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Thick Sets and Dynamics of Operators

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Abstract

Let X be a separable Banach space. For any pair U, V of nonempty open subsets in X, let $N_T(U,V) = \{n \in N : T^nU \cap V \neq \emptyset\}$. In the present paper we show the bounded operator T on X is hereditarily hypercyclic if and only if for every pair U, V of nonempty open subsets in X, $N_T(U,V)$ is a thick set.

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1 Introduction

Let X denote a separable infinite dimensional Banach space. The sequence $\{T_n\}_{n\geq 1}$ of bounded operators on X is said to be universal when there exists a vector x in X such that the set $\{T_n x : n \in N\}$, is dense in X. In the special case, we say that a bounded operator T on X is hypercyclic if the sequence of powers $\{T^n\}_{n\geq 1}$ is universal. It is interesting to know what kind of operators can actually be hypercyclic. The first example of a hypercyclic operator on a Hilbert space was constructed by Rolewicz in 1969 [6]. He showed that if B is the backward shift on $\ell^2(N)$, then any multiple λB, $|\lambda| > 1$, is hypercyclic.

Kitai [5] and Gethner and Shapiro [2], independently, provided a useful sufficient condition for an operator to be hypercyclic. This criterion referred to as the Hypercyclicity Criterion and has been used to determined the hypercyclicity of some classes of operators [4].

Hypercyclicity Criterion. Let T be a bounded operator on X and there exist a sequence (n_k), two dense subsets Y, Z of X and functions $S_k : Z \to X$ such that

(i) $T^{n_k} y \to 0$ for all $y \in Y$,

(ii) $S_k z \to 0$ and $T^{n_k} S_k z \to z$ for all $z \in Z$.

then T is hypercyclic.

Bes and Peris [1] introduced the notion of hereditary hypercyclicity and proved that T satisfies the Hypercyclicity Criterion if and only if T is hereditarily hypercyclic:

Definition 1.2. The bounded operator T on separable Banach space X is called hereditarily hypercyclic with respect to (n_k), provided for all subsequences (n_{k_j}) of (n_k); $\{T^{n_{k_j}} : j \geq 1\}$ is universal. Moreover, T is called hereditarily hypercyclic if T is hereditarily hypercyclic with respect to some sequence (n_k) of positive integers. T is called mixing if it is hereditarily hypercyclic with respect to (k), the entire sequence of positive integers.

Definition 1.3. Let $(n_k), (m_k) \subseteq N$. We say (n_k) is a quasi-subsequence of (m_k), whenever there exists some increasing (not necessary strictly) infinite sequence $\sigma : N \to N$ such that

$$\sup_k |n_k - m_{\sigma(k)}| < +\infty. \tag{1.1}$$

Hence every subsequence of a sequence, is a quasi-subsequence. It is easy to check that, the sequence (n_k) is syndetic if and only if (k) is a quasi-subsequence of (n_k).

In this paper we prove that every bounded operator, hereditarily hypercyclic with respect to a sequence, is hereditarily hypercyclic with respect to its all quasi-subsequences. As a consequence, we show a bounded operator T on a separable Banach space X is hereditarily hypercyclic if and only if for every pair U, V of nonempty open subsets in X, the set $N_T(U, V) := \{n \in N : T^n U \cap V \neq \emptyset\}$ is a thick set. Here a subset S of positive integer is said to be thick set if for every $n \in N$, there exists some $l \in N$ such that $l, l+1, \cdots, l+n \in S$

2 Main Results

In what follows, X will denote a separable Banach space of infinite dimension and $B(X)$ the set of all bounded linear operators on X. Also, $(n_k) \subseteq N$ will always refer to an increasing sequence of positive integers. It is well known that T is hypercyclic if and only if it is transitive, i.e, for any given two open sets U, V there is some positive integer n such that $T^n(U) \cap V$ is nonempty (see Theorem 1.2 in [3]). A hereditarily hypercyclic operator satisfies a much stronger condition:

Proposition 2.1. The bounded operator T on separable Banach space X is hereditarily hypercyclic with respect to (n_k) if and only if for any pair (U, V) of nonempty open subsets of X, there is some positive integer N such that

$$T^{n_k}U \cap V \neq \emptyset \text{ for all } k > N,$$

or equivalently $n_k \in N_T(U, V)$ for $k > N$.

