mathbb{E}(F|\mathcal{T})(\omega)) \quad \text{a.s.}$$

where $\mathbb{E}(X|\mathcal{T})$ denotes the mutivalued conditional expectation of F over the sub-σ-field $\mathcal{T}$ of θ-invariant sets. Accordingly, we present a version of the strong law of large numbers for independent identically distributed random sets initially shown by Hess [13] and Hiai [17] with respect to Mosco convergence, by Hess [15] with respect to Wijsman-topology and by Hess [14] with respect to Slice-topology.

The layout of this paper is as follows. In Section 2, we give some basic definitions and properties, and the Birkhoff ergodic theorem in different forms is proposed in Section 3. In the last section we investigated the relationship between this result and strong law of large numbers for random sets.

2. Notations and preliminaries

Throughout this paper, let $(\Omega, \mathcal{A}, P)$ denotes a probability space and $(E, \|.\|)$ is a separable Banach space with the dual space E'. $\mathcal{C}(E)$ (resp. $\mathcal{C}_c(E)$) is the space of all closed (resp. closed convex) nonempty subsets of E. A space of subsets is often called an hyperspace.

For each $A \subset E$, $\overline{A}$ (resp. $\overline{\text{co}}A$) denotes the norm-closure (resp. the closed convex hull) of A. The support function $\delta^*(., A)$ and the function distance $\mathrm{d}(., A)$, are defined respectively by

$$\delta^*(x', A) = \sup\{\langle x', x\rangle : x \in A\},\ x' \in E'$$

$$\mathrm{d}(x, A) = \inf\{\|x - y\| : y \in A\},\ x \in E.$$

The Wijsman topology on $\mathcal{C}(E)$ is the topology determined by the family of distance functions $\{A \longrightarrow \mathrm{d}(x, A) : x \in E\}$ and denoted τ_w (see Wijsman [27, 28]).

Let (A_n) be a sequence of closed sets in $\mathcal{C}(E)$. Let $\text{s-li}\, A_n$ be the set of all $a \in E$ such that $\|a_n - a\|$ tends to 0 for some $a_n \in A_n$, and let $\text{w-ls}\, A_n$ be the set of all $a \in E$ such that $\|a_k - a\|$ tends weakly (i.e. under the topology $\sigma(E, E')$) to 0, for some $a_k \in A_{n_k}$ and some subsequence (A_{n_k}) of (A_n). It is easily seen that $\text{s-li}\, A_n \subset \text{w-ls}\, A_n$. Thus we say A_n converges to A in the Mosco sense and denote

$$\text{M-lim}\, A_n = A$$

if and only if

$$\text{w-ls}\, A_n \subset A \subset \text{s-li}\, A_n.$$

It has been known that when E is reflexive, the Wijsman convergence is weaker than the Mosco convergence and is equivalent to it when, in addition, the norm of E is Frechet differentiable (see [3, 25])

The gap between two nonempty subsets A and B of E is denoted $\mathrm{D}(A, B)$, and defined by

$$\mathrm{D}(A, B) = \inf\{\|x - y\| : x \in A,\ y \in B\}.$$

For each $x' \in E'$ and each $\alpha \in \mathbb{R}$, we denote the half space $F(x',\alpha)$ by $\{x \in E : \langle x', x\rangle \geq \alpha\}$ and

$$\mathcal{S}_b = \{B(x_0,r) \cap F(x',\alpha) : x_0 \in E;\ x' \in E',\ x' \neq 0;\ \alpha \in \mathbb{R};\ r > 0\},$$

where $B(x_0,r)$ is the closed ball of center x_0 and radius r.
The Slice-topology on $\mathcal{C}(E)$ will be designated τ_S. Recall that this topology, which was defined by Sonntag and Zalinescu [24] and Beer [5, 6], is that of pointwise convergence of functions $\mathrm{D}(C,.)$ For $C \in \mathcal{S}_b$.
According to a result of Beer ([6], Theorem 5.2), the topology τ_S on $\mathcal{C}_c(E)$ coincides with the topology of pointwise convergence of functions $\mathrm{D}(C,.)$ for $C \in \mathcal{C}_{cb}(E)$, where $\mathcal{C}_{cb}(E)$ denotes the space of nonempty closed bounded convex subsets of E.
We introduce the family of subsets of E, denoted $\mathcal{R}_{cwk}(E)$ defined by

$$\mathcal{R}_{cwk}(E) = \{C \in \mathcal{C}_c(E) : C \cap B(0,r) \in \mathcal{C}_{cwk}(E),\ \forall r > 0\},$$

where $\mathcal{C}_{cwk}(E)$ denotes the space of weakly compact convex subsets of E. Moreover, in ([14] prop. 3.3), Hess showed that for every $R \in \mathcal{R}_{cwk}(E)$, the restriction of the topology τ_S on the subset $\mathcal{R}_{cwk}(R)$ defined by $\mathcal{R}_{cwk}(R) = \{C \in \mathcal{C}_c(E) : C \subset R\}$ is that of pointwise convergence of functions $\mathrm{D}(C,.)$ for $C \in \mathcal{C}_{cwk}(R)$, where $\mathcal{C}_{cwk}(R)$ denotes the subset $\mathcal{C}_{cwk}(R) = \{C \in \mathcal{C}_{cwk}(E) : C \subset R\}$.
Define the Minkowski addition and scalar multiplication, respectively, in $\mathcal{C}(E)$ by

$$A + B = \{a + b : a \in A, b \in B\}$$
$$\lambda A = \{\lambda a : a \in A\}$$

where $A, B \in \mathcal{C}(E)$ and λ is a real number.
For any finite sequence $A_1, \cdots, A_n$ of $\mathcal{C}_c(E)$, we set

$$F_1 \dot{+} F_2 = \overline{F_1 + F_2} \quad \text{and} \quad \sum_{i=1}^{n} F_i = F_1 \dot{+} \cdots \dot{+} F_n.$$

Further, the Effros-σ-field on $\mathcal{C}(E)$ is denoted by $\mathcal{E}$ and is generated by the subsets

$$U^- = \{F \in \mathcal{C}(E) : F \cap U \neq \emptyset\},$$

where U ranges over the open subsets of E. It is obvious that the restriction of $\mathcal{E}$ to E (identified with the hyperspace of singletons) coincide with the Borel σ-field.
A map X from Ω in $\mathcal{C}(E)$ is called multifunction with nonempty closed values in E. The multifunction X is said to be measurable if for any open set U of E, the subset

$$X^- U = \{\omega \in \Omega : X(\omega) \cap U \neq \emptyset\}$$

is in $\mathcal{A}$.
According to the definition of the Effros-σ-field mentioned above the multifunction X is measurable if and only if the map X is Effros-measurable.

