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On Properties of Systems of Root Functions of Well-Posed Boundary Value Problems for the Second Order Differential Operator

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Abstract

In this paper, we consider the second order differential operator of L_σ with nonlocal boundary conditions in the functional space $\mathbf{L}_2(0, 1)$. We construct an explicit system of root functions of L_σ. We study the biorthogonal of properties the systems of root functions of L_σ. We develop a method for constructing biorthogonal systems of root functions of well-posed boundary value problems for the second order differential operator with nonlocal boundary conditions.

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1 Introduction

Let $\sigma(\cdot)$ be arbitrary function from the functional space $\mathbf{L}_2(0,1)$. We introduce the entire function with respect to λ

$$\Delta(\lambda) = 1 - \lambda \int_0^1 cos\sqrt{\lambda}\, x\, \overline{\sigma(x)}\, dx \tag{1.1}$$

Denote by $\Lambda = \{\lambda_1, \lambda_2, \cdots\}$ sequence of zeros of entire function $\Delta(\lambda)$. Each zero of λ_n the function $\Delta(\lambda)$ has a some multiplicity m_n. In this paper, for clarity all results are illustrated of $m_n = 2$. In this case $\Delta(\lambda_n) = 0$, $\Delta'(\lambda_n) = 0$, $\Delta^{(2)}(\lambda_n) \neq 0$. We introduce the chain of functions

$$E_n = \{cos\sqrt{\lambda_n}\, x\,, -\frac{x\, sin\sqrt{\lambda_n}\, x}{2\sqrt{\lambda_n}}\}$$

The system of functions

$$E = \{E_n : \lambda_n \text{ are zeros of the function } \Delta(\lambda)\}$$

called the union of all such chains.

Main purpose: Constructively to build an adjoint system of functions to the system of functions E in the functional space $\mathbf{L}_2\,(0.1)$ (Theorem 4.1). Note that the system of function E is a system of root functions of second order differential operator, where the function $\sigma\,(\cdot)$ is a boundary function. Details are described in the section 2 below. In the case of the differentiation operator as the system of root functions there arises a system of exponentials that studied in detail in [4].

2 Boundary value problems and auxiliary notation

In [2] proved the following statement

Theorem (M. Otelbaev) a) For any choice of functions $\sigma_\nu(x), \nu = 1, 2$ from the space $L_2(0, 1)$ to the nonlocal boundary value problem

$$-y''(x) = f(x), 0 < x < 1, \tag{2.1}$$

$$y^{(\nu-1)}(0) - \int_0^1 (-y''(x))\overline{\sigma_\nu(x)}dx = 0, \nu = 1, 2. \tag{2.2}$$

corresponds to the operator L in the functional space $\mathbf{L}_2(0, 1)$, where L has completely continuous inverse of L^{-1}.

b) Assume that the nonhomogeneous equation (2.1) with some additional conditions for any right side $f(x) \in \mathbf{L}_2(0, 1)$ has a unique solution $y(x)$ in the functional space $\mathbf{W}_2^2[0, 1]$, where $y(x)$ has the a priori estimate

$$\| y \|_{L_2(0,1)} \leq c \| f \|_{L_2(0,1)}$$

Then there exists a unique set of functions $\{\sigma_\nu(x)\}, \nu = 1, 2$ from the functional space $\mathbf{L}_2(0, 1)$ that the additional conditions are equivalent to (2.2).

It follows from Theorem (M. Otelbaev) that the nonlocal boundary conditions (2.2) for all possible $\{\sigma_\nu(x)\}, \nu = 1, 2$ from the functional space $\mathbf{L}_2(0, 1)$

describe everything well-posed solvable boundary value problems corresponding to expression of $\ell(\cdot)$.

Without loss of generality in can be assumed that in the problem (2.1), (2.2) the function $\sigma_2(\cdot) = 0$. Thus, we consider the operator L_σ in the functional space $\mathbf{L}_2\,(0,1)$ corresponding to the following nonlocal boundary value problem

$$-y''(x) = f(x), 0 < x < 1, \tag{2.1}$$

$$y(0) - \int_0^1 (-y''(x))\overline{\sigma(x)}\,dx = 0, \tag{2.3}$$

$$y'(0) = 0, \tag{2.4}$$

where $\sigma(x) \in \mathbf{L}_2(0,1)$.