Proof. Without loss of generality suppose that $n_k = k$. For the proof of the sufficiency by way of contradiction, suppose that for a pair (U, V) of nonempty open subsets of X, $T^{n_k}U \cap V = \emptyset$ for some subsequence (n_k) of (k). By definition, for a vector $x \in X$, the set $\{T^{n_k}x : k \geq 1\}$ is dense in X whence $T^i x \in U$ for some integer i. Since T has dense rang, the set $\{T^{n_k}(T^i x) : k \geq 1\}$ is also dense in X and consequently $T^{n_k}(T^i x) \in V$ for some k large enough, a contradiction. For the prove the necessity fix a countable basis $\{V_i\}$ for the topology of X and a subsequence (n_k) of (k). By assumption, each of the open sets $\bigcup_k T^{-n_k}V_i$ is dense in X for any integer $i \geq 0$ and so is $\bigcap_i \bigcup_k T^{-n_k}V_i$. It is east to check that for any vector x in the recent set, $\{T^{n_k}x : k \geq 1\}$ is dense in X. This completes the proof.

From now on we write $T \in \mathcal{H}_h(n_k)$, when T is hereditarily hypercyclic with respect to (n_k).

The first observation of this section show that if (n_k) is a quasi-subsequence of (m_k), then every bounded operator which is hereditarily hypercyclic with respect to (m_k), is also hereditarily hypercyclic with respect to (n_k). For our purpose, the following Lemma is needed which in turn is a direct consequence of Proposition 2.1:

Lemma 2.2. Suppose that X_1, X_2,...,X_N are separable Banach spaces and $T_i \in B(X_i)$ for $i = 1, 2, .., N$. For given $(n_k) \subseteq N$, if $T_i \in \mathcal{H}_h(n_k)$ on X_i, then $T_1 \oplus T_2 ... \oplus T_N \in \mathcal{H}_h(n_k)$ on $X_1 \oplus X_2 ... \oplus X_N$.

Theorem 2.3. Let (n_k) be a quasi-subsequence of (m_k) and $T \in B(X)$. If $T \in \mathcal{H}_h(m_k)$, then $T \in \mathcal{H}_h(n_k)$.

Proof. Assume that for some increasing infinite sequence $\sigma : N \to N$, (1.1) holds. Let $M = \sup_k |n_k - m_{\sigma(k)}|$. Then for each $k \in N$, there exits some integer j_k with $|j_k| \leq M - 1$ and

$$n_k = m_{\sigma(k)} + j_k \tag{2.1}$$

Now suppose $T \in \mathcal{H}_h(m_k)$. Fix two open subsets U, V in X. By Proposition 2.1, it suffices to show that $T^{n_k}U \cap V$ is nonempty for sufficiently large k. To do this, let $\hat{X} = X \oplus X \oplus \cdots \oplus X$ and $\hat{T} = T \oplus T \oplus \cdots \oplus T$, be $2M - 1$ copies of X and T, respectively. By Lemma 2.2, $\hat{T} \in \mathcal{H}_h(m_k)$ on $\hat{X}$. Consider the open subsets $\hat{U}, \hat{V}$ of $\hat{X}$ as the following:

$$\begin{aligned} \hat{U} &:= \underbrace{U \oplus U \oplus \cdots \oplus U}_{M \text{ copies}} \oplus T^{-1}U \cdots \oplus T^{-M+2}U \oplus T^{-M+1}U \quad \text{and} \\ \hat{V} &:= T^{-M+1}V \oplus T^{-M+2}V \oplus \cdots T^{-1}V \oplus \underbrace{V \oplus V \cdots \oplus V}_{M \text{ copies}}. \end{aligned}$$

By Proposition 2.1, $\hat{T}^{m_k}\hat{U} \cap \hat{V} \neq \emptyset$ for sufficiently large k. In particular, for some positive integer N, $\hat{T}^{m_{\sigma(k)}}\hat{U} \cap \hat{V} \neq \emptyset$ for any $k > N$. The definitions of $\hat{T}, \hat{U}$ and $\hat{V}$ imply that for each $k > N$, there exists some integer j_k with $|j_k| \leq M - 1$, satisfying

$$T^{m_{\sigma(k)}+j_k}(U) \cap V \neq \emptyset. \tag{2.2}$$

We conclude from (2.1) and (2.2) that $T^{n_k}U \cap V \neq \emptyset$ for any $k > N$. Therefore $T \in \mathcal{H}_h(n_k)$ on X.

