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On Absolutely n-th Continuous Functions

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Abstract

There are two different approaches in the definition of absolutely n-th continuous function. It is shown that these two approaches give the same class of functions on the closed intervals. Necessary and sufficient conditions under which a function is absolutely n-th continuous function on closed intervals are obtained for each approach. It is also shown that the two n-th variation norms give the same topology on the space of all absolutely n-th continuous functions.

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1 Introduction

There are two different approaches to introduce the concepts of functions of bounded variation and absolute continuity of higher order. One approach is based on the concept of higher order divided difference. This was followed by Russell [5, 6], Das and Lahiri [2] and others (see, for example, [3] and [9]). Another approach is based on the notion of generalized derivatives. This was followed by Sargent [11], Mukhopadhyay and Sain [4]. Though, the two approaches are different, it is proved in [4] that the two definitions of function

of bounded n-th variation are equivalent on closed interval. Therefore, it is natural to search a connection between the two approaches of absolute continuity of higher order. In this paper a connection between the two approaches is established by obtaining necessary and sufficient conditions under which a function is absolutely n-th continuous. Actually, it is shown that these definitions when considered on closed interval are equivalent.
It may be recalled that each absolutely n-th continuous function is also a function of bounded n-th variation and it was pointed out in [8, p.233] that no relation is known between the two variations of the same function, one introduced by Russell [5] and the other introduced by Mukhopadhyay and Sain [4]. In this paper a relation between the two variations of an absolutely n-th continuous function is obtained and it is shown that the two n-th variation norms define the same topology on the space of all absolutely n-th continuous functions.

2 Preliminary Notes

Let f be a real-valued function defined in some neighbourhood of the point x on the real line. If there are real numbers $\alpha_0(= f(x)), \alpha_1, ..., \alpha_r$ depending on x but not on h such that

$$f(x+h) = \sum_{i=0}^{r} \alpha_i \frac{h^i}{i!} + o(h^r), \qquad (h \longrightarrow 0)$$

then α_r is called the Peano derivative of f at x of order r and is denoted by $f_{(r)}(x)$. Clearly, if $f_{(r)}(x)$ exists then $f_{(i)}(x)$ exists for all i, $1 \le i < r$. Also, if the ordinary r^{th} derivative $f^{(r)}(x)$ exists, then $f_{(r)}(x)$ exists and equal to $f^{(r)}(x)$. The converse is true only for $r = 1$.
Let $f_{(r)}(x)$ exist. If

$$\frac{(r+1)!}{(t-x)^{r+1}} \left[f(t) - \sum_{i=0}^{r} f_{(i)}(x) \frac{(t-x)^i}{i!} \right],$$

tends to a limit as $t \to x^+$, then this limit is called the right-hand Peano derivative of f at x of order $(r+1)$ and is denoted by $f^+_{(r+1)}(x)$. The left-hand Peano derivative $f^-_{(r+1)}(x)$ is defined similarly.
Let $n \ge 1$ be a fixed integer and let $f^+_{(n)}(x)$ exist and be finite. Define

$$\epsilon^+_n(f, x, t) = \begin{cases} \frac{n!}{(t-x)^n} \left[f(t) - \sum_{i=0}^{n-1} \frac{(t-x)^i}{i!} f_{(i)}(x) - \frac{(t-x)^n}{n!} f^+_{(n)}(x) \right] & \text{if } t \ne x \\ 0 & \text{if } t = x \end{cases}$$

