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New Extensions of Kannan Fixed-Point Theorem on Complete Metric and Generalized Metric Spaces

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Abstract

We obtain sufficient conditions for existence of unique fixed point of Kannan type mappings on complete metric spaces and on generalized complete metric spaces depended an another function.

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1 Introduction

The fixed point theorem most be frequently cited in Banach condition mapping principle (see [3] or [5]), which asserts that if (X, d) is a complete metric space and $S : X \longrightarrow X$ is a contractive mapping (S is contractive if there exists $k \in [0, 1)$ such that for all $x, y \in X$, $d(Sx, Sy) \leq kd(x, y)$) then S has a unique fixed point.
In 1968 [4] Kannan established a fixed point theorem for mapping satisfying:

$$d(Sx, Sy) \leq \lambda[d(x, Sx) + d(y, Sy)] \quad (x, y \in X) \tag{1}$$

where $\lambda \in [0, \frac{1}{2})$.
Kannan's paper [4] was followed by a spate of papers containing a variety of

contractive definitions in metric spaces.
Rhoades [8] in 1977 considered some type of contractive definitions and analyzed the relationship among them.
In 2000 Branciari [2] introduced a class of generalized metric spaces by replacing triangular inequality by similar ones which involve four or more points instead of three and improved Banach contraction mapping principle.
In 2008 Azam and Arshad [1] extended the Kannan's theorem for this kind of generalized metric spaces.
In 2010 Moradi and Beiranvand [6], and Moradi and Omid [7] introduced new classes of contractive functions and established the Banach contractive principle.
In the present paper at first we extend the Kannan's theorem [4] and then extend the theorem due to Azam and Arshad [1] for these new classes of functions. In the end of paper we introduce another extension of Kannan's theorem.

From the main results we need some new definitions.

Definition 1.1 [2] *Let (X, d) be a metric space. A mapping $T : X \longrightarrow X$ is said to be sequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergence then $\{y_n\}$ also is convergence. T is said to be subsequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergence then $\{y_n\}$ has a convergent subsequence.*

Definition 1.2 ([1] *or* [2]) *Let X be a nonempty set. Suppose that the mapping $d : X \longrightarrow X$, satisfies:*
(i) $d(x, y) \geq 0$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(ii)$d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii)$d(x, y) \leq d(x, w) + d(w, z) + d(z, y)$ for all $x, y \in X$ and for all distinct points $w, z \in X \backslash \{x, y\}$[rectangular property].
Then d is called a generalized metric and (X, d) is a generalized metric space.

For more information about generalized metric spaces see for example [1] and [2].

2 Main Results

These are the main results of the paper.

In this section at first we extend the Kannan's theorem [4] and then extend the Azam and Arshad theorem [1].

Theorem 2.1 *Let (X, d) be a complete metric space and $T, S : X \longrightarrow X$ be mappings such that T is continuous, one-to-one and subsequentially convergent. If $\lambda \in [0, \frac{1}{2})$ and*

$$d(TSx, TSy) \leq \lambda[d(Tx, TSx) + d(Ty, TSy)] \ \ (x, y \in X) \tag{2}$$

then S has a unique fixed point. Also if T is sequentially convergent then for every $x_0 \in X$ the sequence of iterates $\{S^n x_0\}$ converges to this fixed point.

Proof 1 *Let x_0 be an arbitrary point in X. We define the iterative sequence $\{x_n\}$ by $x_{n+1} = Sx_n$ (equivalently, $x_n = S^n x_0$), $n = 1, 2, \dots$. Using (2), we have*

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &= d(TSx_{n-1}, TSx_n) \\ &\leq \lambda[d(Tx_{n-1}, TSx_{n-1}) + d(Tx_n, TSx_n)], \end{aligned} \tag{3}$$

so,

$$d(Tx_n, Tx_{n+1}) \leq \frac{\lambda}{1-\lambda} d(Tx_{n-1}, Tx_n) \tag{4}$$

Using induction and (4),

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &\leq \frac{\lambda}{1-\lambda} d(Tx_{n-1}, Tx_n) \leq (\frac{\lambda}{1-\lambda})^2 d(Tx_{n-2}, Tx_{n-1}) \\ &\leq \dots \leq (\frac{\lambda}{1-\lambda})^n d(Tx_0, Tx_1) \end{aligned} \tag{5}$$

