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Robust Exponential Stability for Uncertain Linear Non-Autonomous Systems with Discrete and Distributed Time-Varying Delays

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Abstract

This paper investigates the problems of robust exponential stability and stabilization for uncertain linear non-autonomous control systems with discrete and distributed time-varying delays. Based on combination of the Riccati differential equation approach, linear matrix inequality (LMI) techinque and the use of suitable Lyapunov-Krasovskii functional, new sufficient conditions for the robust exponential stability and stabilization are obtained in terms of the solution of Riccati differential equations (RDEs) and LMIs.

Keywords: Robust exponential stability, Stabilization, Distributed delay, Riccati differential equation, Non-autonomous control delayed system

1 Introduction

Over the past decades, uncertain systems with state delays have been a topic of recurrent interest since time delays and uncertainties are frequently a source of instability or poor performances in various systems such as electric, chemical processes, long transmission line in pneumatic systems, and so on [8]. The problems of robust stability and stabilization for uncertain dynamical systems with or without state delays have been intensively studied in the past years by many researchers mathematics and control communities [1, 24]. Stability criteria for dynamical systems with time delay is generally divided into two classes: delay-independent ones and delay-dependent ones. Delay-independent stability criteria tends to be more conservative, especially for small size delay, such criteria do not give any information on the size of the delay. On the other hand, delay-dependent stability criteria is concerned with the size of the delay and usually provide a maximal delay size.

Recently, many researchers have been studied the problem of stability and stabilization for time delay systems with distributed delays such as [7] and [14] studied the problem of stability for the type of linear switching systems with discrete and distributed delays. In [24], they presented some stability conditions for uncertain neutral systems with discrete and distributed delays. The robust stability of uncertain linear neutral systems with discrete and distributed delays has studied in [9]. Numerous researchers deal with the stability of linear time-invariant (LTI) delayed systems [6, 7, 10, 11, 13, 15, 17, 22, 24] while only few paper [3, 16, 19, 21] studied some stability conditions for linear time-varying (LTV) delayed systems such as [16] presented a new sufficient delay dependent exponential stability condition for a class of linear time-varying systems with nonlinear delayed perturbations by using a suitable Lyapunov-Krasovskii functional. In [20], they presented sufficient conditions for the exponential stability and stabilization of uncertain linear time-varying systems by using parameter dependent Lyapunov function.

In this paper, we shall investigate the problems of robust exponential stability and stabilization for uncertain linear non-autonomous (linear time-varying) control system with discrete and distributed time-varying delays. Based on combination of the Riccati equation approach, linear matrix inequality (LMI) techinque and the use of suitable Lyapunov-Krasovskii functional, new sufficient conditions for the robust exponential stability will be obtained in terms of the solution of RDEs and LMIs.

2 Problem formulation and preliminaries

We introduce some notations and definitions that will be used throughout the paper. $\mathbb{R}^+$ denotes the set of all real non-negative numbers; $\mathbb{R}^n$ denotes the n-dimensional space with the vector norm $\|\cdot\|$; $\|x\|$ denotes the Euclidean vector norm of $x \in \mathbb{R}^n$; $M^{n\times r}$ denotes the space of all matrices of $(n \times r)$-dimensions; A^T denotes the transpose of the matrix A; A is symmetric if $A = A^T$; I denotes the identity matrix; $\lambda(A)$ denotes the set of all eigenvalues of A; $\lambda_{\max}(A) = \max\{\mathrm{Re}\,\lambda : \lambda \in \lambda(A)\}$; Matrix A is called semi-positive definite ($A \geq 0$) if $x^TAx \geq 0$, for all $x \in \mathbb{R}^n$; A is positive definite ($A > 0$) if $x^TAx > 0$ for all $x \neq 0$; Matrix B is called semi-negative definite ($B \leq 0$) if $x^TBx \leq 0$, for all $x \in \mathbb{R}^n$; B is negative definite ($B < 0$) if $x^TBx < 0$ for all $x \neq 0$; $A > B$ means $A - B > 0$; $A \geq B$ means $A - B \geq 0$; $C([-h, 0], \mathbb{R}^n)$ denotes the space of all piecewise continuous vector functions mapping $[-h, 0]$ into $\mathbb{R}^n$; $BM^+(0,\infty)$ denotes the set of all symmetric semi-positive definite matrix functions bounded on $[0,\infty)$; $*$ represents the elements below the main diagonal of a symmetric matrix.