3 Resolvent of the operator L_σ

In this section we compute an explicit solution of the nonlocal boundary value problem

$$-y''(x) = \lambda y(x) + f(x), 0 < x < 1, \tag{3.1}$$

$$y(0) - \int_0^1 (-y''(x))\overline{\sigma(x)}\,dx = 0, \tag{2.3}$$

$$y'(0) = 0. \tag{2.4}$$

The solution of this nonlocal boundary value problem is called a resolvent of L_σ. The explicit form of the resolvent has a significant meaning for the study properties of biorthogonal systems of root functions of L_σ.

Theorem 3.1 A resolvent of the operator L_σ is determined by the formula

$$y(x) = (L_\sigma - \lambda I)^{-1} f(x) = \frac{< f(t),\, M_{\overline{\lambda}}(t) >}{\Delta(\lambda)} cos\sqrt{\lambda}\, x + \int_0^x \frac{sin\sqrt{\lambda}\,(t-x)}{\sqrt{\lambda}} f(t)\,dt, \tag{3.2}$$

where

$$M_{\overline{\lambda}}(t) = \sigma(t) + \overline{\lambda} \int_t^1 \frac{sin\sqrt{\overline{\lambda}}\,(t-x)}{\sqrt{\overline{\lambda}}} \sigma(x)\,dx, \tag{3.3}$$

and the entire function $\Delta(\lambda)$ is defined by formula (1.1).

Proof General solution of differential equations (3.1) is a function of

$$y(x) = c_1 cos\sqrt{\lambda}\, x + c_2 \frac{sin\sqrt{\lambda}\, x}{\sqrt{\lambda}} + \int_0^x \frac{sin\sqrt{\lambda}\,(t-x)}{\sqrt{\lambda}} f(t)\,dt, \tag{3.4}$$

where $\{cos\sqrt{\lambda}\, x, \frac{sin\sqrt{\lambda}\, x}{\sqrt{\lambda}}\}$ is a fundamental system of solutions for the homogeneous differential equation (3.1).

We substitute equation (3.4) also its first and second order derivatives on the boundary conditions (2.3) (2.4). Consequently, we have $c_1 = \frac{<f(t), M_{\overline{\lambda}}(t)>}{\Delta(\lambda)}$, $c_2 = 0$. Using the values of c_1, c_2 in equation (3.4), we obtain (3.2).

The proof is complete.

The entire function $\Delta(\lambda)$ is called the characteristic function of L_σ. We formulate as a lemma some basic properties of functions $\Delta(\lambda)$.

Lemma 3.1 For any eigenvalues of λ_n of multiplicity $m_n = 2$ of the operator L_σ following properties hold:

$$1) \int_0^1 cos\sqrt{\lambda_n}\, x\, \overline{\sigma(x)}\, dx = \frac{1}{\lambda_n}; \;\; 2) \int_0^1 \frac{x\, sin\sqrt{\lambda_n}\, x}{\sqrt{\lambda_n}} \overline{\sigma(x)}\, dx = \frac{2}{\lambda_n^2}.$$

These relations are obtained directly from (1.1) for $\Delta(\lambda)$ taking account of multiplicity of eigenvalues.

4 System of root functions of L_σ and the corresponding adjoint system

In [3, p. 445] give a decomposition theorem. It follows from that for some $\delta > 0$ the projector $P_n : \mathbf{L}_2(0,1) \to Ker(L_\sigma - \lambda)^{m_n}$ is a residue of the resolvent at the singular point λ_n

$$(P_n f)(x) = -\frac{1}{2\pi i} \oint_{|\lambda - \lambda_n| = \delta} (L_\sigma - \lambda I)^{-1} f(x) d\lambda.$$

Recalling of representation (3.2) for resolvent from Theorem 3.1 and basis properties of residue form of the projector P_n can be refined

$$(P_n f)(x) =< f(t), -\lim_{\overline{\lambda} \to \overline{\lambda_n}} \frac{d}{d\overline{\lambda}} \frac{(\overline{\lambda} - \overline{\lambda_n})^2 M_{\overline{\lambda}}(t)}{\overline{\Delta(\lambda)}} > \, cos\sqrt{\lambda_n}\, x +$$

$$+ < f(t), -\lim_{\overline{\lambda} \to \overline{\lambda_n}} \frac{(\overline{\lambda} - \overline{\lambda_n})^2 M_{\overline{\lambda}}(t)}{\overline{\Delta(\lambda)}} > \left(-\frac{x\, sin\sqrt{\lambda_n}\, x}{2\sqrt{\lambda_n}} \right) \tag{4.1}$$

Let us remark that $\int_0^x \frac{sin\sqrt{\lambda}(t-x)}{\sqrt{\lambda}} f(t)\, dt$ is an entire function in λ. We give certain properties of systems of functions E as a lemma.