Similarly if $f^-_{(n)}(x)$ exists (always in finite sense), define $\epsilon^-_n(f, x, t)$ as above replacing $f^+_{(n)}(x)$ by $f^-_{(n)}(x)$. If however $f_{(n)}(x)$ exists, we shall write $\epsilon_n(f, x, t)$

for $\epsilon_n^+(f,x,t)$ and $\epsilon_n^-(f,x,t)$.
Let us suppose that f be defined in the closed interval $[a,b]$ and let $a \leq c < d \leq b$. If $f_{(n)}^+(c)$ and $f_{(n)}^-(d)$ exist, define

$$\begin{aligned}
\bar{\omega}_n^*(f,\ [c,d]) &= \max[\sup_{c\leq t\leq d} \epsilon_n^+(f,c,t),\ \ \sup_{c\leq t\leq d}\{-\epsilon_n^-(f,d,t)\}] \\
\underline{\omega}_n^*(f,\ [c,d]) &= \min[\inf_{c\leq t\leq d} \epsilon_n^+(f,c,t),\ \ \inf_{c\leq t\leq d}\{-\epsilon_n^-(f,d,t)\}] \\
\omega_n^*(f,\ [c,d]) &= \bar{\omega}_n^*(f,\ [c,d]) - \underline{\omega}_n^*(f,\ [c,d]).
\end{aligned}$$

Supposing the existence of $f_{(n)}(c)$ and $f_{(n)}(d)$, Sargent [11] defined

$$\omega_n(f,\ [c,d]) = \max[\sup_{c\leq t\leq d} |\epsilon_n(f,c,t)|,\ \ \sup_{c\leq t\leq d} |\epsilon_n(f,d,t)|].$$

Definition 2.1 ([4, p.193]) *Let f_n^+, f_n^+ exist on a set $E \subset [a,b]$. The strong n-th variation of f on E, denoted by $V_n^*(f,E)$, is the upper bound of the sums $\sum_i \omega_n^*(f,[c_i,d_i])$ where the summation is taken over all sequences $\{[c_i,d_i]\}$ of non-overlapping intervals whose end points belong to E. If $V_n^*(f,E) < \infty$, then f is said to be of bounded n-th variation in the restricted sense, briefly V_nB^*, on E and the class of all functions of bounded n-th variations on E is denoted by $V_nB^*(E)$.*

Definition 2.2 ([8, p.232]) *A function f is said to be absolutely n-th continuous in the restricted sense, briefly AC_n^*, on a set $E \subset [a,b]$, written $f \in AC_n^*(E)$, if $f_{(n)}$ exists on E and for every $\epsilon > 0$ there is a $\delta > 0$ such that $\sum_i \omega_n^*(f,\ [c_i,d_i]) < \epsilon$ for every sequence of non-overlapping intervals $\{[c_i,d_i]\}$ with end points on E such that $\sum_i (d_i - c_i) < \delta$.*

Definition 2.3 ([11, p.366]) *A function f is said to have an n-th generalized derivative $f_{(n)}(x)$ which is V_n-AC^* over a bounded set E, written $f \in V_n$-$AC^*(E)$, if $f_{(n)}$ exists at all points of an interval containing E and to each positive number ϵ, there corresponds a number $\delta > 0$ such that*

$$\sum_{i=1}^{m} \omega_n(f,\ [a_i,b_i]) < \epsilon$$

for all finite set of non-overlapping intervals $(a_1,b_1),\ (a_2,b_2),\ (a_3,b_3), ..., (a_m,b_m)$ with end points on E and such that $\sum_{i=1}^m (b_i - a_i) < \delta$.

Definition 2.4 *Let f be defined on a subset E of real numbers. Let $x_0,\ x_1, ...,\ x_n$ be $n+1$ distinct points in E, not necessarily in the linear order. The n-th divided difference of f at these points is defined by*

$$Q_n(f,\ x_0,\ x_1, ..., x_n) = \sum_{i=0}^{n} [f(x_i) / \prod_{j=0, j\neq i}^{n} (x_i - x_j)].$$

Definition 2.5 ([5, p.160]) *Let f be defined on $[a,b]$ and let $a \leq x_0 < x_1 < x_2 < ... < x_m \leq b$, $m \geq n$ be any subdivision of $[a,b]$. The total n-th variation of f on $[a,b]$ is defined by*

$$V_n[f,\ a,b] = \sup \sum_{i=0}^{m-n} (x_{i+n} - x_i)|Q_n(f, x_i, ..., x_{i+n})|$$

where sup *is taken over all such sub-divisions. If $V_n[f,a,b] < \infty$, then f is said to be of bounded n-th variation on $[a,b]$ and is written $f \in BV_n[a,b]$.*