Consider the uncertain linear non-autonomous control system with discrete and distributed time-varying delays of the form

$$\begin{cases} \dot{x}(t) = \widetilde{A}(t)x(t) + \widetilde{B}(t)x(t-h(t)) + \widetilde{C}(t)\int_{t-r(t)}^{t} x(s)ds \\ \qquad +D(t)u(t), \qquad\qquad\qquad\qquad\qquad\qquad t \geq 0; \\ x(t_0+\theta) = \phi(\theta), \qquad\qquad\qquad\qquad \forall\theta \in [-\max\{r,h\},0], \\ \widetilde{A}(t) = \Big[A(t) + \Delta A(t)\Big], \widetilde{B}(t) = \Big[B(t) + \Delta B(t)\Big], \\ \widetilde{C}(t) = \Big[C(t) + \Delta C(t)\Big], \end{cases} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^n$ is the control, $A(t)$, $B(t)$, $C(t), D(t) \in \mathbb{R}^{n\times n}$ are matrices continuous function and bounded in $t \geq 0$. $h(t)$ is a given time-varying delay function and $r(t)$ is a given time-varying distributed delay function satisfy

$$0 \leq h(t) \leq h, \quad \dot{h}(t) \leq \delta < 1, \quad 0 \leq r(t) \leq r, \tag{2.2}$$

where h, r, δ are positive real numbers. The initial condition function $\phi(t) \in C([-\max\{r,h\}\ \ ,0], \mathbb{R}^n)$ denotes a continuous vector-valued initial function of $t \in [-\max\{r,h\},0]$ with the norm $\|\phi\| = \sup_{s\in[-\max\{r,h\},0]} \|\phi(s)\|$. The uncertainties $\Delta A(t)$, $\Delta B(t)$ and $\Delta C(t)$ are time varying matrices and satisfy the condition

$$\Delta A(t) = E_1(t)F(t)M_1(t), \quad \Delta B(t) = E_2(t)F(t)M_2(t),$$

$$\Delta C(t) = E_3(t)F(t)M_3(t),$$

where $E_i(t), M_i(t), i = 1,2,3$ are matrices continuous function and bounded in $t \geq 0$. The uncertain matrix $F(t)$ satisfies

$$F(t)^T F(t) \leq I. \tag{2.3}$$

Definition 2.1 The system (2.1) where $u(t) = 0$ is robust exponentially stable, if there exist positive real numbers α and M such that for each $\phi(t) \in C([-\max\{r,h\},0], R^n)$, the solution $x(t,\phi)$ of the system (2.1) where $u(t) = 0$ satisfies

$$\|x(t,\phi)\| \leq M\|\phi\|e^{-\alpha t}, \quad \forall t \in \mathbb{R}^+.$$

If there exists the state feedback controller $u(t) = Kx(t)$ in (2.1) where $K \in \mathbb{R}^{n\times n}$ is a constant gain matrix, the closed-loop system (2.1) is robust exponentially stable, then we say system (2.1) is robust exponentially stabilizable.

Lemma 2.1 *[20] Let A, P, G, H, F be real matrices of appropriate dimensions and $P > 0$, $F^T F \leq I$. Then* *(i) For any* $\epsilon > 0 : GFH + H^T F^T G^T \le \epsilon^{-1} GG^T + \epsilon H^T H$.

(ii) For any $\epsilon > 0$ *such that* $\epsilon I - HH^T > 0$,
$(A+GFH)P(A+GFH)^T \le APA^T + APH^T(\epsilon I - HPH^T)^{-1}HPA^T + \epsilon^{-1}GG^T$.

Lemma 2.2 (Schur complement lemma) *[24] Given constant symmetric matrices* X, Y, Z *where* $Y > 0$. *Then* $X + Z^T Y^{-1} Z < 0$ *if and only if*

$$\begin{pmatrix} X & Z^T \\ Z & -Y \end{pmatrix} < 0 \quad or \quad \begin{pmatrix} -Y & Z \\ Z^T & X \end{pmatrix} < 0.$$

3 Stability conditions

3.1 Exponential stability conditions

In this section, we first present the stability criteria for the systems (2.1) without uncertainties and control by combination of the Riccati differential equation approach and the use of suitable Lyapunov-Krasovskii functional. We introduce the following notations for later use.

