Int. Journal of Math. Analysis, Vol. 5, 2011, no. 45, 2229 - 2235

Error Estimation for Some Modified Szasz-Mirakjan-Beta Operators

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Abstract

In this paper, we present a modification of a sequence of mixed summation-integral type operators having Sz´asz and Beta basis functions in summation and integration, the so called Sz´asz -Mirakjan-Beta operators. Then we study the approximation properties of modified operators by means of the elements of the Lipschitz class functional.

Keywords: Sz´asz -Mirakjan-Beta operators, Lipschitz class functional, Modulus of continuity, Lattice homomorphism, Cofinal subspace

1 Introduction

The classical Sz´asz -Mirakjan-Beta operators are defined by

$$S_n(f;x) = e^{-nx}\sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f(\frac{k}{n}) \ ,$$

where $f \in C_\gamma[0,\infty), x \geq 0$ and $n \in N$

Some approximation properties of the Sz´asz -Mirakjan-Beta operators and their modification were studied by Becker[1], Srivastava and Gupta [2], Duman and Ozarslan[3], Agrawal and Kasana[4], Gupta, Agarwal, Sahai and Sinha[5], Gupta and Noor [6],. Gupta, Noor and beniwal [7], Totik[8], Finta, Govil and Gupta[9]. In 2006 Gupta and Noor[6] have proposed a sequence of mixed summation integral type operators, the so called Sz´asz -Mirakjan-Beta operators, as follows:

$$(1.1) \quad U_n(f;x) = e^{-nx}\sum_{k=1}^{\infty} \frac{(nx)^k}{k!B(n+1,k)}\int_0^{\infty} f(t)\frac{t^{k-1}}{(a+t)^{n+k+1}}dt + e^{-nx}f(0),$$

Where $f \in C[0,\infty)$ such that $|f(t)| \leq M(1+t)^\gamma$ for some M>0, $\gamma > 0$

Now for the operators U_n given by (1.1), the following lemma follows:

Lemma A.[6] Let $e_i(x) = x^i$, = 0,1,2. Then for each x$\geq$0 and n>1, we have

(a) $R_n(e_0;x) = 1$

(b) $R_n(e_1;x) = x$

(c) $R_n(e_2;x) = \dfrac{nx^2+2x}{n-1}$

Lemma A shows that the operators U_n preserve the test functions $e_0(x) = 1$ and $e_1(x) = x$. Many well-known approximating operators preserve these test functions, such as Bernstein polynomials, Meyer-KÄonig and Zeller operators, Sz´asz -Mirakjan operators, Baskakov operators etc. Observe that these operators do not preserve the test function $e^2(x) = x^2$. However, by modifying the Bernstein polynomials, King [10] presented a non-trivial sequence of positive linear operators which approximate each continuous function on [0, 1] while preserving the functions e_0 and e_2*:* Then it is proved that these modified operators have a better rate of convergence than the classical Bernstein polynomials on the interval [0,1⁄3]. Thus a natural question arises: can we construct a sequence of positive linear operators preserving the test functions e_0 and e_2 so that our modified operators have better error estimation than Sz´asz-Mirakjan-Beta operators. In the present paper we mainly focus on this problem.

2 Definition of operator

Let $\{q_n^*(x)\}$ be sequence of real valued continuous functions defined on $[0,\infty]$ with $0 \leq q_n^*(x) < \infty$. Then we define the following positive operator

$$\textbf{(1.2)}\ R_n^*(f;x) = a^{n+1}[e^{-nq_n^*(x)}\sum_{k=1}^{\infty} \frac{(nq_n^*(x))^k}{k!B(n+1,k)}\int_0^{\infty} f(t)\frac{t^{k-1}}{(a+t)^{n+k+1}}dt] + e^{-nq_n^*(x)}f(0)$$

where

(1.3) $q_n^*(x) = \frac{1}{n}(-1+\sqrt{1+n(n-\frac{2}{3})x^2}\ )\quad \forall\ x \geq 0, n \in N$

Where $f \in C[0,\infty)$ such that $|f(t)| \leq M(a+t)^\gamma$ for some M>0, $\gamma > 0$,x$\geq$0, $a \in (0,\infty)$

Remark: For a=1 and $q_n^*(x)$=x operator defined by (1.2) will reduce to (1.1)
Thus (1.2) is generalized form of (1.1).