Definition 2.6 ([2, p.160]) *The real valued function $f(x)$ defined on $[a,b]$ is said to be absolutely n-th continuous on $[a,b]$, written $f \in AC_n([a,b])$, if for any arbitrary $\epsilon > 0$ there is $\delta > 0$ such that for any sub-division of the form $a \leq x_{1,0} < x_{1,1} < .... < x_{1,n-1} < x_{1,n} \leq x_{2,0} < x_{2,1} < .... < x_{2,n-1} < x_{2,n} \leq ... \leq x_{m,0} < x_{m,1} < ... < x_{m,n-1} < x_{m,n} \leq b$ of $[a,b]$ with $\sum_{i=1}^{m}(x_{i,n} - x_{i,0}) < \delta$, the inequality*

$$\sum_{i=1}^{m} (x_{i,n} - x_{i,0})|Q_n(f, x_{i,0}, x_{i,1}, ..., x_{i,n})| < \epsilon$$

holds.

Definition 2.7 ([3, p.92]) *Let f be a real-valued function defined on a set E. Let $c,\ d \in E$ and $c < d$. The oscillation of f on $[c,d] \cap E$ of order n is defined to be*

$$O_n(f,\ [c,d] \cap E) = \sup |(d-c)Q_n(f, c, x_1, ...x_{n-1}, d)|$$

where sup is taken over all possible choices of the points $x_1, x_2, ..., x_{n-1}$ in $(c,d) \cap E$.

Definition 2.8 ([3, p.92]) *If for every $\epsilon > 0$, there is a $\sigma > 0$ such that for every sequence of non-overlapping intervals $\{(c_\nu, d_\nu)\}$ with end points on E and with $\sum_\nu (d_\nu - c_\nu) < \sigma$, we have*

$$\sum_\nu O_n(f\ , [c_\nu, d_\nu] \cap E) < \epsilon$$

then f is said to be absolutely n-th continuous on E in the wide sense and is written $f \in AC_n^\omega(E)$.

Remark 2.9 *These definitions of bounded variation and absolute continuity of higher order are analogous to those defined in [10, pp 221-231].*

3 Class of absolutely continuous functions of higher order defined in terms of generalized derivatives.

Theorem 3.1 *A function f is V_n-AC^* over $[a,b]$ if and only if f is AC_n^* on $[a,b]$.*

Proof 3.2 *Clearly for $a \leq c < d \leq b$,*

$$\omega_n^*(f,\ [c,d]) \leq 2\omega_n(f,\ [c,d]) \tag{1}$$

Again

$$\begin{aligned}\sup_{c\leq t\leq d} |\epsilon_n(f,c,t)| &= \max[\sup_{c\leq t\leq d} \epsilon_n(f,c,t),\ \sup_{c\leq t\leq d}\{-\epsilon_n(f,c,t)\}] \\ &\leq \sup_{c\leq t\leq d} \epsilon_n(f,c,t) + \sup_{c\leq t\leq d}\{-\epsilon_n(f,c,t)\} \\ &\leq \bar{\omega}_n^*(f,\ [c,d]) - \underline{\omega}_n^*(f,\ [c,d]) \\ &= \omega_n^*(f,\ [c,d])\end{aligned}$$

Similarly

$$\sup_{c\leq t\leq d} |\epsilon_n(f,d,t)| \leq \omega_n^*(f,\ [c,d]).$$

Hence

$$\omega_n(f,\ [c,d]) \leq \omega_n^*(f,\ [c,d]) \tag{2}$$

Combining (1) and (2), we get

$$\omega_n(f,\ [c,d]) \leq \omega_n^*(f,\ [c,d]) \leq 2\omega_n(f,\ [c,d]) \tag{3}$$