Lemma 4.1 The elements of the chain E_n satisfy the differential equations

$$-y''_{n,1}(x) = \lambda_n\, y_{n,1}(x) + y_{n,0}(x), \tag{4.2}$$

$$-y''_{n,0}(x) = \lambda_n y_{n,0}(x) \tag{4.3}$$

and nonlocal boundary conditions (2.3), (2.4).

Proof We check that the functions $y_{n,0}(x)$, $y_{n,1}(x)$ satisfy the conditions of the lemma. To do we find the first order and second order of derivatives of these functions. We have

$$y'_{n,0}(x) = -\sqrt{\lambda_n}\, sin\sqrt{\lambda_n}\, x, \quad y''_{n,0}(x) = -\lambda_n\, cos\sqrt{\lambda_n}\, x,$$

$$y'_{n,1}(x) = -\frac{sin\sqrt{\lambda_n}x}{2\sqrt{\lambda_n}} - \frac{x\, cos\sqrt{\lambda_n}x}{2}, \quad y''_{n,1}(x) = -cos\sqrt{\lambda_n}\, x + \frac{\sqrt{\lambda_n}\, x\, sin\sqrt{\lambda_n}\, x}{2}.$$

We calculate the linear combination $\lambda_n\, y_{n,1}(x) + y_{n,0}(x) = -\frac{\lambda_n\, x\, sin\sqrt{\lambda_n}\, x}{2\sqrt{\lambda_n}} +$ $+\, cos\sqrt{\lambda_n}\, x = -y''_{n,1}(x)$. Directly, $-y''_{n,0}(x) = -\lambda_n\, cos\sqrt{\lambda_n}\, x = \lambda_n\, y_{n,0}(x)$.

We check the boundary conditions (2.3) (2.4). It is obvious that $y'_{n,0}(0) = 0$, $y'_{n,1}(0) = 0$. Respectively,
$y_{n,0}(0) - \int_0^1 (-y''_{n,0}(x))\overline{\sigma(x)}dx = 1 - \lambda_n \int_0^1 cos\sqrt{\lambda_n}\, x\, \overline{\sigma(x)}dx = 0$ since is true the first property of Lemma 3.1. Also it follows from Lemma 3.1 that $y_{n,1}(0) -$ $- \int_0^1 (-y''_{n,1}(x))\overline{\sigma(x)}dx = -\int_0^1 cos\sqrt{\lambda_n}\, x\, \overline{\sigma(x)}dx + \frac{\lambda_n}{2}\int_0^1 \frac{x\, sin\sqrt{\lambda_n}\, x}{\sqrt{\lambda_n}}\overline{\sigma(x)}dx = 0.$

The proof is complete.

It follows from Lemma 4.1 that the system of function E is the system of root functions of L_σ.

We shall investigate biorthogonal of properties the systems of functions E. In a study of this question we need the following lemma.

Lemma 4.2 For arbitrary complex numbers λ, μ the rightly identity:

$$< cos\,\sqrt{\lambda}\, x, M_{\overline{\mu}}(x) > \equiv -\frac{\Delta(\lambda) - \Delta(\mu)}{\lambda - \mu} \qquad (4.4)$$

Proof We can write for arbitrary λ, μ the scalar product of $\lambda\ < cos\,\sqrt{\lambda}\, x, M_{\overline{\mu}}(x) >$ taking into account relations (3.3), (4.3) in the following form

$$\lambda < cos\,\sqrt{\lambda}\, x, M_{\overline{\mu}}(x) > =$$

$$= \lambda\ < cos\,\sqrt{\lambda}\, x, \sigma(x) > -\mu \int_0^1 \frac{d^2}{dx^2} cos\,\sqrt{\lambda}\, x \left(\int_x^1 \frac{sin\sqrt{\mu}\,(x-t)}{\sqrt{\mu}} \overline{\sigma(t)}dt \right) dx$$

We use formula for integration by parts to the second term of the last relation.