$$P_\epsilon(t) = P(t) + \epsilon I, \quad Q(t) = 2\alpha P_\epsilon(t) + \varrho I,$$

$$R(t) = \frac{e^{2\alpha h}}{\epsilon_1(1-\delta)} B(t)B^T(t) + \frac{re^{2\alpha r}}{\epsilon_2} C(t)C^T(t),$$

where $\varrho = \epsilon_1 + \epsilon_2 r + \gamma$. Consider the Riccati differential equation

$$\dot{P}_\epsilon(t) + P_\epsilon(t)A(t) + A^T(t)P_\epsilon(t) + P_\epsilon(t)R(t)P_\epsilon(t) + Q(t) = 0. \tag{3.1}$$

Theorem 3.1 *The system* (2.1) *without uncertainties and control, is exponentially stable if there exist positive real numbers* $\alpha, \epsilon, \epsilon_1, \epsilon_2, \gamma$ *and a matrix function* $P(t) \in BM^+(0, \infty)$ *and the RDE* (3.1) *holds. Moreover, the solution* $x(t, \phi)$ *satisfies the inequality*

$$\|x(t,\phi)\| \le \sqrt{\frac{N}{\epsilon}} \|\phi\| e^{-\alpha t}, \quad t \in \mathbb{R}^+,$$

where

$$N = \lambda_{max} P(0) + \epsilon + \epsilon_1 \frac{(1 - e^{-2\alpha h})}{2\alpha} + 2\epsilon_2 r^2.$$

Proof. Let $P_\epsilon(t) \in BM^+(0, \infty)$, $t \in \mathbb{R}^+$, be a solution of the RDE (3.1). We define the following Lyapunov-Krasovskii function for system (2.1) without uncertainties and control of the form

$$\begin{aligned} V(t,x(t)) &= x^T(t)P_\epsilon(t)x(t) + \epsilon_1 \int_{t-h(t)}^{t} e^{2\alpha(s-t)}x^T(s)x(s)ds \\ &\quad +\epsilon_2 \int_{-r}^{0}\int_{t+s}^{t} e^{2\alpha(s-t)}x^T(\theta)x(\theta)d\theta ds. \end{aligned}$$

The derivative of $V(.)$ along the trajectories of system (2.1) without uncertainties and control is given by

$$\begin{aligned} \dot V(t,x(t)) &= x^T(t)\dot P(t)x(t) + \dot x^T(t)P_\epsilon(t)x(t) + x^T(t)P_\epsilon(t)\dot x(t) \\ &\quad +\epsilon_1 x^T(t)x(t) - \epsilon_1(1-\dot h(t))e^{-2\alpha h(t)}x^T(t-h(t))x(t-h(t)) \\ &\quad +\epsilon_2 r x^T(t)x(t) - \epsilon_2 \int_{-r}^{0} e^{2\alpha s}x^T(t+s)x(t+s)ds \\ &\quad -2\alpha\epsilon_1 \int_{t-h(t)}^{t} e^{2\alpha(s-t)}x^T(s)x(s)ds \\ &\quad -2\alpha\epsilon_2 \int_{-r}^{0}\int_{t+s}^{t} e^{2\alpha(s-t)}x^T(\theta)x(\theta)d\theta ds. \\ &\leq x^T(t)\dot P(t)x(t) + x^T(t)\Big[A^T(t)P_\epsilon(t) + P_\epsilon(t)A(t)\Big]x(t) \\ &\quad +x^T(t-h(t))B^T(t)P_\epsilon(t)x(t) + x^T(t)P_\epsilon(t)B(t)x(t-h(t)) \\ &\quad +\int_{t-r(t)}^{t}\Big[x^T(t)P_\epsilon(t)C(t)x(s) + x^T(s)C^T(t)P_\epsilon(t)x(t)\Big]ds \\ &\quad +\epsilon_1 x^T(t)x(t) - \epsilon_1(1-\delta)e^{-2\alpha h}x^T(t-h(t))x(t-h(t)) \\ &\quad +\epsilon_2 r x^T(t)x(t) - \epsilon_2 e^{-2\alpha r}\int_{t-r(t)}^{t} x^T(s)x(s)ds \\ &\quad -2\alpha\epsilon_1 \int_{t-h(t)}^{t} e^{2\alpha(s-t)}x^T(s)x(s)ds \\ &\quad -2\alpha\epsilon_2 \int_{-r}^{0}\int_{t+s}^{t} e^{2\alpha(s-t)}x^T(\theta)x(\theta)d\theta ds. \end{aligned} \tag{3.2}$$

