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A Note on Classes of Analytic Functions Concerned with Uniformly Starlike and Convex Functions

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Abstract. In this paper, using the generalized Al-Oboudi operator we introduce the new subclass $\mathcal{H}_{\lambda}^{\mu}(\alpha,\beta)$ of analytic functions. We also consider the subclass $\mathcal{H}_{\lambda}^{\mu^{\star}}(\alpha,\beta)$ of $\mathcal{H}_{\lambda}^{\mu}(\alpha,\beta)$. Coefficient inequalities and convolution conditions are investigated for these classes.

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1. Introduction

Let $\mathcal{A}$ be a class of analytic functions f of the form

$$f(z) = z + \sum_{m=2}^{\infty} a_m \, z^m. \tag{1.1}$$

defined in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Sălăgean [6] has introduced the following operator $D^n : \mathcal{A} \longrightarrow \mathcal{A}$ defined by $D^0 f(z) = f(z)$, $D^1 f(z) = D(f(z)) = zf'(z)$ and $D^n f(z) = D(D^{n-1} f(z))$, for $n \in \mathbb{N} = \{1, 2, 3, ...\}$.

Definition 1.1. [1] *For* $n \in \mathbb{N}$ *and* $\lambda \geq 0$, D_λ^n *denotes the Al-Oboudi operator defined by* $D_\lambda^n : \mathcal{A} \longrightarrow \mathcal{A}$, $D^0 f(z) = f(z)$, $D_\lambda^1 f(z) = (1 - \lambda) f(z) + \lambda z f'(z) = D_\lambda f(z)$ *and* $D_\lambda^n f(z) = D_\lambda(D_\lambda^{n-1} f(z))$.

Definition 1.2. *Let* $\mu, \lambda \in \mathbb{R}, \mu \geq 0, \lambda \geq 0$ *and* $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$. *We denote by* D_λ^μ *the linear operator defined by* $D_\lambda^\mu : \mathcal{A} \longrightarrow \mathcal{A}$,

$$D_\lambda^\mu f(z) = z + \sum_{m=2}^{\infty} [1 + (m-1)\lambda]^\mu a_m z^m.$$

Definition 1.3. *A function* $f \in \mathcal{A}$ *is said to be in the class* $\mathcal{H}_\lambda^\mu(\alpha, \beta)$ *if*

$$\Re\left\{\frac{D_\lambda^{\mu+1} f(z)}{D_\lambda^\mu f(z)}\right\} < \alpha \left|\frac{D_\lambda^{\mu+1} f(z)}{D_\lambda^\mu f(z)} - 1\right| + \beta, \quad (z \in \mathcal{U}) \tag{1.2}$$

for some $\alpha \leq 0$, $\beta > 1$, $\mu \geq 0$ *and* $\lambda \geq 0$.

We note that $\mathcal{H}_1^0(\alpha, \beta) \equiv \mathcal{MD}(\alpha, \beta)$, $\mathcal{H}_1^1(\alpha, \beta) \equiv \mathcal{ND}(\alpha, \beta)$, for $n \in \mathbb{N}_0 = \{0, 1, 2, 3, ...\}$. The classes $\mathcal{MD}(\alpha, \beta)$ and $\mathcal{N}D(\alpha, \beta)$ are introduced by Owa [5].

2. Coefficient inequalities for the class $\mathcal{H}_\lambda^\mu(\alpha, \beta)$

We derive sufficient conditions for f which are given by using coefficient inequalities.

Theorem 2.1. *If* $f \in \mathcal{A}$ *satisfies*

$$\sum_{m=2}^{\infty} [1 + (m-1)\lambda]^\mu \{|(m-1)\lambda - \beta + 2| + |(m-1)\lambda - \beta| - 2\alpha(m-1)\} |a_m| \leq \beta - |2 - \beta|$$

for some $\alpha(\alpha \leq 0)$, $\beta > 1$, $\mu \geq 0$ *and* $\lambda \geq 0$, *then* $f \in \mathcal{H}_\lambda^\mu(\alpha, \beta)$.