Recall that $f:(\Omega,\mathcal{A})\longrightarrow E$ is called a measurable selection of X, if f is measurable and $f(\omega)\in X(\omega)$ a.s.. Furthermore, a countable family of measurable selections (f_n) such that for each $\omega\in\Omega: X(\omega)=\overline{\{f_n(\omega):n\geq 1\}}$ is called a castaing representation of X. A multifunction X with nonempty closed values in E is measurable if and only if $\mathrm{d}(x,X(.))$ is measurable for all x in E or if and only if X has a Castaing representation (see [7], Theorem III.9) ; In this case X is also called closed random set (or simply random set). Denotes by $\mathcal{A}_X$, the σ-field generated by X on Ω.
Denote by $L^1_E(\Omega,\mathcal{A},P)$, or simply by L^1_E the space (of equivalence classes) of Bochner-integrable functions f from $(\Omega,\mathcal{A},P)$ to E with the norm defined by $\|f\|_1=\int_\Omega\|f(\omega)\|\,dP$.
Given a multifunction X, we define the set $S^1_X(\mathcal{A})$ of its selections $\mathcal{A}$-measurable and Bochner-integrable by

$$S^1_X(\mathcal{A})=\left\{f\in L^1_E(\Omega,\mathcal{A},P):f(\omega)\in X(\omega)\,\text{a.s.}\right\}.$$

$\mathcal{A}$ can optionally be replaced by a sub-σ-field $\mathcal{B}$ of $\mathcal{A}$. Note that if X has its values in $\mathcal{C}(E)$ then $S^1_X(\mathcal{A})$ is closed in L^1_E and $S^1_X(\mathcal{A})\neq\emptyset$ if and only if $\mathrm{d}(0,X(.))\in L^1_{\mathbb{R}^+}$; we say that in this case X is integrable. The multivalued integral of X (Aumann integral) is the subset of E defined by

$$\mathbb{E}(X)=\int_\Omega X\,\mathrm{d}\,P=\left\{\int_\Omega f\,dP:f\in S^1_X(\mathcal{A})\right\}.$$

Like for real or vector valued random variables, the distribution μ_X of the random set X can be defined on the measurable space $(\mathcal{C}(E),\mathcal{E})$ by

$$\mu_X(A)=P(X^{-1}(A)),\quad\forall A\in\mathcal{E}.$$

A finite family $\{X_i:i\in I\}$ of random sets is called independent if

$$P\big(\bigcap_{i\in I}X_i^{-1}(A_i)\big)=\prod_{i\in I}P(X_i^{-1}(A_i))$$

for all $A_i\in\mathcal{E}\,(i\in I)$. An infinite family $\{X_i:i\in I\}$ of random sets is called independent if every finite subfamily is independent.
Let $\mathcal{B}$ be a sub-σ-field of $\mathcal{A}$ and X be an integrable random set. The conditional expectation of X over $\mathcal{B}$ is the unique (a.s.) random set $\mathcal{B}$-measurable, denoted $\mathbb{E}(X|\mathcal{B})$ such that

$$S^1_{\mathbb{E}(X|\mathcal{B})}(\mathcal{B})=\overline{\{\mathbb{E}(f|\mathcal{B}):f\in S^1_X(\mathcal{A})\}}^{\|.\|_1}.\ \text{(see [18], Theorem 5.1)}$$

We say that a set of measurable functions M is decomposable (with respect to $\mathcal{A}$) if $f_1,f_2\in M$ and $A\in\mathcal{A}$ imply $\mathbf{1}_A\,f_1+\mathbf{1}_{\Omega\setminus A}\,f_2\in M$, where $\mathbf{1}_A$ denotes the indicator function of A.

The following proposition specifies a result due to Hiai and Umegaki (see [18] theorem 3.1). We give a proof for convenience.

Proposition 2.1. *Let M be a nonempty decomposable subset of L^1_E. Then there exists an unique (a.s.) integrable random set X such that $S^1_X = \overline{M}^{\|\cdot\|_1}$ and that X admits a Castaing representation in M.*

Proof. The space E is separable, let $(f_i)_{\geq 1}$ be a dense sequence (of constant functions) in E. For every integer $i \geq 1$ put $\alpha_i = \inf\{\|f_i - g\|_1 : g \in M\}$. Then there exists a minimizing sequence $(g_{ij})_{j\geq 1}$ of M such that

$$\lim_{j\to\infty} \|f_i - g_{ij}\|_1 = \alpha_i.$$

We define the multifunction X by $X(\omega) = \overline{\{g_{ij}(\omega) : i, j \geq 1\}}$. (g_{ij}) is thus the desired Castaing representation.
First show that $S^1_X \subset \overline{M}_1^{\|\cdot\|}$. Let $f \in S^1_X$ and $\varepsilon > 0$. It follows from lemma 1.3 of [18] that there exists a finite partition $\{A_1, \cdots, A_n\}$ of Ω in $\mathcal{A}$ and $\{h_1, \cdots, h_n\}$ in $\{g_{ij} : i, j \geq 1\}$ such that

$$\left\| f - \sum_{k=1}^{n} \mathbf{1}_{A_k} h_k \right\|_1 < \varepsilon. \tag{2.1}$$

Now, since M is decomposable, $\sum_{k=1}^{n} \mathbf{1}_{A_k} h_k \in M$ and inequality (2.1) implies $f \in \overline{M}_1^{\|\cdot\|}$ and $S^1_X \subset \overline{M}_1^{\|\cdot\|}$.
The proof of $\overline{M}_1^{\|\cdot\|} \subset S^1_X$ is similar to that of Theorem 3.1 of [18]. □

The following lemma is already stated in ([29], Lemma 2.5) is easily deduced from Proposition 2.1. Its importance lies in the fact that to treat the conditional expectation of a closed random set X relative to $\mathcal{B}$ of just treating its selections of the form $\mathbb{E}(f|\mathcal{B})$, where f ranges over the set S^1_X.