We consider

$$\begin{aligned} &-\frac{\epsilon_1(1-\delta)}{e^{2\alpha h}}x^T(t-h(t))x(t-h(t)) + x^T(t-h(t))B^T(t)P_\epsilon(t)x(t) \\ &+x^T(t)P_\epsilon(t)B(t)x(t-h(t)) \\ = \;& -\Big[\frac{\epsilon_1(1-\delta)}{e^{2\alpha h}}x^T(t-h(t)) - x^T(t)P_\epsilon(t)B(t)\Big]\frac{e^{2\alpha h}}{\epsilon_1(1-\delta)}\Big[\frac{\epsilon_1(1-\delta)}{e^{2\alpha h}}x(t-h(t)) \end{aligned}$$

$$
\begin{aligned}
& -B^T(t)P_\epsilon(t)x(t)\Big] + \frac{e^{2\alpha h}}{\epsilon_1(1-\delta)}x^T(t)P_\epsilon(t)B(t)B^T(t)P_\epsilon(t)x(t) \\
\leq\ & \frac{e^{2\alpha h}}{\epsilon_1(1-\delta)}x^T(t)P_\epsilon(t)B(t)B^T(t)P_\epsilon(t)x(t), \qquad (3.3)
\end{aligned}
$$

and

$$
\begin{aligned}
& -\epsilon_2 e^{-2\alpha r}\int_{t-r(t)}^{t} x^T(s)x(s)ds \\
& + \int_{t-r(t)}^{t}\Big[x^T(t)P_\epsilon(t)C(t)x(s) + x^T(s)C^T(t)P_\epsilon(t)x(t)\Big]ds \\
\leq\ & -\int_{t-r(t)}^{t}\Big[\epsilon_2 e^{-2\alpha r}x^T(s) - x^T(t)P_\epsilon(t)C(t)\Big]\epsilon_2^{-1}e^{2\alpha r}\Big[\epsilon_2 e^{-2\alpha r}x(s) \\
& -C^T(t)P_\epsilon(t)x(t)\Big]ds + \int_{t-r(t)}^{t}\epsilon_2^{-1}e^{2\alpha r}x^T(t)P_\epsilon(t)C(t)C^T(t)P_\epsilon(t)x(t)ds \\
\leq\ & \frac{re^{2\alpha r}}{\epsilon_2}x^T(t)P_\epsilon(t)C(t)C^T(t)P_\epsilon(t)x(t). \qquad (3.4)
\end{aligned}
$$

From (3.2), (3.3) and (3.4), we obtain

$$
\begin{aligned}
\dot{V}(t,x(t)) + 2\alpha V(t,x(t)) \ \leq\ & x^T(t)\Big[\dot{P}(t) + A^T(t)P_\epsilon(t) + P_\epsilon(t)A(t) + 2\alpha P_\epsilon(t) \\
& +\epsilon_1 I + \epsilon_2 rI + \frac{e^{2\alpha h}}{\epsilon_1(1-\delta)}P_\epsilon(t)B(t)B^T(t)P_\epsilon(t) \\
& +\frac{re^{2\alpha r}}{\epsilon_2}P_\epsilon(t)C(t)C^T(t)P_\epsilon(t)\Big]x(t).
\end{aligned}
$$

Since $P(t)$ is the solution of (3.1). Therefore,

$$\dot{V}(t,x(t)) + 2\alpha V(t,x(t)) \leq 0, \quad \forall t \in \mathbb{R}^+,$$

which gives

$$V(t,x(t)) \leq V(0,x(0))e^{-2\alpha t}, \quad \forall t \in \mathbb{R}^+. \qquad (3.5)$$

From (3.5), it is easy to see that

$$\epsilon\|x(t)\|^2 \leq V(t,x(t)) \leq V(0,x(0))e^{-2\alpha t} \leq N\|\phi\|^2 e^{-2\alpha t}, \qquad (3.6)$$

where $N = \lambda_{max}P(0) + \epsilon + \epsilon_1\frac{(1-e^{-2\alpha h})}{2\alpha} + 2\epsilon_2 r^2$. From (3.6), we get

$$\|x(t)\| \leq \sqrt{\frac{N}{\epsilon}}\|\phi\|e^{-\alpha t}, \quad \forall t \in \mathbb{R}^+. \qquad (3.7)$$

