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A Unique Common Fixed Point Theorem for Six Non Self-Maps in Metric Spaces

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Abstract

In this paper, we prove a unique common fixed point theorem for six non self-maps in a metric space. Which generalizes and extends the results of S.L.Singh, Apichai Hematulin and Rajendra Pant [16].

Keywords: Coincidence point; Fixed Point ;Banach Contraction; Asymptotic regularity

1. Introduction

The well known Banach Fixed Point Theorem has been generalized and extended by many authors in various ways. In 2006 Proinov[12] has obtained two types of generalizations of Banach fixed point theorem. The first type involves Meir-Keeler type conditions (See, for instance, Cho et al.[3], Jachymski[4], Lim[8], Matkowski[9], Park and Rhoades[11]) and second type involves contractive guage functions (see, for instance ,Boyd and Wong [1] and Kim et al. [7]). Proinov [12] obtained equilance between these two types of contractive conditions and also obtained a new fixed point theorem. Recently S.L.Singh et al.[16] have extended Proinov [12]Theorem 4.1 for three non self-maps. In this paper we extend the Theorem 2.7 of S.L.Singh et al.[16] for six non self-maps.

In all that follows Y is an arbitrary non-empty set, (X,d) a metric space and N={1,2,3,……}. For T, f : $Y \to X$, let C (T, f) denote the set of coincidence points of T and f, that is $C(T, f) = \{z \in Y : Tz = fz\}$.

We define common asymptotic regularity of two functions in the following way.

Definition1.1. Let A,B,S,T and f, g be maps on Y with values in a meric space (X, d).The pairs (A,B) and (S,T) are said to be a common asymptotically regular with respect to f and g respectively at $x_0 \in Y$ if there exists $\{x_n\}$ in Y Such that

$$fx_{2n+1} = Ax_{2n} = Sx_{2n+2} = gx_{2n+3},$$
$$fx_{2n+2} = Bx_{2n+1} = Tx_{2n+3} = gx_{2n+4}, \quad n= \ 0,1,2,3\ldots\ldots$$

and $\lim_{n\to\infty} d(fx_n, fx_{n+1}) = 0 = \lim_{n\to\infty} d(gx_n, gx_{n+1})$.

Definition 1.2 .(see[16]). Let T, f: X→X.Then the pair (T ,f) is (IT)-Commuting at $z \in X$ if Tfz = fTz with Tz = fz . They are (IT)-Commuting on X (also called weakly compatible,by Jungck and Rhoades[6]) if Tfz =fTz for all $z \in X$ such that Tz = fz.

Definition 1.3: (see [12], Definition 2.1(i)).Let Φ denote the class of all functions $\varphi: R^+ \to R^+$ satisfying: for any $\varepsilon > 0$, there exists $\delta > \varepsilon$ such that $\varepsilon < t < \delta$ implies $\varphi(t) \leq \varepsilon$.

2. Main Result

The following Theorem improves and extends the Theorem 2.7 of [16].

Theorem 2.3. : Let A,B,S,T and f, g be maps on an arbitrary non-empty set Y with values in a metric space(X, d).Let the pairs (A,B) and (S,T) be a common asymptotically regular with respect to f and g respectively at $x_0 \in Y$ and the following conditions are satisfied:

(E1) : $A(Y) \cup B(Y) \cup S(Y) \cup T(Y) \subseteq f(Y)$ (=g(Y));

(E2): $d(Ax, By) \leq \varphi(h_1(x, y))$ for all $x, y \in Y$,

Where $h_1(x, y) = d(f x, f y) + \gamma[d(Ax, f x) + d(By, f y)]$,for some $\gamma (0 \leq \gamma \leq 1)$ and $\varphi: R^+ \to R^+$ a Continuous function with $\varphi(t) < t$ for all $t > 0$.

(E3): $d(S x, Ty) \leq \varphi(h_2(x, y))$ for all $x, y \in Y$,

Where $h_2(x, y) = d(g x, g y) + \gamma[d(S x, g x) + d(Ty, g y)]$.