Lemma 2.2. *Let $\mathcal{B}$ be a sub-σ-field of $\mathcal{A}$ and X be an integrable closed random set. Then, there exists a sequence $(f_m)_{m\geq 1}$ in S^1_X such that $\mathbb{E}(X|\mathcal{B})$ admits $(\mathbb{E}(f_m|\mathcal{B}))_{m\geq 1}$ as a representation of Castaing.*

Proof. Just apply Proposition 2.1 to the set $M = \{\mathbb{E}(f|\mathcal{B}) : f \in S^1_X\}$ who is nonempty and decomposable with respect to $\mathcal{B}$. □

Let $\theta : \Omega \longrightarrow \Omega$ be an $\mathcal{A}$-measurable transformation. We said that θ is a measure-preserving transformation or, equivalently, P is said to be θ-invariant measure, if $P(\theta^{-1}(A)) = P(A)$ for all $A \in \mathcal{A}$. The sets $A \in \mathcal{A}$ that satisfy $\theta^{-1}(A) = A$ are called invariant sets and constitute a sub-σ-field $\mathcal{T}$ of $\mathcal{A}$. We say that θ is an ergodic transformation if $\mathcal{T}$ is trivial i.e. whenever $A \in \mathcal{T}$ then $P(A) = 0$ or 1. Checking that a given transformation is ergodic is often a non-trivial task. Some stronger properties that a transformation may enjoy in some cases are easier to check : for example the Bernoulli shift as we see later.

3. Birkhoff ergodic theorem for random sets

3.1. **Non-convex case.** The crucial step for showing the multivalued Birkhoff ergodic theorem for non-convex case consits of the following proposition. This result is given under an assumption of ergodicity of the measure-preserving transformation.

Proposition 3.1. *Let F be an integrable random set with values in $\mathcal{C}(E)$ and let θ be a measure-preserving transformation of $(\Omega, \mathcal{A}, P)$ such that, for each integer $m \geq 1$, θ^m is ergodic. Then*

$$\overline{\text{co}}\, \mathbb{E}(F) \subset \text{s-li}\, \frac{1}{n}\left(\overline{F(\theta\omega) + \cdots + F(\theta^n\omega)}\right) \quad a.s.$$

Proof. Let $\varepsilon > 0$. For each $x \in \overline{\text{co}}\, \mathbb{E}(F)$, x can be approximated by a convex combination of $\mathbb{E}(f_j)$, then we can approximate the coefficients of the combination by rational numbers with the same denominator m. So let $f_1, \cdots, f_m \in S^1_F$ such that

$$\left\| \frac{1}{m} \sum_{j=1}^{m} \mathbb{E}(f_j) - x \right\| < \varepsilon.$$

Thus, we define the sequence $(h_n)_{n\geq 1}$ by $h_{(i-1)m+j}(\omega) = f_j(\theta^{(i-1)m+j}\omega)$ for each integer $i \geq 1$ and $j = 1, \cdots, m$.
For $n = (k-1)m + l$ where $1 \leq l \leq m$, we have

$$\begin{aligned}
\left\| \frac{1}{n} \sum_{p=1}^{n} h_p(\omega) - \frac{1}{m} \sum_{j=1}^{m} \mathbb{E}(f_j) \right\| &= \left\| \frac{1}{n} \sum_{j=1}^{m} \sum_{i=1}^{k} f_j(\theta^{(i-1)m+j}\omega) \right. \\
&\quad \left. - \frac{1}{n} \sum_{j=l+1}^{m} f_j(\theta^{(k-1)m+j}\omega) - \frac{1}{m} \sum_{j=1}^{m} \mathbb{E}(f_j) \right\| \\
&\leq \frac{k}{n} \sum_{j=1}^{m} \left\| \frac{1}{k} \sum_{i=1}^{k} f_j(\theta^{(i-1)m+j}\omega) - \mathbb{E}(f_j) \right\| \\
&\quad + \frac{k}{n} \sum_{j=1}^{m} \frac{1}{k} \left\| f_j(\theta^{(k-1)m+j}\omega) \right\| \\
&\quad + \left(\frac{k}{n} - \frac{1}{m} \right) \left\| \sum_{j=1}^{m} \mathbb{E}(f_j) \right\|.
\end{aligned}$$

For any integer $m \geq 1$, define the sub-σ-field $\mathcal{T}_m$ of θ^m-invariant sets. Since the measure preserving transformation θ^m is ergodic, then $\mathcal{T}_m$ is trivial σ-field and for any $j = 1, \cdots, m$ we have

$$\mathbb{E}(f_j \circ \theta^j) = \int_\Omega f_j \circ \theta^j \, dP = \int_\Omega f_j \, dP = \mathbb{E}(f_j) = \mathbb{E}(f_j | \mathcal{B}_m).$$

According to Birkhoff ergodic theorem for Bochner integrable functions (see e.g. [19], Theorem 2.1 p. 167) we have

$$\lim_{k\to\infty}\left\|\frac{1}{k}\sum_{i=1}^{k} f_j(\theta^{(i-1)m+j}\omega) - \mathbb{E}(f_j)\right\| = 0 \quad \text{a.s.}$$

Secondly,

$$\lim_{k\to\infty}\frac{1}{k}\left\|f_j(\theta^{(k-1)m+j}\omega)\right\| = 0 \quad \text{a.s.} \quad \text{for } j = 1, \cdots, m.$$

Since n and k tend to $+\infty$ simultaneously, with $\frac{k}{n} = \frac{1}{m} + \frac{1}{n} - \frac{l}{mn}$ we have $\lim_{k\to\infty}\left(\frac{k}{n} - \frac{1}{m}\right) = 0$ and

$$\lim_{k\to\infty}\left(\frac{k}{n} - \frac{1}{m}\right)\left\|\sum_{j=1}^{m}\mathbb{E}(f_j)\right\| = 0.$$

Note that from the inclusion

$$\frac{1}{n}\sum_{p=1}^{n} h_p(\omega) \in \frac{1}{n}\left(F(\theta\omega) + \cdots + F(\theta^n\omega)\right) \quad \text{a.s.}$$

it follows that

$$\frac{1}{m}\sum_{j=1}^{m}\mathbb{E}(f_j) \in \text{s-li}\,\frac{1}{n}\left(\overline{F(\theta\omega) + \cdots + F(\theta^n\omega)}\right) \quad \text{a.s.}$$

and therefore

$$\overline{\text{co}}\,\mathbb{E}(F) \subset \text{s-li}\,\frac{1}{n}\left(\overline{F(\theta\omega) + \cdots + F(\theta^n\omega)}\right) \quad \text{a.s.}$$

□

Without assuming ergodicity on the measure-preserving transformation θ, we have the following proposition.