Proof. Let

$$\sum_{m=2}^{\infty} [1 + (m-1)\lambda]^\mu \{|(m-1)\lambda - \beta + 2| + |(m-1)\lambda - \beta| - 2\alpha(m-1)\} |a_m| \leq \beta - |2 - \beta|$$

for $f \in \mathcal{A}$. It suffices to show that

$$\left|\frac{\left(\frac{D_\lambda^{\mu+1}f(z)}{D_\lambda^{\mu}f(z)} - \alpha\left|\frac{D_\lambda^{\mu+1}f(z)}{D_\lambda^{\mu}f(z)} - 1\right| - \beta\right) + 1}{\left(\frac{D_\lambda^{\mu+1}f(z)}{D_\lambda^{\mu}f(z)} - \alpha\left|\frac{D_\lambda^{\mu+1}f(z)}{D_\lambda^{\mu}f(z)} - 1\right| - \beta\right) - 1}\right| < 1, \quad (z \in \mathcal{U}). \tag{2.1}$$

We have,

$$\left|\frac{\left(\frac{D_\lambda^{\mu+1}f(z)}{D_\lambda^{\mu}f(z)} - \alpha\left|\frac{D_\lambda^{\mu+1}f(z)}{D_\lambda^{\mu}f(z)} - 1\right| - \beta\right) + 1}{\left(\frac{D_\lambda^{\mu+1}f(z)}{D_\lambda^{\mu}f(z)} - \alpha|\frac{D_\lambda^{\mu+1}f(z)}{D_\lambda^{\mu}f(z)} - 1| - \beta\right) - 1}\right|$$

$$= \left|\frac{D_\lambda^{\mu+1}f(z) - \alpha e^{i\theta}|D_\lambda^{\mu+1}f(z) - D_\lambda^{\mu}f(z)| - \beta D_\lambda^{\mu}f(z) + D_\lambda^{\mu}f(z)}{D_\lambda^{\mu+1}f(z)) - \alpha e^{i\theta}|D_\lambda^{\mu+1}f(z) - D_\lambda^{\mu}f(z)| - \beta D_\lambda^{\mu}f(z) - D^{\mu}f(z)}\right|$$

$$< \frac{|2-\beta| + \sum_{m=2}^{\infty}[1+(m-1)\lambda]^{\mu}|(m-1)\lambda - \beta + 2||a_m| - \alpha\sum_{m=2}^{\infty}[1+(m-1)\lambda]^{\mu}(m-1)\lambda|a_m|}{\beta - \sum_{m=2}^{\infty}[1+(m-1)\lambda]^{\mu}|(m-1)\lambda - \beta||a_m| + \alpha\sum_{m=2}^{\infty}[1+(m-1)\lambda]^{\mu}(m-1)\lambda|a_m|}.$$

The last expression is bounded above by 1 if

$$|2-\beta| + \sum_{m=2}^{\infty}[1+(m-1)\lambda]^{\mu}|(m-1)\lambda - \beta + 2||a_m| - \alpha\sum_{m=2}^{\infty}[1+(m-1)\lambda]^{\mu}(m-1)\lambda|a_m|$$

$$\leq \beta - \sum_{m=2}^{\infty}[1+(m-1)\lambda]^{\mu}|(m-1)\lambda - \beta||a_m| + \alpha\sum_{m=2}^{\infty}[1+(m-1)\lambda]^{\mu}(m-1)\lambda|a_m|$$

which is equivalent to our condition

$$\sum_{m=2}^{\infty}[1+(m-1)\lambda]^{\mu}\{|(m-1)\lambda - \beta + 2| + |(m-1)\lambda - \beta| - 2\alpha(m-1)\}|a_m| \leq \beta - |2-\beta|$$