This means that the system (2.1) without uncertainties and control is exponentially stable. The proof of the theorem is complete. □

Remark 1. When the system (2.1) without uncertainties and control is time-invariant, using the Schur complement lemma, the RDE (3.1) can be rewritten in terms of the LMI:

$$\begin{pmatrix} \Delta_{11} & P_\epsilon B & P_\epsilon C \\ B^T P_\epsilon & -\epsilon_1(1-\delta)e^{-2\alpha h} I & 0 \\ C^T P_\epsilon & 0 & -\frac{\epsilon_2}{r} e^{-2\alpha r} I \end{pmatrix} < 0,$$

where

$$\Delta_{11} := P_\epsilon A + A^T P_\epsilon + 2\alpha P_\epsilon + \varrho I.$$

3.2 Robust Exponential Stability Conditions

We introduce the following notations for later use.

$$\begin{aligned} P_\epsilon(t) &= P(t) + \epsilon I, \quad \overline{Q}(t) = 2\alpha P_\epsilon(t) + \epsilon_3 {M_1}^T(t) M_1(t) + \varrho I, \\ \overline{R}(t) &= \frac{e^{2\alpha h}}{\epsilon_1(1-\delta)} \Big[B(t)B^T(t) + B(t){M_2}^T(t)\big[\epsilon_4 I - M_2(t){M_2}^T(t)\big]^{-1} \\ & \quad \times M_2(t)B^T(t) + \epsilon_4^{-1} E_2(t){E_2}^T(t)\Big] + \frac{re^{2\alpha r}}{\epsilon_2}\Big[C(t)C^T(t) + C(t){M_3}^T(t) \\ & \quad \times \big[\epsilon_5 I - M_3(t){M_3}^T(t)\big]^{-1} M_3(t)C^T(t) + \epsilon_5^{-1} E_3(t){E_3}^T(t)\Big] \\ & \quad + \epsilon_3^{-1} E_1(t)E_1^T(t). \end{aligned}$$

Consider the Riccati differential equation

$$\dot{P}_\epsilon(t) + P_\epsilon(t)A(t) + A^T(t)P_\epsilon(t) + P_\epsilon(t)\overline{R}(t)P_\epsilon(t) + \overline{Q}(t) = 0. \tag{3.8}$$

Theorem 3.2 *The system* (2.1) *where* $u(t) = 0$ *is robust exponentially stable if there exist positive real numbers* $\gamma, \alpha, \epsilon, \epsilon_i, i = 1, 2, ..., 5$ *and a matrix function* $P(t) \in BM^+(0, \infty)$ *such that* $\epsilon_4 I - M_2(t){M_2}^T(t) > 0$, $\epsilon_5 I - M_3(t){M_3}^T(t) > 0$ *and the RDE* (3.8) *holds. Moreover, the solution* $x(t, \phi)$ *satisfies the inequality*

$$\|x(t,\phi)\| \le \sqrt{\frac{N}{\epsilon}} \|\phi\| e^{-\alpha t}, \quad t \in \mathbb{R}^+,$$

where

$$N = \lambda_{max} P(0) + \epsilon + \epsilon_1 \frac{(1 - e^{-2\alpha h})}{2\alpha} + 2\epsilon_2 r^2.$$

Proof. We prove this theorem by similarly to prove in Theorem 3.1. We consider Lyapunov-Krasovskii function in Theorem 3.1 for system (2.1) where

$u(t) = 0$ and we have $\widetilde{A}(t) = \Big[A(t) + E_1(t)F(t)M_1(t)\Big], \widetilde{B}(t) = \Big[B(t) + E_2(t)F(t)M_2(t)\Big]$ and $\widetilde{C}(t) = \Big[C(t) + E_3(t)F(t)M_3(t)\Big]$. Consider

$$\begin{aligned}
\widetilde{A}^T(t)P_\epsilon(t) + P_\epsilon(t)\widetilde{A}(t) &= P_\epsilon(t)A(t) + A^T(t)P_\epsilon(t) + P_\epsilon(t)E_1(t)F(t)M_1(t) \\
&\quad + {M_1}^T(t)F^T(t){E_1}^T(t)P_\epsilon(t), \\
\widetilde{B}(t)\widetilde{B}^T(t) &= [B(t) + E_2(t)F(t)M_2(t)][B(t) + E_2(t)F(t)M_2(t)]^T, \\
\widetilde{C}(t)\widetilde{C}^T(t) &= [C(t) + E_3(t)F(t)M_3(t)][C(t) + E_3(t)F(t)M_3(t)]^T.
\end{aligned}$$