Also observe that every R_n^* maps $C_B[0,\infty)$, the space of all bounded and continuous function on [0,∞), into itself.

In our paper we shall study approximation properties of operators (1.2). We shall prove approximation theorems which are similar to some results given for operators (1.1)

We obtain the following result for this operator

3 Auxiliary Results

Lemma 2.1 Let $e_i(x) = x^i$,i = 0, 1, 2. Then for each x$\geq$0 and n>1, we have

(a) $R_n^*(e_0;x) = 1$

(b) $R_n^*(e_1;x) = a\ q_n^*(x) = a\frac{1}{n}(-1+\sqrt{1+n(n-\frac{2}{3})x^2}\ )$

(c) $R_n^*(e_2;x) = a^2x^2$

Proof:

(a) $R_n^*(e_0;x)) = a^{n+1}[e^{-nq_n^*(x)}\sum_{k=1}^{\infty}\frac{(nq_n^*(x))^k}{k!B(n+1,k)}\int_0^\infty \frac{t^{k-1}}{(a+t)^{n+k+1}}dt]$

$= a^{n+1}[e^{-nq_n^*(x)}\sum_{k=1}^{\infty}\frac{(nq_n^*(x))^k}{k!B(n+1,k)}\frac{1}{a^{n+1}}B(n+1,k)]$

=1

(b) $R_n^*(e_1;x) = a^{n+1}[e^{-nq_n^*(x)}\sum_{k=1}^{\infty}\frac{(nq_n^*(x))^k}{k!B(n+1,k)}\int_0^\infty t\frac{t^{k-1}}{(a+t)^{n+k+1}}dt]$

$= a^{n+1}[e^{-nq_n^*(x)}\sum_{k=1}^{\infty}\frac{(nq_n^*(x))^k}{k!B(n+1,k)}\frac{1}{a^n}B(k+1,n)]$

$=a\,q_n^*(x) = a\frac{1}{n}(-1+\sqrt{1+n(n-\frac{2}{3})x^2}\ )$

(c) $R_n^*(e_2;x) = a^{n+1}[e^{-nq_n^*(x)}\sum_{k=1}^{\infty}\frac{(nq_n^*(x))^k}{k!B(n+1,k)}\int_0^\infty t^2\frac{t^{k-1}}{(a+t)^{n+k+1}}dt]$

$$= a^{n+1}[e^{-nq_n^*(x)}\sum_{k=1}^{\infty} \frac{(nq_n^*(x))^k}{k!B(n+1,k)} \frac{1}{a^{n-1}} B(k+2,n-1)]$$

$$= a^2x^2$$

This implies that the operator (2.1) preserve the test function e_0 and e_2

Lemma 2.2 Let ψ_x (t) = t-x, for every x≥ 0, we have

(a) $R_n^*(\psi_x;x) = -x + a\frac{1}{n}(-1+\sqrt{1+n(n-\frac{2}{3})x^2}\ )$

(b) $R_n^*(\psi_x^2;x) = \left[\frac{a^2}{(n-1)}\left(n-\frac{2}{3}\right)+1\right]x^2 - 2axq_n^*(x)$

Proof (a):

$$R_n^*(\psi_x;x)) = a^{n+1}[e^{-nq_n^*(x)}\sum_{k=1}^{\infty} \frac{(nq_n^*(x))^k}{k!B(n+1,k)}\int_0^{\infty} (t-x)\frac{t^{k-1}}{(a+t)^{n+k+1}}dt] + e^{-nq_n^*(x)}f(0)$$

$$= a^{n+1}[e^{-nq_n^*(x)}\sum_{k=1}^{\infty} \frac{(nq_n^*(x))^k}{k!B(n+1,k)}\int_0^{\infty} (\frac{t^k}{(a+t)^{n+k+1}} - x\frac{t^{k-1}}{(a+t)^{n+k+1}})dt]\text{-x}\,e^{-nq_n^*(x)}$$