If A(Y) or B(Y) or S(Y) or T(Y) or f(Y)(=g(Y)) is a complete subspace of X, then

(i) : C(A, f) is non-empty.

(ii) : C(S, g) is non-empty.

(iii) : C(B, f) is non-empty.

(iv): C(T, g) is non-empty.

Further if Y=X, then

(v) : A and f have a common fixed point provided that A and f are (IT)-commuting at a point $u \in C(A, f)$.

(vi) : B and f have a common fixed point provided that B and f are(IT)-commuting at a point $v \in C(B, f)$.

(vii) : S and g have a common fixed point provided that S and g are (IT)-commuting at a point $u^1 \in C(S, g)$.

(viii) : T and g have a common fixed point provided that T and g are (IT)-commuting at a point $v^1 \in C(T, g)$.

(ix) : A,B,S,T and f, g have a unique common fixed point provided that (v), (vi), (vii) and (viii) all are true.

Proof: Let x_0 be an arbitrary point in Y. Since the pairs (A,B) and (S,T) are a common asymptotically regular with respect to f and g respectively at $x_0 \in Y$. Then there exists a sequence $\{x_n\}$ in Y such that.

$$f x_{2n+1} = Ax_{2n} = Sx_{2n+2} = gx_{2n+3},$$
$$fx_{2n+2} = Bx_{2n+1} = Tx_{2n+3} = gx_{2n+4}, \quad n = 0,1,2,\ldots..$$

and $\lim_{n\to\infty} d(fx_n, fx_{n+1}) = 0 = \lim_{n\to\infty} d(gx_n, gx_{n+1})$.

First we shall show that $\{fx_n\}$ is Cauchy sequence. Suppose $\{fx_n\}$ is not Cauchy sequence. Then there exists $\mu>0$ and increasing Sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that m_k even and n_k odd and for all k, $m_k < n_k$,

$$d(fx_{m_k}, fx_{n_k}) \geq \mu \text{ and } d(fx_{m_k}, fx_{n_k-1}) < \mu \qquad (2.1)$$

By the triangle inequality,

$$d(fx_{m_k}, fx_{n_k}) \leq d(fx_{m_k}, fx_{n_k-1}) + d(fx_{n_k-1}, fx_{n_k}).$$

Letting $k\to\infty$, we get that

$$\lim_{k\to\infty} d(fx_{m_k}, fx_{n_k}) < \mu .$$

Therefore there exists k_0 such that

$$d(fx_{m_k}, fx_{n_k}) < \mu \ \forall k \geq k_0 \qquad (2.2)$$

By (2.1) and (2.2), we get that

$$\mu \leq d(fx_{m_k}, fx_{n_k}) < \mu \ \forall k \geq k_0 \text{ implies } \lim_{k\to\infty} d(fx_{m_k}, fx_{n_k}) = \mu .$$

By (E2), we have

$$d(fx_{m_k+1}, fx_{n_k+1}) = d(Ax_{m_k}, Bx_{n_k}) \leq \varphi(h_1(x_{m_k}, x_{n_k}))$$
$$= \varphi(d(fx_{m_k}, fx_{n_k}) + \gamma[d(Ax_{m_k}, fx_{m_k}) + d(Bx_{n_k}, fx_{n_k})])$$

That is $d(fx_{m_k+1}, fx_{n_{k+1}}) \leq \varphi(d(fx_{m_k}, fx_{n_k}) + \gamma[d(fx_{m_k+1}, fx_{m_k}) + d(fx_{n_k+1}, fx_{n_k})])$.

Letting $k\to\infty$, we get that

$$\mu \leq \varphi(\mu) < \mu \text{ , a contradiction.}$$

Thus $\{fx_n\}$ is Cauchy sequence.

Similarly we can prove that $\{gx_n\}$ is Cauchy sequence.