Applying Lemma 2.1, we obtain

$$\begin{aligned}
\widetilde{A}^T(t)P_\epsilon(t) + P_\epsilon(t)\widetilde{A}(t) &\leq P_\epsilon(t)A(t) + A^T(t)P_\epsilon(t) + \epsilon_3^{-1}P_\epsilon(t)E_1(t){E_1}^T(t)P_\epsilon(t) \\
&\quad + \epsilon_3 {M_1}^T(t)M_1(t), \\
\widetilde{B}(t)\widetilde{B}^T(t) &\leq B(t){M_2}^T(t)[\epsilon_4 I - M_2(t){M_2}^T(t)]^{-1}M_2(t)B^T(t) \\
&\quad + B(t)B^T(t) + \epsilon_4^{-1}E_2(t){E_2}^T(t), \\
\widetilde{C}(t)\widetilde{C}^T(t) &\leq C(t){M_3}^T(t)[\epsilon_5 I - M_3(t){M_3}^T(t)]^{-1}M_3(t)C^T(t) \\
&\quad + C(t)C^T(t) + \epsilon_5^{-1}E_3(t){E_3}^T(t).
\end{aligned}$$

By Theorem 3.1 and (3.8), the system (2.1) where $u(t) = 0$ is robust exponentially stable. The proof of the theorem is complete. □

As an application, we consider the uncertain linear autonomous system with discrete and distributed time-varying delays of the form

$$\begin{cases}
\dot{x}(t) = A(t)x(t) + B(t)x(t - h(t)) + C(t)\int_{t-r(t)}^{t} x(s)ds, & t \geq 0; \\
x(t_0 + \theta) = \phi(\theta), & \forall\theta \in [-\max\{r, h\}, 0], \\
A(t) = \Big[A + E_1F(t)M_1\Big], B(t) = \Big[B + E_2F(t)M_2\Big], & \\
C(t) = \Big[C + E_3F(t)M_3\Big], &
\end{cases} \tag{3.9}$$

where $A, B, C, E_i, M_i,\ i = 1, 2, 3$ are constant matrices of appropriate dimensions and the uncertainty $F(t)$ satisfies (2.3). Therefore, we obtain the result.

Corollary 3.3 *The system* (3.9) *is robust exponentially stable if there exist symmetric positive definite matrices P and positive real numbers $\gamma, \alpha, \epsilon, \epsilon_i, i = 1, 2, ..., 5$ such that $\epsilon_4 I - M_2{M_2}^T > 0$, $\epsilon_5 I - M_3{M_3}^T > 0$ and the following LMI hold.*

$$\begin{bmatrix}
\Delta_{11} & P_\epsilon B & P_\epsilon C & P_\epsilon B M_2^T & P_\epsilon C M_3^T & P_\epsilon E_1 & P_\epsilon E_2 & P_\epsilon E_3 \\
* & -\omega I & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & -\mu I & 0 & 0 & 0 & 0 & 0 \\
* & * & * & \Delta_{44} & 0 & 0 & 0 & 0 \\
* & * & * & * & \Delta_{55} & 0 & 0 & 0 \\
* & * & * & * & * & -\omega\epsilon_3 I & 0 & 0 \\
* & * & * & * & * & * & -\omega\epsilon_4 I & 0 \\
* & * & * & * & * & * & * & -\mu\epsilon_5 I
\end{bmatrix} < 0, \tag{3.10}$$

where $\Delta_{11} = P_\epsilon A + A^T P_\epsilon + 2\alpha P_\epsilon + \epsilon_3 {M_1}^T M_1 + \varrho I$, $\Delta_{44} = -\frac{\epsilon_1(1-\delta)}{e^{2\alpha h}}(\epsilon_4 I - M_2 {M_2}^T)$ *and* $\Delta_{55} = -\frac{\epsilon_2}{re^{2\alpha r}}(\epsilon_5 I - M_3 {M_3}^T)$.