of the Theorem. □

3. Relation for $\mathcal{H}_{\lambda}^{\mu^{\star}}(\alpha,\beta)$

Definition 3.1. *By Theorem (2.1), the class* $\mathcal{H}_{\lambda}^{\mu^{\star}}(\alpha,\beta)$*is considered as the subclass of* $\mathcal{H}_{\lambda}^{\mu}(\alpha,\beta)$ *consisting of* f *satisfying:*

$$\sum_{m=2}^{\infty}[1+(m-1)\lambda]^{\mu}\{|(m-1)\lambda-\beta+2|+|(m-1)\lambda-\beta|-2\alpha(m-1)\}|a_m|\leq\beta-|2-\beta| \tag{3.1}$$

for some $\alpha\leq 0,\ \beta>1$ *and* $n\in\mathbb{N}_0$.

We note that $\mathcal{H}_1^{0^{\star}}(\alpha,\beta)\equiv\mathcal{MD}^{\star}(\alpha,\beta),\ \mathcal{H}_1^{1^{\star}}(\alpha,\beta)\equiv\mathcal{ND}^{\star}(\alpha,\beta)$, $\mathcal{H}_1^{n^{\star}}(\alpha,\beta)\equiv\mathcal{M}^{\star}(\alpha,\beta,n),\ \ n\in\mathbb{N}_0=\{0,1,2,3,...\}$. The classes $\mathcal{MD}^{\star}(\alpha,\beta)$ and $\mathcal{ND}^{\star}(\alpha,\beta)$ are studied by Owa [5] and $\mathcal{M}^{\star}(\alpha,\beta,n)$ is the class studied by Mahzoon and Latha [4].

Theorem 3.2. *If* $f\in\mathcal{A}$ *, then* $\mathcal{H}_{\lambda}^{\mu^{\star}}(\alpha_1,\beta)\subseteq\mathcal{H}_{\lambda}^{\mu^{\star}}(\alpha_2,\beta)$ *for some* $\alpha_1,\ \alpha_2$*, such that* $\alpha_1\leq\alpha_2\leq 0$.

Proof. For $\alpha_1\leq\alpha_2\leq 0$, we have

$$\sum_{m=2}^{\infty}[1+(m-1)\lambda]^{\mu}\{|(m-1)\lambda-\beta+2|+|(m-1)\lambda-\beta|-2\alpha_2(m-1)\}|a_m|$$

$$\leq\sum_{m=2}^{\infty}[1+(m-1)\lambda]^{\mu}\{|(m-1)\lambda-\beta+2|+|(m-1)\lambda-\beta|-2\alpha_1(m-1)\}|a_m|. \tag{3.2}$$

Therefore, if $f\in\mathcal{H}_{\lambda}^{\mu^{\star}}(\alpha_1,\beta)$ then $f\in\mathcal{H}_{\lambda}^{\mu^{\star}}(\alpha_2,\beta)$. ◻

4. Convolution of the class $\mathcal{H}_{\lambda}^{\star\mu}(\alpha,\beta)$

Definition 4.1. *For analytic function* f_i *given by*

$$f_i(z)=z+\sum_{m=2}^{\infty}a_{m,i}\ z^m\quad(i=1,2,3,...,p). \tag{4.1}$$

the Hadamard product of $f_1,f_2,...,f_p$ *is defined by:*

$$(\ f_1*f_2*...*f_p)(z)=z+\sum_{m=2}^{\infty}\left(\prod_{i=1}^{p}a_{m,i}\right)\ z^m \tag{4.2}$$