$$\lambda < cos\,\sqrt{\lambda}\, x, M_{\overline{\mu}}(x) > = \lambda\ < cos\,\sqrt{\lambda}\, x, \sigma(x) > -$$

$$-\mu \int_0^1 \frac{d}{dx} cos\,\sqrt{\lambda}\, x \left(\int_x^1 cos\sqrt{\mu}(x-t)\overline{\sigma(t)}dt \right) dx$$

Once again, we use the formula for integration by parts to the second term of the last relation. Also, given the first property of Lemma 3.1, we have

$$\lambda < cos\,\sqrt{\lambda}\, x, M_{\overline{\mu}}(x) > = -\Delta(\lambda) + \Delta(\mu) + \mu < cos\,\sqrt{\lambda}\, x, M_{\overline{\mu}}(x) >$$

From the obtained equation it follows the desired relation (4.4).

The proof is complete.

Analysis of (4.1) leads to the following notation:

$$E_n' = \{h_{n,0}(x),\, h_{n,1}(x)\},$$

where

$$h_{n,0}(x) = -\lim_{\lambda\to\overline{\lambda_n}} \frac{d}{d\overline{\lambda}} \frac{(\overline{\lambda}-\overline{\lambda_n})^2 M_{\overline{\lambda}}(x)}{\overline{\Delta(\lambda)}};\; h_{n,1}(x) = -\lim_{\lambda\to\overline{\lambda_n}} \frac{(\overline{\lambda}-\overline{\lambda_n})^2 M_{\overline{\lambda}}(x)}{\overline{\Delta(\lambda)}}.$$

We introduce the following family of functions

$$E' = \{E_n' : \lambda_n \;\text{ is arbitrary eigenvalue of the operator } L_\sigma\}$$

We formulate main result.

Theorem 4.1 The system of function E' is biorthogonal to the system of functions E, i.e.

$$< y_{n,j}(x),\, h_{n,k}(x) >= \begin{cases} 1, & \text{if } (n,j) = (n,k); \\ 0, & \text{if } (n,j) \neq (n,k),\; where\; j,k = 0,1. \end{cases}$$

Proof Let $j = 0$, $k = 0$. Then

$$< y_{n,0}(x),\, h_{n,0}(x) >= -\lim_{\lambda\to\lambda_n} \frac{d}{d\lambda} \frac{(\lambda-\lambda_n)^2}{\Delta(\lambda)} < cos\sqrt{\lambda_n}\,x, M_{\overline{\lambda}}(x) >$$

Considering of relation (4.4), we have

$$< y_{n,0}(x),\, h_{n,0}(x) >= \lim_{\lambda\to\lambda_n} \frac{d}{d\lambda} \frac{(\lambda-\lambda_n)^2}{\Delta(\lambda)} \frac{\Delta(\lambda_n)-\Delta(\lambda)}{\lambda_n-\lambda}$$

Since $\Delta(\lambda_n) = 0$ then the last relation takes the form

$$< y_{n,0}(x),\, h_{n,0}(x) >= \lim_{\lambda\to\lambda_n} \frac{d}{d\lambda}(\lambda-\lambda_n) = 1. \tag{4.5}$$

Let $j = 0$, $k = 1$. Then

$$< y_{n,0}(x),\, h_{n,1}(x) >= -\lim_{\lambda\to\lambda_n} \frac{(\lambda-\lambda_n)^2}{\Delta(\lambda)} < cos\sqrt{\lambda_n}\,x, M_{\overline{\lambda}}(x) >$$

Considering of relation (4.4), we have

$$< y_{n,0}(x),\, h_{n,1}(x) >= \lim_{\lambda\to\lambda_n} \frac{(\lambda-\lambda_n)^2}{\Delta(\lambda)} \frac{\Delta(\lambda_n)-\Delta(\lambda)}{\lambda_n-\lambda}$$

Since $\Delta(\lambda_n) = 0$ then the last relation takes the form

$$< y_{n,0}(x),\, h_{n,1}(x) >= \lim_{\lambda\to\lambda_n} (\lambda - \lambda_n) = 0. \tag{4.6}$$

Let $j = 1$, $k = 0$. Then

$$< y_{n,1}(x),\, h_{n,0}(x) >= \lim_{\lambda\to\lambda_n} \frac{d}{d\lambda} \frac{(\lambda-\lambda_n)^2}{\Delta(\lambda)} < \frac{x\, sin\sqrt{\lambda_n}x}{2\sqrt{\lambda_n}},\, M_{\overline{\lambda}}(x) > \tag{4.7}$$