By (5), for every $m, n \in N$ such that $m > n$ we have,

$$\begin{aligned} d(Tx_m, Tx_n) &\leq d(Tx_m, Tx_{m-1}) + d(Tx_{m-1}, Tx_{m-2}) + \dots + d(Tx_{n+1}, Tx_n) \\ &\leq [(\frac{\lambda}{1-\lambda})^{m-1} + (\frac{\lambda}{1-\lambda})^{m-2} + \dots + (\frac{\lambda}{1-\lambda})^n] d(Tx_0, Tx_1) \\ &\leq [(\frac{\lambda}{1-\lambda})^n + (\frac{\lambda}{1-\lambda})^{n+1} + \dots] d(Tx_0, Tx_1) \\ &= (\frac{\lambda}{1-\lambda})^n \frac{1}{1-(\frac{\lambda}{1-\lambda})} d(Tx_0, Tx_1) \end{aligned} \tag{6}$$

Letting $m, n \longrightarrow \infty$ in (6), we have $\{Tx_n\}$ is a Cauchy sequence, and since X is a complete metric space, there exists $v \in X$ such that

$$\lim_{n\to\infty} Tx_n = v \tag{7}$$

Since T is a subsequentially convergent, $\{x_n\}$ has a convergent subsequence. So there exists $u \in X$ and $\{x_{n(k)}\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} x_{n(k)} = u$. Since T is continuous and $\lim_{k\to\infty} x_{n(k)} = u$, $\lim_{k\to\infty} Tx_{n(k)} = Tu$.
By (7) we conclude that $Tu = v$. So

$$\begin{aligned} d(TSu, Tu) &\leq d(TSu, TS^{n(k)}x_0) + d(TS^{n(k)}x_0, TS^{n(k)+1}x_0) \\ &+ d(TS^{n(k)+1}x_0, Tu) \\ &\leq \lambda[d(Tu, TSu) + d(TS^{n(k)-1}x_0, TS^{n(k)}x_0)] \end{aligned}$$

$$\begin{aligned} &+ \quad (\frac{\lambda}{1-\lambda})^{n(k)} d(TSx_0, Tx_0) + d(Tx_{n(k)+1}, Tu) \\ &\leq \quad \lambda d(Tu, TSu) + \lambda(\frac{\lambda}{1-\lambda})^{n(k)-1} d(Tx_0, Tx_1) \\ &+ \quad (\frac{\lambda}{1-\lambda})^{n(k)} d(Tx_1, Tx_0) + d(Tx_{n(k)+1}, Tu) \qquad (8) \end{aligned}$$

hence,

$$\begin{aligned} d(TSu, Tu) \quad &\leq \quad (\frac{\lambda}{1-\lambda})^{n(k)} d(Tx_0, Tx_1) + \frac{1}{1-\lambda}(\frac{\lambda}{1-\lambda})^{n(k)} d(Tx_1, Tx_0) \\ &+ \quad \frac{1}{1-\lambda} d(Tx_{n(k)+1}, Tu). \qquad (9) \end{aligned}$$

Letting $k \longrightarrow \infty$ *in (9) we get* $d(TSu, Tu) = 0$.
Since T *is one-to-one* $Su = u$. *So* S *has a fixed point.*
Uniqueness of the fixed point follows from (2).
Also if T *is sequentially convergent, by replacing* $\{n\}$ *with* $\{n(k)\}$ *we conclude that* $\lim_{n\to\infty} x_n = u$ *and this shows that* $\{x_n\}$ *converges to the fixed point of* S.

Remark 2.2 *By taking* $Tx \equiv x$ *in Theorem 2.1, we can conclude the Kannan's theorem [4].*

The following example shows that Theorem 2.1 is indeed a proper extension on Kannan's theorem.