Proposition 3.2. *Let F be an integrable random set with values in $\mathcal{C}(E)$ and let θ be a measure-preserving transformation of $(\Omega, \mathcal{A}, P)$. Then*

$$\mathbb{E}(F|\mathcal{T}) \subset \text{s-li}\,\frac{1}{n}\left(\overline{F(\theta\omega) + \cdots + F(\theta^n\omega)}\right) \quad \textit{a.s.}$$

Proof. By virtue of lemma 2.2, we can choose $(f_m)_{m\geq 1}$ as a sequence in S^1_F such that $(\mathbb{E}(f_m|\mathcal{T}))_{m\geq 1}$ be a Castaing representation of $\mathbb{E}(F|\mathcal{T})$. According to the Birkhoff ergodic theorem for Bochner integrable functions we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} f_m(\theta^i\omega) = \mathbb{E}(f_m|\mathcal{T}) \quad \text{a.s.} \tag{3.1}$$

Put $G_i(\omega) = F(\theta^i\omega)$ and $g^i_m(\omega) = f_m(\theta^i\omega)$. it is clear that $g^i_m \in S^1_{G_i}$ for all integer $i \geq 1$ and all integer $m \geq 1$.
Let N_1 be a negligible outside of which $g^i_m(\omega) \in G_i(\omega)$ for all integer $i \geq 1$ and

all integer $m \geq 1$. Denotes by N_2 a negligible outside which the convergence (3.1) holds for all integer $m \geq 1$.
Then for every $\omega \in \Omega \setminus (N_1 \cup N_2)$ and all integer $m \geq 1$, we have

$$\mathbb{E}(f_m|\mathcal{T})(\omega) \in \text{s-li}\, \frac{1}{n}\left(\overline{G_1(\omega) + \cdots + G_n(\omega)}\right).$$

And since the $\mathbb{E}(f_m|\mathcal{T})(\omega)$ are dense in $\mathbb{E}(F|\mathcal{T})(\omega)$, we obtain

$$\mathbb{E}(F|\mathcal{T})(\omega) \in \text{s-li}\, \frac{1}{n}\left(\overline{G_1(\omega) + \cdots + G_n(\omega)}\right) \quad \text{a.s.} \tag{3.2}$$

□

Now, the proof of the "limsup" part of the main result (Theorem 3.4) is given in the following proposition

Proposition 3.3. *Let F be an integrable random set with values in $\mathcal{C}(E)$ and let θ be an ergodic measure-preserving transformation of $(\Omega, \mathcal{A}, P)$. Then*

$$\text{w-ls}\, \frac{1}{n}\left(\overline{F(\theta\omega) + \cdots + F(\theta^n\omega)}\right) \subset \overline{\text{co}}\, \mathbb{E}(F) \quad a.s.$$

Proof. Separability of the space E ensures the existence of a sequence $(x_i)_{i\geq 1}$ dense in $E \setminus \overline{\text{co}}\, \mathbb{E}(F)$. From Hahn-Banach theorem we have for any integer $i \geq 1$ (see e.g. [7], Theorem II.18)

$$\mathrm{d}\left(x_i, \overline{\text{co}}\, \mathbb{E}(F)\right) = \sup_{\|x'\|\leq 1} \left[\langle x', x_i\rangle - \delta^*\left(x', \overline{\text{co}}\, \mathbb{E}(F)\right)\right]. \tag{3.3}$$

Moreover, for any integer i, the function $\langle ., x_i\rangle - \delta^*\left(., \overline{\text{co}}\, \mathbb{E}(F)\right)$ is upper semi-continuous on E' endowed with the weak topology $\sigma(E', E)$, and the closed unit ball $B_{E'} = \{x' \in E' : \|x'\| \leq 1\}$ is $\sigma(E', E)$ compact, then the supremum in (3.3) can be taken on the unit sphere $S_{E'} = \{x' \in E' : \|x'\| = 1\}$. Thus, there exists a sequence $(x'_i)_{i\geq 1}$ in $S_{E'}$ such that

$$\mathrm{d}\left(x_i, \overline{\text{co}}\, \mathbb{E}(F)\right) = \langle x'_i, x_i\rangle - \delta^*\left(x'_i, \overline{\text{co}}\, \mathbb{E}(F)\right). \tag{3.4}$$

holds for any integer $i \geq 1$. We therefore have the following equivalence

$$x \in \overline{\text{co}}\, \mathbb{E}(F) \iff \langle x'_i, x\rangle \leq \delta^*\left(x'_i, \overline{\text{co}}\, \mathbb{E}(F)\right), \quad \forall i \geq 1 \tag{3.5}$$

Indeed, the implication from left to right is obvious. To prove the converse implication, suppose there exists x such that for every integer $i \geq 1$ we have $\langle x'_i, x\rangle \leq \delta^*\left(x'_i, \overline{\text{co}}\, \mathbb{E}(F)\right)$ and $x \notin \overline{\text{co}}\, \mathbb{E}(F)$. Then, one has a sub-sequence of $(x_i)_{i\geq 1}$ denoted by $(x_{i_k})_{k\geq 1}$ which converges to x. It follows from (3.4) that

$$\begin{aligned}
\mathrm{d}\left(x_{i_k}, \overline{\text{co}}\, \mathbb{E}(F)\right) &= \left\langle x'_{i_k}, x_{i_k}\right\rangle - \delta^*\left(x'_{i_k}, \overline{\text{co}}\, \mathbb{E}(F)\right) \\
&\leq \left\langle x'_{i_k}, x_{i_k}\right\rangle - \left\langle x'_{i_k}, x\right\rangle \\
&\leq \|x_{i_k} - x\|.
\end{aligned}$$

holds, for any integer k. When we tend k to $+\infty$, we get $\mathrm{d}\,(x, \overline{\mathrm{co}}\,\mathbb{E}(F)) = 0$ which is absurd because $x \notin \overline{\mathrm{co}}\,\mathbb{E}(F)$.
Moreover, it follows from the equality (3.4) that for every integer $i \geq 1$,

$$\mathbb{E}\left(\delta^*(x_i', F(.))\right) = \delta^*\left((x_i', \overline{\mathrm{co}}\,\mathbb{E}(F)\right) < +\infty.$$

Therefore $\delta^*(x_i', F(.))$ is integrable. Birkhoff's theorem for integrable real-valued functions applied to the function $\delta^*(x_i', F)$ for each integer $i \geq 1$, implies the existence of a negligible N of $\mathcal{A}$ such that for every $\omega \in \Omega \setminus N$ and all $i \geq 1$ we have

$$\begin{aligned}\lim_{n\to\infty} \delta^*\left(x_i', \frac{1}{n}(\overline{F(\theta\omega) + \cdots + F(\theta^n\omega)})\right) &= \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \delta^*\left(x_i', F(\theta^j\omega)\right) \\ &= \mathbb{E}\left(\delta^*(x_i', F(.))\right) \\ &= \delta^*\left(x_i', \overline{\mathrm{co}}\,\mathbb{E}(F)\right).\end{aligned}$$

Put $H_n(\omega) = \frac{1}{n}(\overline{F(\theta\omega) + \cdots + F(\theta^n\omega)})$. If for $\omega \in \Omega \setminus N$, $y \in$ w-ls $H_n(\omega)$, then there exists $y_m \in H_{n_m}(\omega)$ such that

$$\langle x_i', y\rangle = \lim_{m\to\infty} \langle x_i', y_m\rangle \leq \lim_{m\to\infty} \delta^*\left(x_i', H_{n_m}(\omega)\right) = \delta^*\left(x_i', \overline{\mathrm{co}}\,\mathbb{E}(F)\right), \forall i \geq 1.$$

This implies, according to (3.5) that

$$y \in \overline{\mathrm{co}}\,\mathbb{E}(F)$$

and then that

$$\text{w-ls}\, H_n(\omega) \subset \overline{\mathrm{co}}\,\mathbb{E}(F) \quad \text{a.s.}$$

□

By applying Proposition 3.1 and Proposition 3.3 we obtain immediately the following version of multivalued Birkhoff ergodic theorem with respect to the Mosco convergence.

