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A Equivalent Condition for Uniform Domains[1]

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Abstract

Suppose that D is a domain in $\overline{R}^n$, in this paper, we prove that D is a uniform domain if and only if for any $\alpha \in (0,1]$ there exists positive constant $M(\alpha)$ such that, each pair of points $z_1, z_2 \in D \setminus \{\infty\}$ can be joined by a rectifiable arc $\gamma \subset D$ such that for any positive integer n and any $0 \le c_1 < c_2 < \cdots < c_n \le \frac{1}{2}$ yields

$$\sum_{i=1}^{n-1} \frac{1}{c_{i+1}^{\alpha} - c_i^{\alpha}} \int_{\gamma_j[c_i, c_{i+1}]} d(z, \partial D)^{\alpha-1} ds \le (n-1) M(\alpha) |z_1 - z_2|^{\alpha},$$

where $\gamma_j[c_i, c_{i+1}] = \{\gamma(s) : c_i l \le s \le c_{i+1} l\}$ denotes the subcurve of γ which starting from z_j and with arc length s of γ as parameter, $j = 1, 2, i = 1, 2, \cdots, n-1$, $l = l(\gamma)$ is the euclidean length of γ.

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1. Introduction

Suppose that D is a domain in $\overline{R}^n$, we say that D is a uniform domain if there are constants a and b such that each pair of points $z_1, z_2 \in D \setminus \{\infty\}$ can be

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joined by a rectifiable arc $\gamma \subset D$ for which

$$\begin{cases} l(\gamma) \le a|z_1 - z_2|, \\ \min\limits_{j=1,2} l(\gamma(z_j, z)) \le bd(z, \partial D) \quad \text{for all} \quad z \in \gamma. \end{cases} \tag{1.1}$$

Here $l(\gamma)$ denotes the euclidean length of γ, $\gamma(z_j, z)$ the part of γ between z_j and z, and $d(z, \partial D)$ the euclidean distance from z to ∂D.

Uniform domain was first introduced by O. Martio and J. Sarvas [9] when they studied the approximation and injectivity theory. Later, F. W. Gehring and B. G. Osgood [5], P. W. Jones [7], M. Vuorinen [10], V. Lappalainen and A. Lehtonen [8], L. Capogna and P. Q. Tang [2], A. V. Greshnov [6], H. Aikawa and T. Lundh [1], T. Futamura [3] et al. studied the uniform domains extensively, and many interesting and useful results are obtained.

In 1985, F. W. Gehring and O. Martio [4, Theorems 2.2 and 2.24] obtained the following results:

Theorem A If D is a uniform domain in $\overline{R}^n$, then for any $\alpha \in (0, 1]$ there exists constant $M = M(\alpha)$ such that for all $z_1, z_2 \in D$ there is a rectifiable curve γ joining z_1 to z_2 in D with

$$\int_\gamma d(z, \partial D)^{\alpha-1} ds \le M|z_1 - z_2|^\alpha.$$

The main aim of this paper is to prove the following equivalent condition for uniform domain:

Theorem 1.1. If D is a domain in $\overline{R}^n$, then D is a uniform domain if and only if for any $\alpha \in (0, 1]$ there exists positive constant $M(\alpha)$ such that, each pair of points $z_1, z_2 \in D \setminus \{\infty\}$ can be joined by a rectifiable curve $\gamma \subset D$ such that for any positive integer n and any $0 \le c_1 < c_2 < \cdots < c_n \le \frac{1}{2}$ yields

$$\sum_{i=1}^{n-1} \frac{1}{c_{i+1}^\alpha - c_i^\alpha} \int_{\gamma_j[c_i, c_{i+1}]} d(z, \partial D)^{\alpha-1} ds \le (n-1)M(\alpha)|z_1 - z_2|^\alpha, \tag{1.2}$$

where $\gamma_j[c_i, c_{i+1}] = \{\gamma(s) : c_i l \le s \le c_{i+1} l\}$ denotes the subcurve of γ which starting from z_j and with arc length s of γ as parameter, $j = 1, 2, i = 1, 2, \cdots, n-1$, $l = l(\gamma)$ is the euclidean length of γ.