Theorem 4.2. *If* $f_1 \in \mathcal{H}_\lambda^{\mu^\star}(\alpha, \beta_1)$ *and* $f_2 \in \mathcal{H}_\lambda^{\mu^\star}(\alpha, \beta_2)$ *for some* α $(\alpha \leq 2 - \sqrt{5})$ *and* β_1, β_2 $(1 < \beta_1,\ \beta_2 \leq 2)$, $\lambda \geq 1$ *and* $\mu \geq 0$, *then* $(\ f_1 * f_2\ ) \in \mathcal{H}_\lambda^{\mu^\star}(\alpha, \beta)$, *where*

$$\beta = 1 + \frac{(\beta_1 - 1)(\beta_2 - 1)(\lambda - \alpha + 1)}{(\beta_1 - 1)(\beta_2 - 1) + (1+\lambda)^\mu(\lambda + 1 - \alpha - \beta_1)(\lambda + 1 - \alpha - \beta_2)}$$

Proof. From (3.1), for $f \in \mathcal{H}_\lambda^{\mu^\star}(\alpha, \beta)$ with $1 < \beta \leq 2$ and $\lambda \geq 1$, we have

$$\sum_{m=2}^{\infty}[1 + (m-1)\lambda]^\mu\{[(m-1)\lambda - \beta + 2] + [(m-1)\lambda - \beta] - 2\alpha(m-1)\}|a_m|$$

$$\leq \sum_{m=2}^{\infty}[1 + (m-1)\lambda]^\mu\{[(m-1)\lambda - \beta + 2] + |(m-1)\lambda - \beta| - 2\alpha(m-1)\}|a_m|$$

$$\leq 2(\beta - 1).$$

That is, if $f \in \mathcal{H}_\lambda^{\mu^\star}(\alpha, \beta)$, then

$$\sum_{m=2}^{\infty}\frac{[1 + (m-1)\lambda]^\mu[m(\lambda - \alpha) + (1 - \lambda) + (\alpha - \beta)]}{\beta - 1}|a_m| \leq 1. \tag{4.3}$$

Conversely, if f satisfies:

$$\sum_{m=2}^{\infty}\frac{[1 + (m-1)\lambda]^\mu[m(\lambda - \alpha) + (1 - \lambda) + (1 - \beta + \alpha)]}{\beta - 1}|a_m| \leq 1, \tag{4.4}$$

then $f \in \mathcal{H}_\lambda^{\mu^\star}(\alpha, \beta)$. From (4.3), if $f_1 \in \mathcal{H}_\lambda^{\mu^\star}(\alpha, \beta_1)$, then

$$\sum_{m=2}^{\infty}\frac{[1 + (m-1)\lambda]^\mu[m(\lambda - \alpha) + (1 - \lambda) + (\alpha - \beta_1)]}{\beta_1 - 1}|a_{m,1}| \leq 1 \tag{4.5}$$

and also if $f_2 \in \mathcal{H}_\lambda^{\mu^\star}(\alpha, \beta_2)$, then

$$\sum_{m=2}^{\infty}\frac{[1 + (m-1)\lambda]^\mu[m(\lambda - \alpha) + (1 - \lambda) + (\alpha - \beta_2)]}{\beta_2 - 1}|a_{m,2}| \leq 1. \tag{4.6}$$

Applying the Schwarz's inequality and using (4.5) and (4.6), we have the following inequality:

$$\sum_{m=2}^{\infty}\sqrt{\frac{PQR}{S}}\sqrt{|a_{m,1}||a_{m,2}|} \leq 1 \tag{4.7}$$

where $P = [1 + (m-1)\lambda]^{2\mu}$, $Q = \{m(\lambda - \alpha) + (1 - \lambda) + (\alpha - \beta_1)\}$, $R = \{m(\lambda - \alpha) + (1 - \lambda) + (\alpha - \beta_2)\}$ and $S = (\beta_1 - 1)(\beta_2 - 1)$. From (4.4) and (4.7), if the following inequality

$$\sum_{m=2}^{\infty} \frac{[1 + m - 1)\lambda]^{\mu}[m(\lambda - \alpha) + (1 - \lambda) + (1 - \beta + \alpha)]}{\beta - 1} |a_{m,1}||a_{m,2}|$$