Suppose f(Y)(=g(Y)) is a complete sub space of X. Then $\{y_n\}$ where $y_n = \{fx_n\}$, being Cauchy sequence in f(Y) which is complete has a limit in f(Y) say z.

Let $u = f^{-1}z$. Thus fu=z for some $u \in Y$.Note that the sub sequences $\{fx_{2n+1}\}$ and $\{fx_{2n+2}\}$ also converges to z.By E(2),we obtain

$$d(Au, Bx_{2n+1}) \leq \varphi(h_1(u,x_{2n+1}))$$
$$= \varphi(d(fu,fx_{2n+1}) + \gamma[d(Au, fu) + d(Bx_{2n+1}, fx_{2n+2})]).$$

Letting $n\to\infty$, we get that

$d(Au, z) \leq \varphi(d(fu, z) + \gamma[d(Au, fu) + 0])$
$\leq \varphi (d(z, z) + \gamma[d(Au, fu)]),$
$d(Au, fu) \leq \varphi (\gamma d(Au, fu)) < d(Au, fu)$,a contradiction.

Therefore Au=fu=z.Thus C(A, f) is non empty . This proves (i). (2.3)
Since we get $\{gx_n\}$ is a cauchy sequence in f(Y) which is complete.
Hence $\{gx_n\}$ has a limit in f(Y). Note that $\lim(fx_n) = \lim(gx_n)$.
we get $\lim_{n\to\infty} gx_n = z \in g(Y)$ .Therefore there exists $u^1 \in Y$ such that $gu^1= z$ implies $u^1=g^{-1}z$. Note that the subsequences $\{gx_{2n+3}\}$ and $\{gx_{2n+4}\}$are also converges to z. By E(3) ,we obtain

$$d(Su^1, Tx_{2n+3}) \leq \varphi (h_2(u^1, x_{2n+3}))$$
$$= \varphi(d(gu^1, gx_{2n+3}) + \gamma[d(Su^1, gu^1) + d(Tx_{2n+3}, gx_{2n+3})])$$
$$= \varphi (d(gu^1, gx_{2n+3})+\gamma[d (Su^1, gu^1)+ d(gx_{2n+4}, gx_{2n+3})]).$$

Letting $n\to\infty$,we get that

$$d(Su^1, gu^1) \leq \varphi (d(z, z) + \gamma[d (su^1, gu^1) + d(z, z)])$$
$$\leq\varphi(\gamma d (su^1, gu^1))< d (su^1, gu^1),\text{a contradiction.}$$

Therefore $su^1 = gu^1 = z$. Thus C(S, g) is non-empty. This proves (ii). (2.4)
In view of (2.3) and (2.4) it follows that
$Au=fu =Su^1 = gu^1 = z$. (2.5)
Since $A(Y) \cup B(Y) \cup S(Y) \cup T(Y) \subseteq f(Y) (=g(Y))$.
Therefore there exists $\upsilon,\upsilon^1 \in Y$ such that
$Au = f\upsilon$ and $su^1 = g\upsilon^1$. (2.6)
We claim that $f\upsilon = B\upsilon$ and $g\upsilon^1 = T\upsilon^1$.
Using (E2) and (E3) ,we obtain

$$d(f\upsilon, B\upsilon) = d(Au, B\upsilon) \leq \varphi(h_1(u,\upsilon))$$
$$= \varphi (d(fu, f\upsilon)+\gamma[d (Au, fu) + d(B\upsilon,f\upsilon)])$$
$$\leq \varphi (d(Au, Au)+\gamma[(d(Au, Au) + d(f\upsilon, B\upsilon)]),$$