$$= a^{n+1}[e^{-nq_n^*(x)}\sum_{k=1}^{\infty} \frac{(nq_n^*(x))^k}{k!B(n+1,k)}\{\frac{1}{a^n}B(k+1,n) - x\frac{1}{a^{n+1}}B(n+1,k)\}]\text{ - x}\,e^{-nq_n^*(x)}$$

$$= a[e^{-nq_n^*(x)}\sum_{k=1}^{\infty} \frac{(nq_n^*(x))^k}{k!B(n+1,k)}B(k+1,n)] - xe^{-nq_n^*(x)}\sum_{k=1}^{\infty} \frac{(nq_n^*(x))^k}{k!B(n+1,k)}B(n+1,k)\}\text{ -}$$
$$\text{x}\,e^{-nq_n^*(x)}$$

$$= aq_n^*(x) - x\, e^{-nq_n^*(x)}(e^{nq_n^*(x)} - 1)\text{ - x}\,e^{-nq_n^*(x)}$$

$$= \text{-x+}\,a\frac{1}{n}(-1+\sqrt{1+n(n-\frac{2}{3})x^2}\ )$$

Proof (b):

$$R_n^*(\psi^2{}_x;x)) = a^{n+1}[e^{-nq_n^*(x)}\sum_{k=1}^{\infty} \frac{(nq_n^*(x))^k}{k!B(n+1,k)}\int_0^{\infty} (t-x)^2\frac{t^{k-1}}{(a+t)^{n+k+1}}dt]$$
$$+ e^{-nq_n^*(x)}f(0)$$

$$= a^{n+1}[e^{-nq_n^*(x)}\sum_{k=1}^{\infty}\frac{(nq_n^*(x))^k}{k!B(n+1,k)}\int_0^{\infty}(t^2-2tx+x^2)\frac{t^{k-1}}{(a+t)^{n+k+1}}dt]+x^2\, e^{-nq_n^*(x)}$$

$$= a^{n+1}[e^{-nq_n^*(x)}\sum_{k=1}^{\infty}\frac{(nq_n^*(x))^k}{k!B(n+1,k)}\int_0^{\infty}\{\frac{t^{k+1}}{(a+t)^{n+k+1}}-2x\frac{t^k}{(a+t)^{n+k+1}}+x^2\frac{t^{k-1}}{(a+t)^{n+k+1}}\}dt]$$

$$+x^2\, e^{-nq_n^*(x)}$$

$$= a^{n+1}[e^{-nq_n^*(x)}\sum_{k=1}^{\infty}\frac{(nq_n^*(x))^k}{k!B(n+1,k)}\{\frac{1}{a^{n-1}}B(k+2,n-1)-2x\frac{1}{a^n}B(n,k+1)+x^2\frac{1}{a^{n+1}}B(n+1,k)\}]$$

$$+x^2\, e^{-nq_n^*(x)}$$

$$= a^2e^{-nq_n^*(x)}\sum_{k=1}^{\infty}\frac{(nq_n^*(x))^k(k+1)}{n(n-1)(k-1)!}-2xae^{-nq_n^*(x)}\sum_{k=1}^{\infty}\frac{(nq_n^*(x))^k}{n(k-1)!}+x^2e^{-nq_n^*(x)}\sum_{k=1}^{\infty}\frac{(nq_n^*(x))^k}{(k)!}$$

$$+x^2\, e^{-nq_n^*(x)}$$

$$=\frac{a^2}{n(n-1)}[n^2q_n^{*2}(x)+2nq_n^*(x)]-2axq_n^*(x)+x^2e^{-nq_n^*(x)}(e^{nq_n^*(x)}-1)$$

$$+x^2\, e^{-nq_n^*(x)}$$

$$=\left[\frac{a^2}{(n-1)}\left(n-\frac{2}{3}\right)+1\right]x^2-2axq_n^*(x)$$

4 Main Results

Theorem 3.1 $\lim_{n\to\infty} R_n^*(f;x)$ =*f(x)* uniformly with respect to $x\in[0,b]$ provided $f\in C_\gamma[0,\infty)$, $\gamma>0$, b>0.