Theorem 3.4. *Let F be an integrable random set with values in $\mathcal{C}(E)$ and let θ be a measure-preserving transformation of $(\Omega, \mathcal{A}, P)$ such that, for each integer $m \geq 1$, θ^m is ergodic. Then*

$$\text{M-lim}\, \frac{1}{n}\left(\overline{F(\theta\omega) + \cdots + F(\theta^n\omega)}\right) = \overline{\mathrm{co}}\,\mathbb{E}(F) \quad a.s.$$

Remark 3.5. (1) The assumption : "θ^m ergodic for each $m \geq 1$", is required in the convexification technique established in Proposition 3.1. This type of technique is due to Artstein and Hart [1] initially in the proof of the strong law of large numbers for a sequence of an independent and identically distributed random sets in finite-dimentional space and taken up by Hiai [17] and still further by Hess [15] always in the demonstration of the multivalued strong law of large numbers but in infinite dimensional space.

(2) An equivalent condition to the ergodicity of θ is that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}P(\theta^{-n}(A)\cap B)=P(A)P(B) \tag{3.6}$$

holds for all $A,B\in\mathcal{A}$, ie the Cesaro average of the sequence $(P(\theta^{-n}(A)\cap B))_{n\geq 1}$ converges to $P(A)P(B)$. A sufficient condition for (3.6) be realized is of course that the sequence $(P(\theta^{-n}(A)\cap B))_{n\geq 1}$ converges itself to $P(A)P(B)$. We can then find the classical definition as θ be strong-mixing.

(3) It is obvious that

$$\theta \text{ is strong-mixing} \Longrightarrow \forall m\geq 1, \theta^m \text{ is ergodic} \Longrightarrow \theta \text{ is ergodic.}$$

In order to show that a given transformation θ is ergodic it is often easier to show that θ satisfies another property which then implies ergodicity. for examples of transformations that are ergodic but not strong-mixing can refer to Walters [26].

Before giving our version of the Birkhoff ergodic theorem with respect to the Wijsman convergence, we give in the following lemma the proof of the "lim inf" part.

Lemma 3.6. *Let F be an integrable random set with values in $\mathcal{C}(E)$ and let θ be an ergodic measure-preserving transformation of $(\Omega,\mathcal{A},P)$. Then there exists a negligible N such that*

$$\liminf_{n\to\infty} \mathrm{d}\left(x,\frac{1}{n}\left(\overline{F(\theta\omega)+\cdots+F(\theta^n\omega)}\right)\right)\geq \mathrm{d}(x,\overline{\mathrm{co}}\,\mathbb{E}(F))$$

holds, $\forall x\in E$, and $\forall\omega\in\Omega\setminus N$.

Proof. Resume for simplified notation : $H_n(\omega)=\frac{1}{n}(\overline{F(\theta\omega)+\cdots+F(\theta^n\omega)})$. Consider a dense sequence (x_i) in $E\setminus\overline{\mathrm{co}}\,\mathbb{E}(F)$. Let (x'_i) be a sequence in the unit sphere $S_{E'}$ as in (3.4) of the proof of Proposition 3.3 such that

$$\mathrm{d}\left(x_i,\overline{\mathrm{co}}\,\mathbb{E}(F)\right)=\langle x'_i,x_i\rangle-\delta^*\left(x'_i,\overline{\mathrm{co}}\,\mathbb{E}(F)\right).$$

Then for each integer $i\geq 1$, the support function $\delta^*(x'_i,F(.))$ is integrable. According to the Birkhoff theorem for integrable real-valued functions applied to the function $\delta^*(x'_i, F(.))$ for each integer $i \geq 1$, we obtain

$$\begin{aligned}
\mathrm{d}\left(x_i, \overline{\mathrm{co}}\,\mathbb{E}(F)\right) &= \langle x'_i, x_i\rangle - \delta^*\left(x'_i, \overline{\mathrm{co}}\,\mathbb{E}(F)\right) \\
&= \langle x'_i, x_i\rangle - \mathbb{E}\left(\delta^*(x'_i, F(.))\right) \\
&= \lim_{n\to\infty}\left[\langle x'_i, x_i\rangle - \frac{1}{n}\sum_{j=1}^{n}\delta^*\left(x'_i, F(\theta^j\omega)\right)\right] \quad \text{a.s.} \\
&= \lim_{n\to\infty}\left[\langle x'_i, x_i\rangle - \delta^*\left(x'_i, H_n(\omega)\right)\right] \quad \text{a.s.} \\
&\leq \liminf_{n\to\infty} \sup_{x'\in B_{E'}}\left[\langle x', x_i\rangle - \delta^*\left(x', H_n(\omega)\right)\right] \quad \text{a.s.} \\
&= \liminf_{n\to\infty} \mathrm{d}(x_i, \overline{\mathrm{co}} H_n(\omega)) \quad \text{a.s.} \\
&\leq \liminf_{n\to\infty} \mathrm{d}(x_i, H_n(\omega)) \quad \text{a.s.}
\end{aligned}$$

Since the distance function is Lipschitz and that the sequence $(x_i)_{i\geq 1}$ is dense in $E \setminus \overline{\mathrm{co}}\,\mathbb{E}(F)$, the previous inequality can be extended to $E \setminus \overline{\mathrm{co}}\,\mathbb{E}(F)$ and hence to E. □

Now, we present the result which gives a version of multivalued Birkhoff ergodic theorem with respect to Wijsman convergence

Theorem 3.7. *Let F be an integrable random set with values in $\mathcal{C}(E)$ and let θ be a measure-preserving transformation of $(\Omega, \mathcal{A}, P)$ such that, for each integer $m \geq 1$, θ^m is ergodic. Then*

$$\tau_{\mathrm{W}}\text{-lim}\,\frac{1}{n}\left(\overline{F(\theta\omega) + \cdots + F(\theta^n\omega)}\right) = \overline{\mathrm{co}}\,\mathbb{E}(F) \quad a.s.$$