Using formula (3.3), we calculate relation (4.7).

$$I = \lim_{\lambda\to\lambda_n} \frac{d}{d\lambda} \frac{(\lambda-\lambda_n)^2}{\Delta(\lambda)} < \frac{x\, sin\sqrt{\lambda_n}\, x}{2\sqrt{\lambda_n}},\, M_{\overline{\lambda}}(x) >=$$

$$= \lim_{\lambda\to\lambda_n} \frac{d}{d\lambda} \frac{(\lambda-\lambda_n)^2}{\Delta(\lambda)} < \frac{x\, sin\sqrt{\lambda_n}\, x}{2\sqrt{\lambda_n}},\, \sigma(x) > +$$

$$+ \lim_{\lambda\to\lambda_n} \frac{d}{d\lambda} \frac{(\lambda-\lambda_n)^2}{\Delta(\lambda)} < \frac{x\, sin\sqrt{\lambda_n}\, x}{2\sqrt{\lambda_n}},\, \overline{\lambda} \int_x^1 \frac{sin\sqrt{\overline{\lambda}}\,(x-t)}{\sqrt{\overline{\lambda}}} \sigma(t)dt >$$

Given the second property in Lemma 3.1, we calculate the first term of the last relation:

$$I_1 = \frac{1}{2} \lim_{\lambda\to\lambda_n} \frac{d}{d\lambda} \frac{(\lambda-\lambda_n)^2}{\Delta(\lambda)} < \frac{x\, sin\sqrt{\lambda_n}x}{\sqrt{\lambda_n}},\, \sigma(x) >= \frac{1}{\lambda_n^2} \lim_{\lambda\to\lambda_n} \frac{d}{d\lambda} \frac{(\lambda-\lambda_n)^2}{\Delta(\lambda)}$$

We introduce the notation $B(\lambda) = (\lambda - \lambda_n)^2$. Then

$$I_1 = \frac{1}{\lambda_n^2} \lim_{\lambda\to\lambda_n} \frac{d}{d\lambda} \frac{B(\lambda)}{\Delta(\lambda)} = \frac{1}{\lambda_n^2} \lim_{\lambda\to\lambda_n} \frac{B'(\lambda)\Delta(\lambda) - \Delta'(\lambda)B(\lambda)}{\Delta^2(\lambda)} = \left[\frac{0}{0} =?\right]$$

We apply L'Hôpital's rule to the last limit relation thrice. Also, given that $B'(\lambda_n) = 0$, $B^{(2)}(\lambda_n) = 2$, we have

$$I_1 = -\frac{2\,\Delta^{(3)}(\lambda_n)}{3(\lambda_n \Delta^{(2)}(\lambda_n))^2} \tag{4.8}$$

Now we compute the second term of I.

$$I_2 = \lim_{\lambda\to\lambda_n} \frac{d}{d\lambda} \frac{(\lambda-\lambda_n)^2}{\Delta(\lambda)} < \frac{x\, sin\sqrt{\lambda_n}\, x}{2\sqrt{\lambda_n}},\, \overline{\lambda} \int_x^1 \frac{sin\sqrt{\overline{\lambda}}\,(x-t)}{\sqrt{\overline{\lambda}}} \sigma(t)dt >=$$

$$= \frac{1}{2} \lim_{\lambda\to\lambda_n} \frac{d}{d\lambda} \frac{(\lambda-\lambda_n)^2}{\Delta(\lambda)} \lambda \int_0^1 \frac{x\, sin\sqrt{\lambda_n}\, x}{\sqrt{\lambda_n}} \left(\int_x^1 \frac{sin\sqrt{\lambda}\,(x-t)}{\sqrt{\lambda}} \overline{\sigma(t)}dt \right) dx$$

In the last integral we do a permutation of the limits:

$$I_2 = \frac{1}{2} \lim_{\lambda\to\lambda_n} \frac{d}{d\lambda} \frac{(\lambda-\lambda_n)^2}{\Delta(\lambda)} \lambda \int_0^1 \overline{\sigma(t)} \left(\int_0^t \frac{x\, sin\sqrt{\lambda_n}\, x}{\sqrt{\lambda_n}} \frac{sin\sqrt{\lambda}\,(x-t)}{\sqrt{\lambda}} dx \right) dt$$

We use formula of integration by parts to the inner integral the last relation.