Since the entire sequence of positive integers is a quasi-subsequence of any syndetic one, the following Corollary is deduced:

Corollary 2.4. Every operator, hereditarily hypercyclic with respect to a syndetic sequence, is mixing.

Corollary 2.5. Let $T \in B(X)$ and for two nonempty open subsets (U, V) of X, $N_T(U, V) = \{n_k : k \geq 1\}$. If $T \in \mathcal{H}_h(n_k)$, then T is mixing.

Proof. Let $T \in \mathcal{H}_h(n_k)$. Since $(n_k + 1)$ is a quasi-subsequence of (n_k), by Theorem 2.3, $T \in \mathcal{H}_h(n_k + 1)$ and so by Proposition 2.1, there exists some integer N such that $n_k + 1 \in N_T(U, V) = \{n_k : k \geq 1\}$ for any $k > N$. Therefore the sequence (n_k) is syndetic and by the last corollary, T is mixing.

Now we are ready to state an important application of Theorem 2.3. Recall that a subset S of N is called a thick set if for every $n \in N$, there exists some $l \in N$ such that $l, l + 1, \cdots, l + n \in S$

Proposition 2.6. The following statements are equivalent for an operator $T \in B(X)$:

(i) T is hereditarily hypercyclic,

(ii) T satisfies the Hypercyclicity Criterion,

(iii) For every pair (U, V) of non empty open subsets of X, $N_T(U, V)$ is a thick set.

Proof. By Theorem 2.3 in [1], (i) and (ii) are equivalent. In order to prove that (i) implies (iii) suppose $T \in \mathcal{H}_h(n_k)$ for some $(n_k) \subseteq N$. Pick $n \in N$, since $(n_k + i)$ is a quasi-subsequence of (n_k), by Theorem 2.3, $T \in \mathcal{H}_h(n_k + i)$ for $i = 1, 2, ..., n$. Now for two nonempty open subsets U, V of X applying Proposition 2.1 to see $n_k + i \in N_T(U, V)$ for sufficiently large k and $i = 1, 2..., n$. Hence $N_T(U, V)$ is a thick set.

Now we prove that (iii) implies (i). Assume that (iii) holds. By ([1], Theorem 2.3), it is enough to show $T \oplus T$ is hypercyclic or equivalently transitive. Pick the open subsets U_i, V_i in X for $i = 1, 2$. Since $N_T(U_1, U_2)$ and $N_T(V_1, V_2)$ are thick set and so nonempty, there are some positive integers p and $q > p$ such that
$$A := U_1 \cap T^{-p}U_2 \neq \emptyset \quad \text{and} \quad B := V_1 \cap T^{-q}V_2 \neq \emptyset.$$
By the assumptions, $N_T(A, B)$ is a thick set, hence for $n = q - p$, there is some $l \in N$ such that $l + i \in N_T(B, A)$ for $i = 0, 1, 2, ..., n$. Thus

$$T^{l+i}(V_1 \cap T^{-q}V_2) \cap U_1 \cap T^{-p}U_2 \neq \emptyset$$

for $i = 0, 1, 2, ..., n$. In particular, for $i = 0$ we obtain

$$T^{l}(V_1 \cap T^{-q}V_2) \cap U_1 \cap T^{-p}U_2 \neq \emptyset,$$

whence $T^lV_1 \cap U_1 \neq \emptyset$. Also for $i = n = q - p$ we get

$$T^{l+q-p}(V_1 \cap T^{-q}V_2) \cap U_1 \cap T^{-p}U_2 \neq \emptyset.$$

On the other hand

$$\begin{aligned} T^{l+q-p}(V_1 \cap T^{-q}V_2) \cap U_1 \cap T^{-p}U_2 &\subseteq T^{l+q-p}(T^{-q}V_2) \cap T^{-p}U_2 \\ &= T^{-p}(T^lV_2 \cap U_2) \end{aligned}$$

which implies that $T^lV_2 \cap U_2 \neq \emptyset$. Therefore, $T^lV_i \cap U_i \neq \emptyset$ for $i = 1, 2$ and so $T \oplus T$ is transitive on $X \oplus X$. This completes the proof.

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