Proof. From Proposition 3.1, there exists a negligible N such that

$$\overline{\mathrm{co}}\,\mathbb{E}(F) \subset \text{s-li}\, H_n(\omega)$$

holds $\forall\omega \in \Omega \setminus N$, where $H_n(\omega) = \frac{1}{n}(\overline{F(\theta\omega) + \cdots + F(\theta^n\omega)})$.
This implies

$$\limsup_{n\to\infty} \mathrm{d}(x, H_n(\omega)) \leq \mathrm{d}(x, \overline{\mathrm{co}}\,\mathbb{E}(F)), \quad \forall x \in E, \forall\omega \in \Omega \setminus N$$

Lemma 3.6 allows to conclude. □

3.2. **Convex case.** In this section we treat the case of a random set F with values in $\mathcal{C}_c(E)$, replacing the assumption of ergodicity on θ by the following assumption on conditional expectation $\mathbb{E}(F|\mathcal{T})$

Asumption (H)**.** There exists a sequence (x'_k) in the unit ball $B_{E'}$ of E' such that for almost all ω in Ω we have

$$\mathrm{D}(B, \mathbb{E}(F|\mathcal{T})(\omega)) = \sup_{k\geq 1}[-\delta^*(x'_k, \mathbb{E}(F|\mathcal{T})(\omega)) - \delta^*(-x'_k, B)]$$

for all $B \in \mathcal{F}_{cb}(E)$, (the negligible does not depend on B).

Remark 3.10 and Remark 3.11 indicate situations where it is made.

Lemma 3.8. *Let F be an integrable random set with values in $\mathcal{C}_c(E)$ and let θ be a measure-preserving transformation of $(\Omega, \mathcal{A}, P)$. If the assumption (H) is satisfied, then there exists a negligible N of Ω such that*

$$\liminf_{n\to\infty} \mathrm{D}\left(B, \frac{1}{n}\left(F(\theta\omega)\dot{+}\cdots\dot{+}F(\theta^n\omega)\right)\right) \geq \mathrm{D}(B, E(F|\mathcal{T})(\omega))$$

holds, $\forall B \in \mathcal{C}_{cb}(E)$, and $\forall \omega \in \Omega \setminus N$

Proof. Let N_1 be a negligible Ω such that for all $B \in \mathcal{C}_{cb}(E)$ and for all $\omega \in \Omega \setminus N_1$, we have

$$\mathrm{D}(B, \mathbb{E}(F|\mathcal{T})(\omega)) = \sup_{k\geq 1}[-\delta^*(x'_k, \mathbb{E}(F|\mathcal{T})(\omega)) - \delta^*(-x'_k, B)]. \tag{3.7}$$

The functions $\delta^*(x'_k, F(.))$ are quasi-integrable. It follows from Birkhoff ergodic theorem for quasi-integrable functions due to Choirat-Hess-Seri (see [9], Theorem 1), that for every $k \geq 1$ the following limits exist for almost all $\omega \in \Omega$ and verify

$$\begin{aligned} \lim_{n\to\infty} \delta^*\left(x'_k, \frac{1}{n}\left(F(\theta\omega)\dot{+}\cdots\dot{+}F(\theta^n\omega)\right)\right) &= \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \delta^*\left(x'_k, F(\theta^j\omega)\right) \\ &= \mathbb{E}(\delta^*(x'_k, F(.))|\mathcal{T})(\omega) \\ &= \delta^*(x'_k, |\,\mathbb{E}(F|\mathcal{T})(\omega)) \end{aligned} \tag{3.8}$$

Consider a negligible N_2 outside which equalities (3.8) are verified for every integer $k \geq 1$. Since the random sets $\frac{1}{n}\left(F(\theta\omega)\dot{+}\cdots\dot{+}F(\theta^n\omega)\right)$ are convex closed-valued, we deduce from (3.7), that for all $\omega \in \Omega \setminus (N_1 \cup N_2)$ and all $\mathcal{B} \in \mathcal{C}_{cb}(E)$ we have

$$\begin{aligned} &\liminf_{n\to\infty} \mathrm{D}\left(B, \frac{1}{n}\left(F(\theta\omega)\dot{+}\cdots\dot{+}F(\theta^n\omega)\right)\right) \\ &\quad = \liminf_{n\to\infty} \sup_{x'\in B_{E'}} \left[-\delta^*\left(x', \frac{1}{n}\left(F(\theta\omega)\dot{+}\cdots\dot{+}F(\theta^n\omega)\right)\right) - \delta^*(-x', B)\right] \\ &\quad \geq \sup_{x'\in B_{E'}} \liminf_{n\to\infty} \left[-\delta^*\left(x', \frac{1}{n}\left(F(\theta\omega)\dot{+}\cdots\dot{+}F(\theta^n\omega)\right)\right) - \delta^*(-x', B)\right] \\ &\quad \geq \sup_{k\geq 1} \liminf_{n\to\infty} \left[-\delta^*\left(x'_k, \frac{1}{n}\left(F(\theta\omega)\dot{+}\cdots\dot{+}F(\theta^n\omega)\right)\right) - \delta^*(-x'_k, B)\right] \\ &\quad = \sup_{k\geq 1} [-\delta^*(x'_k, \mathbb{E}(F|\mathcal{T})(\omega)) - \delta^*(-x'_k, B)] \\ &\quad = \mathrm{D}(B, \mathbb{E}(F|\mathcal{T})(\omega)). \end{aligned}$$

□

We can now state the main result of this subsection.

Theorem 3.9. *Let F be an integrable random set with values in $\mathcal{C}_c(E)$ and let θ be a measure-preserving transformation of $(\Omega, \mathcal{A}, P)$. If the assumption (H) is satisfied, then there exists a negligible N of Ω such that*

$$\tau_{\mathrm{S}}\text{-}\lim \frac{1}{n}\left(F(\theta\omega)\dot{+}\cdots\dot{+}F(\theta^n\omega)\right) = \mathbb{E}(X|\mathcal{T})(\omega)) \tag{3.9}$$

holds for all $\omega \in \Omega \setminus N$.