Proof. Because (3.9) is time-invariant system with uncertainties, using the Schur complement lemma and Theorem 3.1, 3.2. Then the RDE (3.8) can be rewritten in terms of the LMI (3.10). The proof of the corollary is complete.□

4 Stabilization conditions

Consider the Riccati differential equation of the form

$$\dot{P}_\epsilon(t) + P_\epsilon(t)[A(t) + C(t)K] + [A(t) + C(t)K]^T P_\epsilon(t) + P_\epsilon(t)\overline{R}(t)P_\epsilon(t) + \overline{Q}(t) = 0. \tag{4.1}$$

Theorem 4.1 *The system* (2.1) *is robust exponentially stabilizable if there exist positive real numbers* $\gamma, \alpha, \epsilon, \epsilon_i, i = 1, 2, ..., 5$, *a gain matrix* K *and a matrix function* $P(t) \in BM^+(0, \infty)$ *such that* $\epsilon_4 I - M_2(t){M_2}^T(t) > 0$, $\epsilon_5 I - M_3(t){M_3}^T(t) > 0$ *and the RDE* (4.1) *holds. Moreover, the solution* $x(t, \phi)$ *satisfies the inequality*

$$\|x(t,\phi)\| \leq \sqrt{\frac{N}{\epsilon}}\|\phi\|e^{-\alpha t}, \quad t \in \mathbb{R}^+,$$

where

$$N = \lambda_{max}P(0) + \epsilon + \epsilon_1\frac{(1 - e^{-2\alpha h})}{2\alpha} + 2\epsilon_2 r^2.$$

The feedback controller of (2.1) *is given by* $u(t) = Kx(t)$.

Proof. We prove this theorem by similarly to prove in Theorem 3.1, 3.2. The feedback controller of the system (2.1) is given by $u(t) = Kx(t)$. □

We consider the uncertain linear autonomous control system with discrete and distributed time-varying delays of the form

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)x(t - h(t)) + C(t)\int_{t-r(t)}^{t} x(s)ds + Du(t), t \geq 0; \\ x(t_0 + \theta) = \phi(\theta), \qquad \forall\theta \in [-\max\{r, h\}, 0], \\ A(t) = \Big[A + E_1F(t)M_1\Big], B(t) = \Big[B + E_2F(t)M_2\Big], \\ C(t) = \Big[C + E_3F(t)M_3\Big], \end{cases} \tag{4.2}$$

where $A, B, C, D, E_i, M_i,\ i = 1, 2, 3$ are constant matrices of appropriate dimensions and the uncertainty $F(t)$ satisfies (2.3). Therefore, we obtain the result.

Corollary 4.2 *The system* (4.2) *is robust exponentially stabilizable if there exist symmetric positive definite matrices* P*, a gain matrix* K *and positive real numbers* $\gamma, \alpha, \epsilon, \epsilon_i, i = 1, 2, ..., 5$ *such that* $\epsilon_4 I - M_2 {M_2}^T > 0$*,* $\epsilon_5 I - M_3 {M_3}^T > 0$ *and the following LMI hold.*

$$\begin{bmatrix} \Delta_{11} & P_\epsilon B & P_\epsilon C & P_\epsilon B M_2^T & P_\epsilon C M_3^T & P_\epsilon E_1 & P_\epsilon E_2 & P_\epsilon E_3 \\ * & -\omega I & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -\mu I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Delta_{44} & 0 & 0 & 0 & 0 \\ * & * & * & * & \Delta_{55} & 0 & 0 & 0 \\ * & * & * & * & * & -\omega\epsilon_3 I & 0 & 0 \\ * & * & * & * & * & * & -\omega\epsilon_4 I & 0 \\ * & * & * & * & * & * & * & -\mu\epsilon_5 I \end{bmatrix} < 0,$$

where $\Delta_{11} = P_\epsilon[A + DK] + [A + DK]^T P_\epsilon + 2\alpha P_\epsilon + \epsilon_3 {M_1}^T M_1 + \varrho I$*,* $\Delta_{44} = -\frac{\epsilon_1(1-\delta)}{e^{2\alpha h}}(\epsilon_4 I - M_2 {M_2}^T)$ *and* $\Delta_{55} = -\frac{\epsilon_2}{re^{2\alpha r}}(\epsilon_5 I - M_3 {M_3}^T)$*. The feedback controller of* (4.2) *is given by* $u(t) = Kx(t)$*.*

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