or $d(f\upsilon, B\upsilon) \leq \varphi(d (z, z) + \gamma[(d (z, z) + d (f\upsilon, B\upsilon)]),$
$d(f\upsilon, B\upsilon) \leq \varphi(\gamma d(f\upsilon, B\upsilon))< d(f\upsilon, B\upsilon)$,a contradiction.
Therefore $f\upsilon=B\upsilon$. (2.7)
Inview of (2.7), (2.6) and (2.3) it follows
$B\upsilon = f\upsilon = Au = fu = z$. (2.8)
From (2.5) and (2.8) it follows
$B\upsilon = f\upsilon = Au = fu = Su^1 = gu^1 = z$. (2.9)
Thus C(B, f) is non-empty. This proves (iii).
Now using (E3), we obtain

$$d(g\upsilon^1, T\upsilon^1) = d(su^1, T\upsilon^1) \leq \varphi(h_2(u^1, \upsilon^1))$$
$$= \varphi (d (gu^1, g\upsilon^1) + \gamma [d(su^1, gu^1) + d(T\upsilon^1, g\upsilon^1)])$$
$$= \varphi (d(z, z) + \gamma[d(z, z) + d(g\upsilon^1, T\upsilon^1)]),$$

$d(g\upsilon^1, T\upsilon^1) \leq \varphi (\gamma d(g\upsilon^1, T\upsilon^1)) < d(g\upsilon^1, T\upsilon^1)$,a contradiction.
Therefore $T\upsilon^1 = g\upsilon^1$. (2.10)
In view of (2.5), (2.6) and (2.10) it follows
$T\upsilon^1 = g\upsilon^1 = Su^1 = gu^1 = z$. (2.11)

Thus C(T,g) is non-empty. This proves (iv).
In view of (2.9), (2.11) it follows

$$Au = fu = B\upsilon = f\upsilon = Su^1 = gu^1 = T\upsilon^1 = g\upsilon^1 . \qquad (2.12)$$

Now, if Y=X, (A, f), (B, f) and (S, g),(T, g) are (IT)-Commuting, then

$$\begin{aligned} &Afu = fAu \text{ implies } AAu = Afu = fAu = ffu,\\ &Bfv = fBv \text{ implies } BBv=Bfv=fBv=ffv, \qquad (2.13)\\ &Sgu^1 = gSu^1 \text{ implies } SSu^1 = Sgu^1 = gSu^1 = ggu^1\\ &Tgv^1 = gT\upsilon^1 \text{ implies } TT\upsilon^1 = Tg\upsilon^1 = gT\upsilon^1 = gg\upsilon^1. \end{aligned}$$

In view of (E2) it follows that

$$\begin{aligned} d(AAu, Au) = d(AAu, Bv) &\leq \varphi (h_1(Au,\upsilon))\\ &= \varphi (d(fAu, f\upsilon)+\gamma[d(AAu, fAu)+ d(B\upsilon, f\upsilon)]), \end{aligned}$$

$$d(AAu, Au) \leq \varphi (\gamma\, d(AAu, Au)) < d(AAu, Au), \text{ a contradiction.}$$

Therefore AAu=Au=fAu(=z),
Au is a common fixed point of A and f . (2.14)
Similarly, we get BBυ=Bυ.
Therefore Bv is a common fixed point of B and f. (2.15)
Followed by (2.13)
BBυ = Bυ = fBυ (=z).
Since Aυ=Bυ.From (2.14) and (2.15), we conclude that
Au(=z) is a common fixed point of A,B and f. (2.16)
Now in view of (E3), we obtain

$$\begin{aligned} d(SSu^1, Su^1) = d (SSu^1, T\upsilon) &\leq \varphi (h_2 (Su^1, \upsilon^1))\\ &= \varphi(d(gSu^1, g\upsilon^1) + \gamma[d (SSu^1, gSu^1)+ d(T\upsilon^1, g\upsilon^1)]), \end{aligned}$$

$$d(SSu^1, Su^1) \leq \varphi (\gamma\, d(SSu^1, Su^1)) < d (SSu^1, Su^1), \text{a contradiction.}$$