Proof. First, using the definition of the the gap functional and Proposition 3.2, there exists a negligible N such that for any $\omega \in \Omega \setminus N$ and any $B \in \mathcal{C}_{cb}(E)$, we have

$$\begin{aligned}
\limsup_{n\to\infty} \mathrm{D}\left(B, \frac{1}{n}\left(F(\theta\omega)\dot{+}\cdots\dot{+}F(\theta^n\omega)\right)\right)& \\
&= \limsup_{n\to\infty} \inf_{x\in B} \mathrm{d}\left(x, \frac{1}{n}\left(F(\theta\omega)\dot{+}\cdots\dot{+}F(\theta^n\omega)\right)\right) \\
&\leq \inf_{x\in B} \limsup_{n\to\infty} \mathrm{d}\left(x, \frac{1}{n}\left(F(\theta\omega)\dot{+}\cdots\dot{+}F(\theta^n\omega)\right)\right) \\
&\leq \inf_{x\in B} \mathrm{d}(x, \mathbb{E}(X|\mathcal{T})(\omega)) \\
&= \mathrm{D}(B, \mathbb{E}(F|\mathcal{T})(\omega))
\end{aligned}$$

Now, (3.9) is obtained by Lemma 3.8 and the characterization of the slice-topology τ_S reported in the introduction. □

Remark 3.10. When the space E' is separable, the assumption (H) above is met in the following two cases (i) or (ii):

(i) The random set $E(X|\mathcal{T})$ is almost surely constant in $\mathcal{C}_c(E)$ (see [14], lemma 3.9).

(ii) The values of the random set $E(X|\mathcal{T})$ belong to class $\mathfrak{C}$ defined by

$$\mathfrak{C} = \{C \in \mathcal{C}_c(E) : \exists\, F \text{ a closed half-space of } E\,;\ F \cap C \text{ is borned}\}\,.$$

This will be checked if exists $C \in \mathfrak{C}$ such that $X(\omega) \in C$ for almost all $\omega \in \Omega$. Indeed, recall that the class $\mathfrak{C}$ admits the following dual characterization (see [21] corollary 8.e).

$$\{C \in \mathcal{C}_c(E) : \delta^*(.,C) \text{ is finite and } \|.\| \text{-continuous at some point of } E'\}\,.$$

Remark 3.11. When the space E' is not assumed separable, convergence (3.9) holds if we assume that the random set $E(X|\mathcal{T})$ is almost surely constant in $\mathcal{C}_c(E)$ and there exists a fixed element $R \in \mathcal{R}_{cwk}(E)$ such that for almost all $\omega \in \Omega$, the inclusion $X(\omega) \subset R$ is verified. Indeed, this implies that $E(X|\mathcal{T})$ is a random set almost surely constant and its value belongs to $\mathcal{R}_{cwk}(R)$. Under these conditions, Lemma 3.10 of Hess [14] ensures the existence of a sequence $(x'_k)_{k\geq 1}$ dense in the unit ball $B_{E'}$ for the Mackey-topology $\tau(E', E)$, and satisfying, for almost all $\omega \in \Omega$.

$$\mathrm{D}(B, \mathbb{E}(F|\mathcal{T})(\omega)) = \sup_{k\geq 1}[-\delta^*(x'_k, \mathbb{E}(F|\mathcal{T})(\omega)) - \delta^*(-x'_k, B)],\ \forall B \in \mathcal{C}_{cwk}(R).$$

Taking the same method as in the proof of Theorem 3.9, we can show that there is a negligible N of Ω such that for any $B \in \mathcal{C}_{cwk}(R)$ and that for every $\omega \in \Omega \setminus N$ one has

$$\lim_{n\to\infty} \mathrm{D}\left(B, \frac{1}{n}\left(F(\theta\omega)\dot{+}\cdots\dot{+}F(\theta^n\omega)\right)\right) = \mathrm{D}(B, \mathbb{E}(F|\mathcal{T})(\omega)).$$

This is none other than the almost sure convergence of $\frac{1}{n}\left(F(\theta\omega)\dot{+}\cdots\dot{+}F(\theta^n\omega)\right)$ to $\mathbb{E}(X|\mathcal{T})$, with respect to the Slice-topology τ_S.

4. Application: strong law of large numbers

Before establishing the multivalued strong law of large numbers, we will make some useful notations for simplicity. For the sequence of random sets $(Y_j)_{j\in\mathbb{N}}$ with values in $\mathcal{C}(E)$, we denote by $Y = (Y_0, Y_1, \cdots)$. We can write $Y_j = X_j \circ Y$, where X_j is is the $(j+1)$-th coordinate map defined from $\prod_{j\in\mathbb{N}} \mathcal{C}(E)$ into $\mathcal{C}(E)$ by $X_j(F_0, F_1, \cdots) = F_j$. Y is considered as a measurable map from $(\Omega, \mathcal{A}, P)$ to the measurable space $\left(\prod_{j\in\mathbb{N}} \mathcal{C}(E), \otimes_{j\in\mathbb{N}}\mathcal{E}\right)$ and have the distribution μ_Y uniquely determined by the finitedimensional marginal distributions i.e. μ_Y is the unique measure on the measurable space $\left(\prod_{j\in\mathbb{N}} \mathcal{C}(E), \otimes_{j\in\mathbb{N}}\mathcal{E}\right)$ such that

$$\mu_Y\left(\bigcap_{j\in J} X_j^{-1}(A_j)\right) = P\left(\bigcap_{j\in J} Y_j^{-1}(A_j)\right)$$

holds for all finite $J \subset \mathbb{N}$ and all choices of $A_j \in \mathcal{E}$.

When $(Y_j)_{j\in\mathbb{N}}$ is a sequence of independent random sets having the same distribution μ, the distribution of Y is the product measure $\mu_Y = \otimes_{j\in\mathbb{N}}\,\mu$ i.e. is the unique probability measure on $\prod_{j\in\mathbb{N}} \mathcal{C}(E)$ such that

$$\mu_Y\left(\bigcap_{j\in J} X_j^{-1}(A_j)\right) = \prod_{j\in J} \mu(A_j)$$

The following lemma is an application of the "change of variables" theorem in Multivalued case .

Lemma 4.1. *Consider a sequence $Y = (Y_0, Y_1, \cdots)$ of independent random sets having the same distribution. Suppose that $\mathbb{E}(Y_0) \neq \emptyset$. Then, X_0 is $\otimes_{j\in\mathbb{N}}\,\mu$-integrable and for almost every $\omega \in \Omega$, one has*

$$\overline{\mathrm{co}}\,\mathbb{E}(X_0) = \overline{\mathrm{co}}\,\mathbb{E}(Y_0).$$