Since (3) is true for every sub-interval $[c,d] \subset [a,b]$, it follows that

$$f \in V_n\text{-}AC^*([a,b]) \quad \text{if and only if} \quad f \in AC_n^*([a,b]).$$

Lemma 3.3 *If f is absolutely continuous on $[a,b]$ then $F \in AC_1^*([a,b])$, where $F(x) = \int_a^x f(t)dt$. If $f \in AC_n^*([a,b])$, $n \geq 1$ then $F \in AC_{n+1}^*([a,b])$.*

Proof 3.4 *Clearly F' exists and is equal to f on $[a,b]$. Therefore if $[c,d] \subset [a,b]$ then for $c < x \leq d$ we have*

$$\begin{aligned}\epsilon_1(F,c,x) &= \frac{1}{x-c}[F(x) - F(c) - (x-c)f(c)] \\ &= \frac{1}{x-c}\int_c^x [f(t) - f(c)]dt.\end{aligned}$$

Let $O(f,c,d)$ denote the oscillation of f on $[c,d]$. Then

$$|\epsilon_1(F,c,x)| \leq O(f,c,d).$$

Similarly for $c \leq x < d$ we have

$$|\epsilon_1(F,d,x)| \leq O(f,c,d)$$

and hence

$$\omega_1^*(F,[c,d]) \leq 2O(f,c,d).$$

Let $\epsilon > 0$ be arbitrary. Then there is a $\delta > 0$ such that for every sequence of non-overlapping intervals $\{[c_\nu,d_\nu]\}$ with $\sum_\nu (d_\nu - c_\nu) < \delta$ we have

$$\sum_\nu O(f,c_\nu,d_\nu) < \epsilon/2$$

and so

$$\sum_\nu \omega_1^*(f,c_\nu,d_\nu) < \epsilon$$

Thus the first part is proved. The proof of the second part follows that of [11, Lemma 4] and Theorem 3.1.

Lemma 3.5 *If $f \in AC_n^*([a,b])$ then $f^{(n)}$ exists and is absolutely continuous on $[a,b]$.*

Proof 3.6 *Let $[c,d] \subset [a,b]$. Then proceeding similarly as [4, Lemma 3.1] it can be proved that*

$$|f_{(n)}(d) - f_{(n)}(c)| \leq K\omega_n^*(f,\ [c,d])$$

where K is a constant depending only on n. Hence if $f \in AC_n^*([a,b])$ then it is clear from above that $f_{(n)}$ is absolutely continuous on $[a,b]$. So, by [12, Theorem 1] $f^{(n)}$ exists and is equal to $f_{(n)}$.

The following theorem follows from Lemma 3.5 and repeated application of Lemma 3.3.

Theorem 3.7 *A function $f \in AC_n^*([a,b])$ if and only if $f^{(n)}$ exists and is absolutely continuous on $[a,b]$.*

4 Class of absolutely continuous functions of higher order defined in terms of divided differences

Lemma 4.1 *If $f \in AC_n^\omega([a,b])$ then $f^{(n-1)}$ exists and is absolutely continuous on $[a,b]$.*

The proof of this Lemma follows that of [3, Theorem 6].

Lemma 4.2 *Let $f_{(n-1)}$ exists finitely on $[a,b]$ and $f_{(n)}$ exist, possibly infinite in (a,b), then for every $(n+1)$ tuple of distinct points $x_i,\ 0 \leq i \leq n$ in $[a,b]$ there is a point ξ, $\min_i x_i < \xi < \max_i x_i$ such that*

$$n!Q_n(f, x_0, x_1, ..., x_n) = f_{(n)}(\xi).$$

This is Theorem 8 of [1].