$$\leq \sum_{m=2}^{\infty} \sqrt{\frac{PQR}{S}} \sqrt{|a_{m,1}||a_{m,2}|}$$

is satisfied, then we say that $(f_1 * f_2) \in \mathcal{H}_{\lambda}^{\mu^*}(\alpha, \beta)$. This inequality holds true if

$$\frac{[1 + (m-1)\lambda]^{\mu}[m(\lambda - \alpha) + (1 - \lambda) + (1 - \beta + \alpha)]}{\beta - 1} \sqrt{|a_{m,1}||a_{m,2}|} \leq \sqrt{\frac{PQR}{S}} \tag{4.8}$$

for all $m \geq 2$. Therefore, we have

$$\frac{[1 + (m-1)\lambda]^{\mu}[m(\lambda - \alpha) + (1 - \lambda) + (1 - \beta + \alpha)]}{\beta - 1} \leq \frac{PQR}{S} \tag{4.9}$$

which is equivalent to

$$\beta \geq 1 + \frac{S[m(\lambda - \alpha) + (1 - \lambda) + \alpha]}{S + \sqrt{P}QR}$$

for all $m \geq 2$. Let $\sigma(m)$ be the right hand side of the last inequality. Further, let us define $\zeta(m)$ by numerator of $\sigma'(m)$. Then $\zeta(m)$ gives us that.

$$\begin{aligned}
\zeta(m) = & (\beta_1 - 1)(\beta_2 - 1)\{(\lambda - \alpha)(\beta_1 - 1)(\beta_2 - 1) + (\lambda - \alpha)[1 + (m-1)\lambda]^{\mu}\beta_1\beta_2 \\
& - \mu\lambda[1 + (m-1)\lambda]^{\mu-1}[m(\lambda - \alpha) + (1 - \lambda) + \alpha]^3 \\
& + \mu\lambda[1 + (m-1)\lambda]^{\mu-1}[m(\lambda - \alpha) + (1 - \lambda) + \alpha]^2\beta_1 \\
& + \mu\lambda[1 + (m-1)\lambda]^{\mu-1}[m(\lambda - \alpha) + (1 - \lambda) + \alpha]^2\beta_2 \\
& - \mu\lambda[1 + (m-1)\lambda]^{\mu-1}[m(\lambda - \alpha) + (1 - \lambda) + \alpha]\beta_1\beta_2 \\
& - [1 + (m-1)\lambda]^{\mu}(\lambda - \alpha)[m(\lambda - \alpha) + (1 - \lambda) + \alpha]^2\} \\
\leq & (\beta_1 - 1)(\beta_2 - 1)\{(\lambda - \alpha) + 4(\lambda - \alpha)[1 + (m-1)\lambda]^{\mu} \\
& - \mu\lambda[1 + (m-1)\lambda]^{\mu-1}[m(\lambda - \alpha) + (1 - \lambda) + \alpha]^3 \\
& + 4\mu\lambda[1 + (m-1)\lambda]^{\mu-1}[m(\lambda - \alpha) + (1 - \lambda) + \alpha]^2 \\
& - \mu\lambda[1 + (m-1)\lambda]^{\mu-1}[m(\lambda - \alpha) + (1 - \lambda) + \alpha] \\
& - [1 + (m-1)\lambda]^{\mu}(\lambda - \alpha)[m(\lambda - \alpha) + (1 - \lambda) + \alpha]^2\} \leq 0
\end{aligned} \tag{4.10}$$

for $\alpha \leq 2 - \sqrt{5}$ and $\lambda \geq 1$ which shows that $\sigma(m)$ is decreasing for $m \geq 2$, $\alpha \leq 2 - \sqrt{5}$, $\lambda \geq 1$ and for $\mu \geq 0$. Thus $\sigma(2)$ is the maximum of $\sigma(m)$ for $\alpha \leq 2 - \sqrt{5}$, $\lambda \geq 1$ and for $\mu \geq 0$.