Proof. Choose $f \in S^1_{Y_0}(\mathcal{A}_{Y_0})$. Since f is $\mathcal{A}_{Y_0}$-measurable and E is complete, one can find a map u from $\mathcal{C}(E)$ into E, that is measurable with respect to the σ-field $\mathcal{E}$ and the Borel σ-field $\mathcal{B}(E)$ and satisfies $f = u \circ Y_0$ (see e.g. [10], p.18). Define g from the probability space $\left(\prod_{j\in\mathbb{N}} \mathcal{C}(E), \otimes_{j\in\mathbb{N}} \mathcal{E}, \otimes_{j\in\mathbb{N}} \mu\right)$ into E by $g = u \circ X_0$. It is clear that $f = g \circ Y$. Now, by virtue of the "change of variables" theorem (see e.g. [10], p. 31) we have:

$$\begin{aligned}\int_{\prod_{j\in\mathbb{N}} \mathcal{C}(E)} \mathrm{d}\left(g(.), X_0(.)\right) d\mu_Y &= \int_\Omega \mathrm{d}\left(g \circ Y(.), X_0 \circ Y(.)\right) dP \\ &= \int_\Omega \mathrm{d}\left(f(.), Y_0(.)\right) dP \\ &= 0.\end{aligned}$$

hence

$$\mathrm{d}(g(.), X_0(.)) = 0 \quad \text{a.s.}$$

This implies that $g \in S^1_{X_0}$ and shows that X_0 is $\otimes_{j\in\mathbb{N}} \mu$-integrable.
It follows that for all $x' \in E'$, we have

$$\delta^*\left(x', \overline{\mathrm{co}}\,\mathbb{E}(Y_0)\right) = \sup\left\{\int_\Omega \langle x', f\rangle \, dP : f \in S^1_{Y_0}(\mathcal{A}_{Y_0})\right\} \tag{4.1}$$

$$\leq \sup\left\{\int_{\prod_{j\in\mathbb{N}} \mathcal{C}(E)} \langle x', g\rangle \, d\mu_Y : g \in S^1_{X_0}\right\} \tag{4.2}$$

$$= \delta^*\left(x', \overline{\mathrm{co}}\,\mathbb{E}(X_0)\right). \tag{4.3}$$

The converse inequality of (4.2) is proved similarly as before. Taking g in $S^1_{X_0}(\mathcal{A}_{X_0})$, g can be written as $g = v \circ X_0$, where v is a measurable map from $\mathcal{C}(E)$ into E with respect to the σ-field $\mathcal{E}$ and the Borel σ-field $\mathcal{B}(E)$. We deduce in the same way as before that $f = v \circ Y_0$ belongs to $S^1_{Y_0}$. □

From the results we obtained, we find the multivalued strong law of large numbers obtained by Hess [13, 15] and Hiai [17] in the following theorem

Theorem 4.2. *Consider a sequence $Y = (Y_0, Y_1, \cdots)$ of independent random sets having the same distribution. Suppose that $\mathbb{E}(Y_0) \neq \emptyset$. Then, for almost every $\omega \in \Omega$, one has*

(i) $\text{M-lim}\, \frac{1}{n}\left(\overline{Y_1(\omega) + \cdots + Y_n(\omega)}\right) = \overline{\mathrm{co}}\,\mathbb{E}(Y_0)$.

(ii) $\tau_{\mathrm{W}}\text{-lim}\, \frac{1}{n}\left(\overline{Y_1(\omega) + \cdots + Y_n(\omega)}\right) = \overline{\mathrm{co}}\,\mathbb{E}(Y_0)$.

Proof. We consider first the unilateral Bernoulli shift θ defined on the probability space $\left(\prod_{j\in\mathbb{N}} \mathcal{C}(E), \otimes_{j\in\mathbb{N}} \mathcal{E}, \otimes_{j\in\mathbb{N}} \mu\right)$ by

$$X_k(\theta(F_0, F_1 \cdots)) = X_{k+1}(F_0, F_1 \cdots) = F_{k+1}. \tag{4.4}$$

It follows that θ is strong-mixing (see e.g. [19], p. 24) and thus θ^m is ergodic for every $m \geq 1$.
According to (4.4), we have

$$X_k = X_0 \circ \theta^k \text{ and } \ Y_k = X_k \circ Y. \tag{4.5}$$

Consider the probability space $\left(\prod_{j\in\mathbb{N}} \mathcal{C}(E), \otimes_{j\in\mathbb{N}} \mathcal{E}, \otimes_{j\in\mathbb{N}} \mu\right)$ and apply Theorem 3.4 for X_0. There exists $\mathbf{K} \in \otimes_{j\in\mathbb{N}} \mathcal{E}$ such that $(\otimes_{j\in\mathbb{N}} \mu)(\mathbf{K}) = 1$ and

$$\text{M-lim}\, \frac{1}{n}\overline{(X_0(\theta(F_0, F_1, \cdots)) + \cdots + X_0(\theta^n(F_0, F_1, \cdots)))} = \overline{\text{co}}\, \mathbb{E}(X_0)$$

holds for all $(F_j)_{j\in\mathbb{N}} \in \mathbf{K}$. Now, we have $P(Y^{-1}(\mathbf{K})) = 1$. It follows that for all $\omega \in Y^{-1}(\mathbf{K})$

$$\text{M-lim}\, \frac{1}{n}\overline{(X_0(\theta(Y(\omega))) + \cdots + X_0(\theta^n(Y(\omega)))} = \overline{\text{co}}\, \mathbb{E}(X_0).$$

From (4.5), we obtain

$$\text{M-lim}\, \frac{1}{n}\overline{(Y_1(\omega) + \cdots + Y_n(\omega))} = \overline{\text{co}}\, \mathbb{E}(X_0)$$

for all $\omega \in Y^{-1}(\mathbf{K})$. Lemma 4.1 allows to conclude the proof of (i). The proof of (ii) is identical to that of (i) using this time Theorem 3.7. □

We can establish in the following result, similar as previously, the multivalued strong law of large numbers with respect to the Slice-topology, using the theorem 3.9.

Theorem 4.3. *Consider a sequence $Y = (Y_0, Y_1, \cdots)$ of independent convex random sets having the same distribution. Suppose that $\mathbb{E}(Y_0) \neq \emptyset$ and one of two conditions (i) or (ii) is satisfied:*

(i) E' is separable.

(ii) There exists $R \in \mathcal{R}_{cwk}(E)$ such that for almost all $\omega \in \Omega$, we have $X_1(\omega) \subset R$.

Then, for almost every $\omega \in \Omega$, one has

$$\tau_S\text{-lim}\, \frac{1}{n}\left(Y_1(\omega) \dot{+} \cdots \dot{+} Y_n(\omega)\right) = \overline{\mathbb{E}(Y_0)}.$$

Remark 4.4. Like for real or vector valued random variables, we can say that even in the special case of a Bernoulli shift our version of the multivalued Birkhoff ergodic theorem is strictly more informative than the multivalued strong law of large numbers. The main usefulness of Theorem 3.4, Theorem 3.7 and Theorem 3.9, is due to the fact that there are many processes $Y = (Y_0, Y_1, \cdots)$ for which μ_Y is θ-invariant although the $Y_0, Y_1, \cdots$ are not independant.

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