Example 2.3 *Let* $X = \{0\} \cup \{\frac{1}{4}, \frac{1}{5}, \frac{1}{6}, ...\}$ *endowed with the Euclidean metric. Define* $S : X \longrightarrow X$ *by* $S(0) = 0$ *and* $S(\frac{1}{n}) = \frac{1}{n+1}$ *for all* $n \geq 4$. *Obviously the condition (1) is not true for every* $\lambda > 0$. *So we can not use the Kannan's theorem [5]. By define* $T : X \longrightarrow X$ *by* $T(0) = 0$ *and* $T(\frac{1}{n}) = \frac{1}{n^n}$ *for all* $n \geq 4$ *we have, for* $m, n \in N$ $(m > n)$,

$$\begin{aligned} |TS(\frac{1}{m}) - TS(\frac{1}{n})| \quad &= \quad \frac{1}{(n+1)^{n+1}} - \frac{1}{(m+1)^{m+1}} \\ &< \quad \frac{1}{(n+1)^{n+1}} \leq \frac{1}{3}[\frac{1}{n^n} - \frac{1}{(n+1)^{n+1}}] \\ &\leq \quad \frac{1}{3}[\frac{1}{n^n} - \frac{1}{(n+1)^{n+1}} + \frac{1}{m^m} - \frac{1}{(m+1)^{m+1}}] \\ &= \quad \frac{1}{3}[|T(\frac{1}{n}) - TS\frac{1}{n}| + |T(\frac{1}{m}) - TS\frac{1}{m}|] \qquad (10) \end{aligned}$$

Therefore by Theorem 2.1 S *has a unique fixed point* $u = 0$.

In the following theorem we extend the Azam and Arshad theorem [1].

Theorem 2.4 *Let (X,d) be a complete generalized metric space and $T,S : X \longrightarrow X$ be mappings such that T is continuous, one-to-one and subsequentially convergent. If $\lambda \in [0, \frac{1}{2})$ and*

$$d(TSx, TSy) \leq \lambda[d(Tx, TSx) + d(Ty, TSy)] \quad (x, y \in X) \tag{11}$$

then S has a unique fixed point. Also if T is sequentially convergent then for every $x_0 \in X$ the sequence of iterates $\{S^n x_0\}$ converges to this fixed point.

Proof 2 *With a method similar to that in Theorem 2.1 we can prove this Theorem.*

Remark 2.5 *By taking $Tx \equiv x$ in Theorem 2.4, we can conclude the Azam and Arshad theorem [1].*

The following example shows that Theorem 2.4 is indeed a proper extension on Azam and Arshad theorem.

Example 2.6 [1] *Let $X = \{1, 2, 3, 4\}$. Define $d : X \times X \longrightarrow [0, +\infty)$ as follows:*
$d(1, 2) = d(2, 1) = 3,$
$d(2, 3) = d(3, 2) = d(1, 3) = d(3, 1) = 1,$
$d(1, 4) = d(4, 1) = d(2, 4) = d(4, 2) = d(3, 4) = d(4, 3) = 4.$
Obviously (X, d) is a generalized metric space and is not a metric space.
Now define a mapping $S : X \longrightarrow X$ as follows:

$$Sx = \begin{cases} 2 & ; x \neq 1 \\ 4 & ; x = 1 \end{cases}$$

Obviously the inequality (1) is not holds for S for every $\lambda \in [0, \frac{1}{2})$. So we can not use the Azam and Arshad theorem for S.
By define $T : X \longrightarrow X$ as follows:

$$Tx = \begin{cases} 2 & ; x = 4 \\ 3 & ; x = 2 \\ 4 & ; x = 1 \\ 1 & ; x = 3, \end{cases}$$

we have

$$TSx = \begin{cases} 3 & ; x \neq 1 \\ 2 & ; x = 1. \end{cases}$$

Obviously,

$$d(TSx, TSy) \leq \frac{1}{3}[d(Tx, TSx) + d(Ty, TSy)]. \tag{12}$$

Therefore by Theorem 2.4, S has a unique fixed point.

In the following we have another extension of Kannan's theorem. First let Φ be the class of all nondecreasing continuous functions $F : [0, +\infty) \longrightarrow [0, +\infty)$ such that $F^{-1}(0) = \{0\}$.

Theorem 2.7 *Let the self-mapping S on complete metric space (X, d) satisfying*

$$F(d(Sx, Sy)) \le \lambda[F(d(x, Sx)) + F(d(y, Sy))] \quad (\forall x, y \in X) \tag{13}$$

for some $\lambda \in [0, 1)$ and for some $F \in \Phi$, then S has a unique fixed point and for every $x_0 \in X$ the sequence of iterates $\{S^n x_0\}$ converges to this fixed point.