Therefore $SSu^1 = Su^1$,$SSu^1 = gSu^1 = Su^1 = z$,
Su^1 is a common fixed point of S and g. (2.17)
Similarly we get ,$TT\upsilon^1 = gT\upsilon^1 = T\upsilon^1 = z$,
$T\upsilon^1$ is a common fixed point of T and g. (2.18)
Since $Su^1 = T\upsilon^1 = z$ .From (2.17) and (2.18), we conclude that
$Su^1(= z)$ is common fixed point of S, T and g . (2.19)
Since $Au = Su^1$ (= z) .Therefore from (2.16) and (2.19),we conclude that A,B,S,T and f,g are having a common fixed point.The proof is similar when A(Y) or B(Y) or S(Y) or T(Y) are complete subspaces of X. Since , $A(Y) \cup B(Y) \cup S(Y) \cup T(Y) \subseteq f(Y)$ (=g(Y)).Finally, in order to prove uniqueness, let w be a common fixed point of A,B,S,T and f, g.

$$\begin{aligned} \text{Consider } d(z, w) = d (A z, B w) &\leq \varphi (h_1(z, w))\\ &= \varphi (d(f z, f w) + \gamma[d(f z, A z) + d(f w, B w)]), \end{aligned}$$

$$d(z, w) \leq \varphi ((d(z, w)) < d(z, w), \text{ a contradiction .}$$

Therefore z = w. Hence A,B,S,Tand f, g have unique common fixed point.

References

[1] D.W.Boyd and J.S.W.Wong, on nonlinear contractions, Proc. Amer.Math. Soc. **20** (1966),458-464.

[2] F.E.Browder and W.V.Petershyn, The solution by iteration of nonlinear functional equations in Banach Spaces,Bull.Amer.Math.Soc **72** (1966), 571-575.

[3] Y.J.Cho,P.P.Murthy and G.Jungck ,A theorem of Meir-Keeler type revisited Internat J.Math. Math. Sci. **23** (2000), 507-511.

[4] J.Jachymski,Equivalent conditions and the Meir–Keeler type theorems, J.Math. Anal. Appl.**194** (1995), 293-303.

[5] G.Jungck,Commuting mappings and fixed points,Amer.Math.Monthly **83** (1976),261-263.

[6] G.Jungck and B.E.Rhoades, Fixed points for set – valued functions without continuity Indian J.Pure Appl.Math.**29**,no.3 (1988),227-238.

[7] K.H.Kim,S.M.Kang and Y.J.Cho,Common fixed point of Φ-contractive mappings,East.Asian Math.J.**15** (1999),211-222.

[8] T.C.Lim, on characterization of Meir –Keeler contractive maps, Non linear Anal. **46** (2001), 113-120

[9] J.Matkowski, Fixed point theorems for contractive mappings in metric spaces, Cas. Pest. Mat. **105** (1980),341-344.

[10] A.Meir and E.Keeler, A theorem on contraction mappings, J. Math. Anal. Appl.**28** (1969),326-329.

[11] S.Park and B.E.Rhoades, Meir –Keeler type contractive conditions,Math. Japon. **26** (1981), 13-20.

[12] P.D.Proinov,Fixed point theorems in metric spaces ,Non linear Anal.**64** (2006), 546-557.

[13] B.E.Rhoades ,S.L.Singh and Chitra Kulshrestha,Coincidence theorems for Some multivalued mappings. Internat. J. Math. Math. Sci**. 7**, no.3(1984), 429-434.

[14] K.P.R.Sastry, S.V.R.Naidu, I.H.N.Rao and K.P.R.Rao, Common fixed points for asymptotically regular mappings, Indian J.Pure Appl.Math. **15**, no 8 (1984), 849- 854.

[15] S.L.Singh, K.Ha and Y.J,Cho, Coincidence and fixed point of nonlinear hybrid contractions, Internat. J. Math. Math. Sci. **12**, no.2(1989), 247-256.

[16] S.L.Singh, Apichai Hematulin and Rajendra pant, New coincidence and Common fixed point theorems, Applied General Topology Vol.**10,**No.1(2009),121-130.

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