Lemma 4.3 *If $f \in AC_n^\omega([a,b])$ and $F(x) = \int_a^x f(t)dt$, then $F \in AC_{n+1}^\omega([a,b])$.*

Proof 4.4 *Since $f \in AC_n^\omega([a,b])$, by Lemma 4.1, $f^{(n-1)}$ exists and is absolutely continuous on $[a,b]$. So, given $\epsilon > 0$, there is a $\delta > 0$ such that for every sequence $\{(c_\nu, d_\nu)\}$ of pair wise disjoint intervals with $\sum_\nu (d_\nu - c_\nu) < \delta$ we have*

$$\sum_\nu O(f^{n-1}, \xi_\nu, \eta_\nu) < \epsilon$$

whenever $\xi_\nu, \eta_\nu \in [c_\nu, d_\nu]$. Let $\{(c_\nu, d_\nu)\}$ be such a sequence of intervals and let $c_\nu < x_{\nu,1} < x_{\nu,2} < .... < x_{\nu,n} < d_\nu$. Since $F^{(n)}$ exists on $[a,b]$ therefore by Lemma 4.2 there are $\xi_\nu, \eta_\nu \in (c_\nu, d_\nu)$ such that

$$n!Q_n(F, c_\nu, x_{\nu,1}x_{\nu,2}, ..., x_{\nu,n-1}, x_{\nu,n}) = F^{(n)}(\xi_\nu)$$

and

$$n!Q_n(F, x_{\nu,1}x_{\nu,2}, ..., x_{\nu,n-1}, x_{\nu,n}, d_\nu) = F^{(n)}(\eta_\nu).$$

Since $F^{(n)} = f^{(n-1)}$, we have

$$\begin{aligned}
&|(d_\nu - c_\nu)Q_{n+1}(F, c_\nu, x_{\nu,1}x_{\nu,2}, ..., x_{\nu,n-1}, x_{\nu,n}, d_\nu)| \\
= \ & |Q_n(F, x_{\nu,1}x_{\nu,2}, ..., x_{\nu,n-1}, x_{\nu,n}, d_\nu) - Q_n(F, c_\nu, x_{\nu,1}x_{\nu,2}, ..., x_{\nu,n-1}, x_{\nu,n})| \\
= \ & \frac{1}{n!}|f^{(n-1)}(\eta_\nu) - f^{(n-1)}(\xi_\nu)| \\
\leq \ & O(f^{(n-1)}, c_\nu, d_\nu)
\end{aligned}$$

Hence

$$O_{n+1}(F, [c_\nu, d_\nu]) \leq O(f^{n-1}, c_\nu, d_\nu).$$

Therefore

$$\sum_{\nu} O_{n+1}(F, [c_\nu, d_\nu]) < \epsilon$$

and so

$$F \in AC^{\omega}_{n+1}([a, b]).$$

By repeated application of Lemma 4.3 we can prove

Lemma 4.5 *If $f^{(n-1)}$ exists and is absolutely continuous on $[a, b]$ ten $f \in AC^{\omega}_n([a, b])$.*

Combining Lemma 4.1 and Lemma 4.5 we get

Theorem 4.6 *A function $f \in AC^{\omega}_n([a, b])$ if and only if f^{n-1} exists and is absolutely continuous on$[a, b]$.*

In a similar manner we can prove that

Theorem 4.7 *A function $f \in AC_n([a, b])$ if and only if f^{n-1} exists and is absolutely continuous on$[a, b]$.*

5 Equivalence of the two concepts of absolute continuity of higher order

From Theorems 3.1, 3.7, 4.6,4.7 we have

Theorem 5.1 *For each positive integer n,*

$$AC^*_n([a, b]) = V_n\text{-}AC^*([a, b]) = AC^{\omega}_{n+1}([a, b]) = AC_{n+1}([a, b])$$

Theorem 5.1 shows that when considered on closed intervals the concepts of AC^* and AC of higher order are equivalent. The apparent difference in the order does not matter, since Sargent [11] introduced the scale from $n = 0$ while Das and Lahiri [2] started it from $n = 1$.
The result of Theorem 5.1 is true for the usual AC^* and AC ([10, pp 221,231]).