$$I_2 = \frac{1}{2}\lim_{\lambda\to\lambda_n}\frac{d}{d\lambda}\frac{(\lambda-\lambda_n)^2}{\Delta(\lambda)}\lambda\int_0^1\overline{\sigma(t)}\left(\frac{t}{\lambda_n-\lambda}\,\frac{sin\sqrt{\lambda_n}\,t}{\sqrt{\lambda_n}}+\frac{2\,cos\sqrt{\lambda_n}\,t}{(\lambda_n-\lambda)^2}-\frac{2\,cos\sqrt{\lambda}\,t}{(\lambda_n-\lambda)^2}\right)dt$$

Taking into account Lemma 3.1 and relation (1.1) we have

$$I_2 = \frac{1}{\lambda_n^2}\lim_{\lambda\to\lambda_n}\frac{d}{d\lambda}\left(\frac{2\,\lambda\,\lambda_n-\lambda^2-\lambda_n^2+\lambda_n^2\Delta(\lambda)}{\Delta(\lambda)}\right)$$

We introduce the notation $F(\lambda)=2\,\lambda\,\lambda_n-\lambda^2-\lambda_n^2+\lambda_n^2\Delta(\lambda)$. Then

$$I_2=\frac{1}{\lambda_n^2}\lim_{\lambda\to\lambda_n}\frac{d}{d\lambda}\frac{F(\lambda)}{\Delta(\lambda)}=\frac{1}{\lambda_n^2}\lim_{\lambda\to\lambda_n}\frac{F'(\lambda)\Delta(\lambda)-\Delta'(\lambda)F(\lambda)}{\Delta^2(\lambda)}=\left[\frac{0}{0}=?\right]$$

We use L'Hôpital's rule to the last limit relation thrice. Note that $F'(\lambda_n)=0$, $F^{(2)}(\lambda_n)=-2-\lambda_n^2\Delta^{(2)}(\lambda_n)$, $F^{(3)}(\lambda_n)=-\lambda_n^2\Delta^{(3)}(\lambda_n)$. A result we have

$$I_2=\frac{2\Delta^{(3)}(\lambda_n)}{3(\lambda_n\Delta^{(2)}(\lambda_n))^2} \tag{4.9}$$

Taking (4.7), (4.8), and (4.9) we obtain

$$< y_{n,1}(x), h_{n,0}(x) >= 0 \tag{4.10}$$

Let $j=1$, $k=1$. Then

$$< y_{n,1}(x), h_{n,1}(x) >= \frac{1}{2}\lim_{\lambda\to\lambda_n}\frac{(\lambda-\lambda_n)^2}{\Delta(\lambda)} < \frac{x\,sin\sqrt{\lambda_n}\,x}{\sqrt{\lambda_n}}, M_{\overline{\lambda}}(x) > \tag{4.11}$$

Using formula (3.3), we calculate relation (4.11).

$$C=\frac{1}{2}\lim_{\lambda\to\lambda_n}\frac{(\lambda-\lambda_n)^2}{\Delta(\lambda)} < \frac{x\,sin\sqrt{\lambda_n}\,x}{\sqrt{\lambda_n}}, M_{\overline{\lambda}}(x) >= \frac{1}{2}\lim_{\lambda\to\lambda_n}\frac{(\lambda-\lambda_n)^2}{\Delta(\lambda)} < \frac{x\,sin\sqrt{\lambda_n}\,x}{\sqrt{\lambda_n}}, \sigma(x) > +$$

$$+\frac{1}{2}\lim_{\lambda\to\lambda_n}\frac{(\lambda-\lambda_n)^2}{\Delta(\lambda)} < \frac{x\,sin\sqrt{\lambda_n}\,x}{\sqrt{\lambda_n}}, \overline{\lambda}\int_x^1\frac{sin\sqrt{\overline{\lambda}}\,(x-t)}{\sqrt{\overline{\lambda}}}\sigma(t)\,dt >$$

Given the second property in Lemma 3.1, we calculate the first term of the last relation:

$$C_1=\frac{1}{2}\lim_{\lambda\to\lambda_n}\frac{(\lambda-\lambda_n)^2}{\Delta(\lambda)} < \frac{x\,sin\sqrt{\lambda_n}\,x}{\sqrt{\lambda_n}}, \sigma(x) >= \frac{1}{\lambda_n^2}\lim_{\lambda\to\lambda_n}\frac{(\lambda-\lambda_n)^2}{\Delta(\lambda)}=\left[\frac{0}{0}=?\right]$$