Proof 3 *Since $F^{-1}(0) = \{0\}$, $F(\varepsilon) > 0$ for every $\varepsilon > 0$.*
Let x_0 be an arbitrary point in X. We define the iterative sequence $\{x_n\}$ by $x_{n+1} = Sx_n$ (equivalently, $x_n = S^n x_0$), $n = 1, 2, \ldots$. From (13)

$$\begin{aligned} F(d(x_n, x_{n+1})) &= F(d(Sx_{n-1}, Sx_n)) \\ &\le \lambda[F(d(x_{n-1}, Sx_{n-1})) + F(d(x_n, Sx_n))] \end{aligned} \tag{14}$$

so

$$F(d(x_n, x_{n+1})) \le \frac{\lambda}{1-\lambda} F(d(x_{n-1}, x_n)) \tag{15}$$

Using induction,

$$\begin{aligned} F(d(x_n, x_{n+1})) &\le \frac{\lambda}{1-\lambda} F(d(x_{n-1}, x_n)) \le (\frac{\lambda}{1-\lambda})^2 F(d(x_{n-2}, x_{n-1})) \\ &\le \ldots \le (\frac{\lambda}{1-\lambda})^n F(d(x_0, x_1)). \end{aligned} \tag{16}$$

By (16), for every $m, n \in N$ that $m > n$ we have,

$$\begin{aligned} F(d(x_m, x_n)) &= F(d(Sx_{m-1}, Sx_{n-1})) \\ &\le \lambda[F(d(x_{m-1}, x_m)) + F(d(x_{n-1}, x_n))] \\ &\le \lambda[(\frac{\lambda}{1-\lambda})^{m-1} + (\frac{\lambda}{1-\lambda})^{n-1}] F(d(x_0, x_1)). \end{aligned} \tag{17}$$

Letting $m, n \longrightarrow \infty$ in (17), we get

$$\lim_{n,m\to\infty} F(d(x_m, x_n)) = 0. \tag{18}$$

Since $F \in \Phi$

$$\lim_{n,m\to\infty} d(x_m, x_n) = 0. \tag{19}$$

Hence, $\{x_n\}$ is a Cauchy sequence. So there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$.
Thus by (13) for every $n \in N$

$$\begin{aligned} F(d(Su, x_{n+1})) &= F(d(Su, Sx_n)) \\ &\leq \lambda[F(d(u, Su)) + F(d(x_n, x_{n+1}))]. \end{aligned} \tag{20}$$

Since F is continuous, letting $n \longrightarrow \infty$ in (20) we get

$$F(d(Su, u)) \leq \lambda[F(d(u, Su)) + F(0)]. \tag{21}$$

Since $\lambda \in [0, 1)$ and $F^{-1}(0) = \{0\}$, $d(Su, u) = 0$ and hence, $Su = u$. So S has a fixed point.
Uniqueness of the fixed point follows from (13).

Remark 2.8 *By taking $Tx \equiv x$ in Theorem 2.7, we can conclude the Kannan's theorem [1].*

Remark 2.9 *By a similar method in the proof of Theorem 2.1 and 2.7 and replacing (13) by*

$$F(d(TSx, TSy)) \leq \lambda[F(d(Tx, TSx)) + F(d(Ty, TSy))]\ (x, y \in X) \tag{22}$$

we can extend Theorem 2.1 and Theorem 2.7.

The following example shows that Theorem 2.7 is indeed a proper extension on Kannan's theorem.

Example 2.10 *Let $X = \{1, 2, 3, 4\}$ endowed with the Euclidean metric. Define $S : X \longrightarrow X$ by*

$$Sx = \begin{cases} 3 & ; x \neq 4 \\ 1 & ; x = 4. \end{cases}$$

Obviously (1) is not hold for S. Now we define $F : [0, +\infty) \longrightarrow [0, +\infty)$ by

$$Fx = \begin{cases} \frac{1}{2}x & ; 0 \leq x \leq 2 \\ 7x - 13 & ; 2 \leq x. \end{cases}$$

Obviously the inequality (13) holds for $\lambda = \frac{1}{4}$. Therefore Theorem 2.7 is an extension of Kannan's theorem.

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