6 Space of absolutely n-th continuous functions

The space $AC^*_n([a, b])$ is studied in [8] with respect to the norm $\|.\|^*$ defined by

$$\|f\|^* = \sum_{i=0}^{n} |f^{(i)}_+(a)| + V^*_n(f, [a, b]).$$

On the other hand, in view of Theorem 5.1, one can study the same space under the norm

$$\|f\| = \sum_{i=0}^{n} |f_+^{(i)}(a)| + V_{n+1}[f, a, b]$$

introduced by Russell [7].

In this section, we will show that these two norms define the same topology on $AC_n^*([a,b])$. To show it, we first prove the following lemma.

Lemma 6.1 *If $f \in AC_n^*([a,b])$ then*

$$\frac{1}{2.n!} V_n^*(f, [a,b]) \leq V_{n+1}[f, a, b] \leq \frac{K}{n!} V_n^*(f, [a,b])$$

where $K = 2\sum_{r=0}^{n} \begin{pmatrix} n \\ r \end{pmatrix} \frac{r^n}{n!}$.

Proof 6.2 *Since $f \in AC_n^*([a,b])$, by Lemma 3.5, $f^{(n)}$ exists and is absolutely continuous on $[a,b]$.*

Now by Taylor's Theorem, if $t \in (c,d) \subset [a,b]$ we have

$$\epsilon_n(f, c, t) = f^{(n)}(\xi) - f^{(n)}(c), \qquad c < \xi < t$$

and

$$\epsilon_n(f, d, t) = f^{(n)}(\eta) - f^{(n)}(d), \qquad t < \eta < d$$

and so

$$\omega_n^*(f, [c,d]) \leq 2O(f^{(n)}, c, d).$$

Hence

$$V_n^*(f, [a,b]) \leq 2V(f^{(n)}, a, b) \tag{4}$$

where $V(f^{(n)}, a, b)$ denotes the usual variation of $f^{(n)}$ on $[a,b]$.

Again we have mentioned in the Proof 3.6 of Lemma 3.5 that

$$|f^{(n)}(d) - f^{(n)}(c)| \leq K\omega_n^*(f, [a,b])$$

for every $[c,d] \subset [a,b]$. Hence

$$V(f^{(n)}, a, b) \leq KV_n^*(f, [a,b]) \tag{5}$$

On the other hand, since $f \in AC_n^*([a,b])$, therefore by Theorem 5.1, $f \in AC_{n+1}([a,b])$ and hence by [6, Theorem 8]

$$n!V_{n+1}[f, a, b] = V(f^{(n)}, a, b) \tag{6}$$

Combining (4), (5) and (6) we get

$$\frac{1}{2.n!} V_n^*(f, [a,b]) \leq V_{n+1}[f, a, b] \leq \frac{K}{n!} V_n^*(f, [a,b]).$$

Theorem 6.3 *The two norms* $\|.\|^*$ *and* $\|.\|$ *define the same topology on* $AC_n^*([a,b])$.

Proof 6.4 *By Lemma 6.1, we have*

$$\begin{aligned} \|f\| &= \sum_{i=0}^{n} |f_+^{(i)}(a)| + V_{n+1}[f,a,b] \\ &\le \sum_{i=0}^{n} |f_+^{(i)}(a)| + \frac{K}{n!} V_n^*(f,[a,b]) \\ &\le K\|f\|^* \end{aligned}$$

Again using the Lemma 6.1, we have

$$\begin{aligned} \|f\| &\ge \sum_{i=0}^{n} |f_+^{(i)}(a)| + \frac{1}{2.n!} V_n^*(f,[a,b]) \\ &\ge \frac{1}{2.n!}\|f\|^* \end{aligned}$$

Hence

$$\frac{1}{2.n!}\|f\|^* \le \|f\| \le K\|f\|^*.$$

Therefore the two norms are equivalent and so they define the same topology on $AC_n^*([a,b])$.

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