Proof: for b>0, consider the lattice homomorphism $T_b: C[0,\infty)\to C[0,b]$ defined by $T_b(f):=f|_{[0,b]}$ for every $f\in C[0,\infty)$. In this case, we see that, for each i=0, 1, 2,

(3.1) $\lim_{n\to\infty} T_b(R_n^*(e_i))=T_b(e_i)$ Uniformly on [0, b],

Thus with the universal Korovkin-type property with respect to monotone operators (Theorem 4.1.4(vi) of [2,p. 199]) we have the following: “Let X be a compact set and H be a cofinal subspace of C(X). If E is a Banach lattice, S:C(X)$\rightarrow$E is a lattice homomorphism and if $\{L_n\}$ is a sequence of positive linear operators from C(X) into E such that $\lim_{n\to\infty} L_n(h)=S(h)$ for all $h \in H$, then $\lim_{n\to\infty} L_n(h)=f$ provided that f belongs to the Korovkin closure of H."

Hence, by using (3.1) and the above property we obtain the Korovkin-type approximation result i.e.

$\lim_{n\to\infty} R_n^*(f;x)=f(x)$ uniformly with respect to $x \in [0,b]$ provided $f \in C_\gamma[0,\infty)$, $\gamma > 0$, b>0.

Theorem 3.2. For every $f \in C_B[0,\infty), x \geq 0$ and n>1, we have

$|R_n^*(f;x) - f(x)| \leq 2w(f,\delta_{n,x})$,

where $\delta_{n,x} := \sqrt{\left[\frac{a^2}{(n-1)}\left(n-\frac{2}{3}\right)+1\right]x^2 - 2axq_n^*(x)}$ and $q_n^*(x)$ is defined by (1.3)

Proof:

Let $f \in c_B[0,\infty)$ and x$\geq$0. Then, the modulus of continuity of f denoted by $w(f,\delta)$, is defined to be

$$w(f,\delta) = \sup_{|y-x|\leq\delta; x,y\in[0,\infty)} |f(y) - f(x)|$$

Now, let $f \in C_B[0,\infty), x \geq 0$. Using linearity and monotonicity of U_n^* we easily get, for every $\delta > 0$ and n$\in$N, that

$|R_n^*(f;x) - f(x)| \leq w(f,\delta)\left\{1+\frac{1}{\delta}\sqrt{R_n^*(\psi^2;x)}\right\}$

Now applying lemma 2.2(b) and choosing $\delta = \delta_{n,x}$ the proof is completed.

Now we can also compute the rate of convergence of the operators R_n^* by means of the elements of the Lipschitz class $Lip_M(\alpha), \alpha \in (0,1]$. To get this, we recall that a function

$f \in C_B[0,\infty)$ belongs to $Lip_M(\alpha)$. if the following inequality holds

(3.2) $|f(y) - f(x)| \leq M|y-x|^\alpha$ Where x, y$\in[0,\infty)$

Theorem 3.3 . For every $f\in Lip_M(\alpha)$., $x \geq 0$ and n>1, we have

$$|R_n^*(f;x) - f(x)| \leq M\left[\left\{\frac{a^2}{(n-1)}\left(n-\frac{2}{3}\right)+1\right\}x^2 - 2axq_n^*(x)\right]^{\frac{\alpha}{2}}$$

Where $q_n^*(x)$ is given by (1.3)

Proof: Since $f \in Lip_M(\alpha).$, $x \geq 0$, using inequality (3.2) and then applying the holder inequality with p= $\frac{2}{\alpha}, q = \frac{2}{2-\alpha}$

$$
\begin{aligned}
| R_n^*(f;x) - f(x) | &\leq R_n^*(| f(y) - f(x) |; x) \\
&\leq M\ R_n^*(| y - x |^{\alpha}; x) \\
&\leq M\ \{R_n^*(\psi^2; x)\}^{\frac{\alpha}{2}} \\
&\leq M\ \{R_n^*(\psi^2; x)\}^{\frac{\alpha}{2}} \\
&\leq M\ \{R_n^*(\psi^2; x)\}^{\frac{\alpha}{2}}
\end{aligned}
$$

Hence the result.

Acknowledgment

The author is thankful to the reviewer(s) for making valuable suggestions leading to the better presentation of the paper.

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Received: May, 2011