We apply L'Hôpital's rule to the last limit relation twice. We have

$$C_1=\frac{2}{\lambda_n^2\Delta^{(2)}(\lambda_n)} \tag{4.12}$$

Now we compute the second term of C.

$$C_2 = \frac{1}{2} \lim_{\lambda \to \lambda_n} \frac{\lambda\,(\lambda - \lambda_n)^2}{\sqrt{\lambda\,\lambda_n}\,\Delta(\lambda)} \int_0^1 x\, sin\sqrt{\lambda_n}\, x \left(\int_x^1 sin\,\sqrt{\lambda}\,(x-t)\,\overline{\sigma(t)}\,dt \right) dx$$

In the last integral we do a permutation of the limits:

$$C_2 = \frac{1}{2} \lim_{\lambda \to \lambda_n} \frac{\lambda\,(\lambda - \lambda_n)^2}{\sqrt{\lambda\,\lambda_n}\,\Delta(\lambda)} \int_0^1 \overline{\sigma(t)} \left(\int_0^t x\, sin\sqrt{\lambda_n}\, x\ sin\,\sqrt{\lambda}\,(x-t)\,dx \right) dt$$

We use formula of integration by parts to the inner integral the last relation.

$$C_2 = \frac{1}{2} \lim_{\lambda \to \lambda_n} \frac{\lambda\,(\lambda - \lambda_n)^2}{\Delta(\lambda)} \int_0^1 \overline{\sigma(t)} \left(\frac{t}{\lambda_n - \lambda} \frac{sin\,\sqrt{\lambda_n}\,t}{\sqrt{\lambda_n}} + \frac{2\,cos\,\sqrt{\lambda_n}\,t}{(\lambda_n - \lambda)^2} - \frac{2\,cos\,\sqrt{\lambda}\,t}{(\lambda_n - \lambda)^2} \right) dt$$

Taking into account Lemma 3.1 and relation (1.1) we have

$$C_2 = \frac{1}{\lambda_n^2} \lim_{\lambda \to \lambda_n} \frac{2\lambda\,\lambda_n - \lambda^2 - \lambda_n^2 + \lambda_n^2 \Delta(\lambda)}{\Delta(\lambda)}$$

We introduce the notation $N(\lambda) = 2\lambda\,\lambda_n - \lambda^2 - \lambda_n^2 + \lambda_n^2 \Delta(\lambda)$. Then

$$C_2 = \frac{1}{\lambda_n^2} \lim_{\lambda \to \lambda_n} \frac{N(\lambda)}{\Delta(\lambda)} = \left[\frac{0}{0} = ? \right]$$

We use L'Hôpital's rule to the last limit relation thrice. Note that $N'(\lambda_n) = 0$, $N''(\lambda_n) = -2 + \lambda_n^2 \Delta^{(2)}(\lambda_n)$. We obtain

$$C_2 = -\frac{2}{\lambda_n^2\, \Delta^{(2)}(\lambda_n)} + 1. \tag{4.13}$$

Taking (4.11), (4.12), and (4.13) we have

$$< y_{n,1}(x), h_{n,1}(x) >= 1 \tag{4.15}$$

It follows from (4.5), (4.6), (4.10) and (4.15) that the main result.

The proof is complete.

It follows from Theorem 4.1 that the system of E' is biorthogonal to the system of E. Consequently, the system of functions E is a minimal system of functions [1, p. 171].

References

[1] N.K. Nikol'skií, Lektsii ob operatore sdviga. Nauka, Moscow, 1980 [English translation: N.K. Nikol'skií, Treatise on the Shift Operator. Spectral Function Theory, By Springer - Verlag Berlin Heidelberg, Germany, 1986].

[2] M. Otelbaev and A.N. Shynybekov, Well-posed problems of Bitsadze-Samarskii type, Dokl. Akad. Nauk SSSR, **265**(1982), no.4, 815—919.

[3] F. Riesz and B.Sz.-Nagy, Functional analysis. Mir, Moscow, 1979.

[4] A.M. Sedletskii, Biorthogonal expansions of functions in exponential series on intervals of the real axis, Uspekhi Mat. Nauk (Russian), **37**(1982), no.5, 51—95 [English translation: Russian Mathematical Surveys, **37**(1982), no.5, 57—108].

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