2. Proof of Theorem 1.1

Proof of Theorem 1.1. The necessary. If D is a uniform domain, then there exist constants a and b such that each pair of points $z_1, z_2 \in D \setminus \{\infty\}$ can be joined by a rectifiable curve $\gamma \subset D$ for which (1.1) holds. Making use of (1.1) and straightforward computations reveal that

$$\begin{aligned}
&\frac{1}{c_{i+1}^{\alpha}-c_i^{\alpha}}\int_{\gamma_j[c_i,c_{i+1}]} d(z,\partial D)^{\alpha-1}ds \\
\leq &\frac{b^{1-\alpha}}{c_{i+1}^{\alpha}-c_i^{\alpha}}\int_{\gamma_j[c_i,c_{i+1}]} \left[\min_{j=1,2} l(\gamma(z_j,z))\right]^{\alpha-1} ds \\
= &\frac{b^{1-\alpha}}{c_{i+1}^{\alpha}-c_i^{\alpha}}\int_{c_i l}^{c_{i+1} l} s^{\alpha-1}ds \\
= &\frac{b^{1-\alpha}}{\alpha} l^{\alpha} \\
\leq &\frac{a^{\alpha}b^{1-\alpha}}{\alpha}|z_1-z_2|^{\alpha}, i=1,2,\cdots,n-1.
\end{aligned}$$

This gives

$$\sum_{i=1}^{n-1}\frac{1}{c_{i+1}^{\alpha}-c_i^{\alpha}}\int_{\gamma_j[c_i,c_{i+1}]} d(z,\partial D)^{\alpha-1}ds \leq (n-1)\frac{a^{\alpha}b^{1-\alpha}}{\alpha}|z_1-z_2|^{\alpha}.$$

Hence (1.2) holds with $M(\alpha)=\frac{a^{\alpha}b^{1-\alpha}}{\alpha}$.

The sufficiency. Firstly, taking $n=2, \alpha=1, c_1=0$ and $c_2=\frac{1}{2}$ in (1.2), straightforward computations reveal that

$$l(\gamma)\leq M(1)|z_1-z_2|. \tag{2.1}$$

Secondly, taking $n=2$, for any $0\leq c_1<c_2\leq \frac{1}{2}$, inequality (1.2) implies that

$$\begin{aligned}
&\frac{1}{c_2^{\alpha}-c_1^{\alpha}}\int_{\gamma_j[c_1,c_2]} d(z,\partial D)^{\alpha-1}ds \\
\leq &M(\alpha)|z_1-z_2|^{\alpha}\leq M(\alpha)l^{\alpha} \\
= &\frac{\alpha M(\alpha)}{c_2^{\alpha}-c_1^{\alpha}}\int_{\gamma_j[c_1,c_2]} \left[\min_{j=1,2} l(\gamma(z_j,z))\right]^{\alpha-1} ds.
\end{aligned} \tag{2.2}$$

Because of the randomicity of $0\leq c_1<c_2\leq\frac{1}{2}$ and the continuity of $d(z,\partial D)^{\alpha-1}$ and $\left[\min_{j=1,2} l(\gamma(z_j,z))\right]^{\alpha-1}$ on $\gamma_j[c_1,c_2]$ $(j=1,2)$, from (2.2) we can get

$$d(z,\partial D)^{\alpha-1}\leq \alpha M(\alpha)\left[\min_{j=1,2} l(\gamma(z_j,z))\right]^{\alpha-1} \tag{2.3}$$

for all $z\in\gamma$.

In particular, if we take $\alpha = \frac{1}{2}$ in (2.3), then we have

$$\min_{j=1,2} l(\gamma(z_j, z)) \leq \left[\frac{1}{2}M(\frac{1}{2})\right]^{\frac{1}{2}} d(z, \partial D). \tag{2.4}$$

Therefore, D is a uniform domain follows from (2.1) and (2.4).

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