□

Theorem 4.3. *If* $f_i \in \mathcal{H}_{\lambda}^{\mu^\star}(\alpha, \beta_i)$, $(i = 1, 2, 3, ..., p)$ *for some* α $(\alpha \leq 2 - \sqrt{5})$ β_i $(1 < \beta_i \leq 2)$, $\mu \geq 0$ *and* $\lambda \geq 1$, *then* $(f_1 * f_2 * f_3 * ... * f_p) \in \mathcal{H}_{\lambda}^{\mu^\star}(\alpha, \beta)$, *where*

$$\beta = 1 + \frac{A_p}{B_p - C_p D_p + E_p} \quad (p \geq 2), \tag{4.11}$$

$$A_p = \prod_{i=1}^{p} (\beta_i - 1)(\lambda + 1 - \alpha)^{p-1}, \tag{4.12}$$

$$B_p = (\lambda+1-\alpha)^{p-2} \prod_{i=1}^{p}(\beta_i - 1), \tag{4.13}$$

$$C_p = \sum_{q=1}^{p-2}[(\lambda+1)^{\mu}]^q \ (\lambda+1-\alpha)^{p-q-2} \ (\lambda-\alpha)^{q-1}, \tag{4.14}$$

$$D_p = \prod_{i=1}^{p-q}(\beta_i - 1) \prod_{L=p-q+1}^{p} (\lambda+1-\alpha-\beta_L), \tag{4.15}$$

$$\textit{and } E_p = [(\lambda+1)^{\mu}]^{p-1} \ (\lambda-\alpha)^{p-2} \prod_{i=1}^{p}(\lambda+1-\alpha-\beta_i). \tag{4.16}$$

Proof. When $p = 2$, we have

$$\beta = 1 + \frac{(\beta_1-1)(\beta_2-1)(\lambda-\alpha+1)}{(\beta_1-1)(\beta_2-1)+(1+\lambda)^{\mu}(\lambda+1-\alpha-\beta_1)(\lambda+1-\alpha-\beta_2)} \tag{4.17}$$

Let us suppose that $(f_1 * f_2 * f_3 * ... * f_k) \in \mathcal{H}_{\lambda}^{\mu^\star}(\alpha, \beta_0)$ and $f_{k+1} \in \mathcal{H}_{\lambda}^{\mu^\star}(\alpha, \beta_{k+1})$ where

$$\beta_0 = 1 + \frac{A_k}{B_k - C_k D_k + E_k} \quad (k \geq 2).$$

Using Theorem (4.1) and replacing β_1 by β_0 and β_2 by β_{k+1}, we see that

$$\beta = 1 + \frac{(\beta_0-1)(\beta_{k+1}-1)(\lambda-\alpha+1)}{(\beta_0-1)(\beta_{k+1}-1)+(1+\lambda)^{\mu}(\lambda+1-\alpha-\beta_0)(\lambda+1-\alpha-\beta_{k+1})}$$

$$= 1 + \frac{A_{k+1}}{B_{k+1} - \{(1+\lambda)^{\mu}(\lambda+1-\alpha-\beta_{k+1})B_k + (1+\lambda)^{\mu}(\lambda-\alpha)(\lambda+1-\alpha-\beta_{k+1})\mathcal{C}_k D_k\} + E_{k+1}}$$

$$= 1 + \frac{A_{k+1}}{B_{k+1} - \{(1+\lambda)^{\mu}(\lambda+1-\alpha-\beta_{k+1})B_k + C_k^{+} D_{k+1}\} + E_{k+1}}$$

$$= 1 + \frac{A_{k+1}}{B_{k+1} - C_{k+1}D_{k+1} + E_{k+1}}$$

Where

$$C_k^{+} = \sum_{q=2}^{k-1} [(1+\lambda)^{\mu}]^q \ (1+\lambda-\alpha)^{k-q-1} \ (\lambda-\alpha)^{q-1}.$$

$\square$

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