Int. Journal of Math. Analysis, Vol. 5, 2011, no. 48, 2413 - 2420

A Note on Bernstein Polynomials Associated with q-Euler Numbers and Polynomials

T. Kim

Division of General Education-Mathematics
Kwangwoon University, Seoul 139-701, Republic of Korea
tkkim@kw.ac.kr

J. Choi

Division of General Education-Mathematics
Kwangwoon University, Seoul 139-701, Republic of Korea
jeschoi@kw.ac.kr

Y. H. Kim

Division of General Education-Mathematics
Kwangwoon University, Seoul 139-701, Republic of Korea
yhkim@kw.ac.kr

Abstract

In this paper, we give some identities on the q-Euler numbers and polynomials associated with Bernstein polynomials. By using the fermionic invariant integral on $\mathbf{Z}_p$, we derive some relations between q-Euler polynomials and Bernstein polynomials.

Mathematics Subject Classification: 11B68, 41A30

Keywords: q-Euler numbers and polynomials, fermionic p-adic integral, Bernstein polynomials

1 Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbf{Z}_p$, $\mathbf{Q}_p$ and $\mathbf{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbf{Q}_p$, respectively. Let $\mathbf{N}$ be the set of natural numbers and $\mathbf{Z}_+ = \mathbf{N} \cup \{0\}$. The p-adic absolute value is defined by $|x|_p = 1/p^r$ where $x = p^r s/t$ with $(p,s) = (p,t) = (s,t) = 1$ and $r \in \mathbf{Q}$. In this paper, we assume that $q \in \mathbf{C}_p$ with $|1-q|_p < 1$ and $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q \to 1}[x]_q = x$.

Let $C(\mathbf{Z}_p)$ be the space of continuous functions on $\mathbf{Z}_p$. For $f \in C(\mathbf{Z}_p)$, the fermionic p-adic invariant integral on $\mathbf{Z}_p$ is defined by Kim as follows:

$$I(f) = \int_{\mathbf{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \quad \text{(see [6]).} \tag{1}$$

Let $f_1(x) = f(x+1)$. Then we see from (1) that

$$I(f_1) + I(f) = 2f(0), \quad \text{(see [6]).} \tag{2}$$

As well known, the Euler polynomials $E_n(x)$ are given by

$$\frac{2}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad \text{(see [1-12]).} \tag{3}$$

In the special case, $x = 0$, $E_n(0) = E_n$ are called the n-th Euler numbers. From (2) and (3), we note that

$$e^{xt} \int_{\mathbf{Z_p}} e^{yt} d\mu_{-1}(y) = \int_{\mathbf{Z_p}} e^{(x+y)t} d\mu_{-1}(y) = \frac{2}{e^t+1} e^{xt}. \tag{4}$$

By (3) and (4), we get the Witt's formula for the Euler polynomials as follows:

$$\int_{\mathbf{Z}_p} (x+y)^n d\mu_{-1}(x) = E_n(x), \quad (n \in \mathbf{Z}_+). \tag{5}$$

In the viewpoint of q-extension of Euler polynomials, we consider q-Euler polynomials as follows:

$$\int_{\mathbf{Z_p}} q^y e^{(x+y)t} d\mu_{-1}(y) = \frac{2}{qe^t+1} e^{xt} = \sum_{n=0}^{\infty} \widetilde{E}_{n,q}(x) \frac{t^n}{n!}. \tag{6}$$

Note that $\lim_{q \to 1} \widetilde{E}_{n,q}(x) = E_n(x)$. In the special case, $x = 0$, $\widetilde{E}_{n,q}(0) = \widetilde{E}_{n,q}$ are called the n-th q-Euler numbers (see [4]).

In [4], it is known that

$$(-1)^n q \widetilde{E}_{n,q}(1-x) = \widetilde{E}_{n,q^{-1}}(x), \quad \text{for } n \in \mathbf{Z}_+. \tag{7}$$

From (2), (6) and (7), we can derive the following equation:

$$q^2 \widetilde{E}_{n,q}(2) = \begin{cases} 2q - \frac{2q}{1+q}, & \text{if } n = 0, \\ 2q + \widetilde{E}_{n,q}, & \text{if } n > 0, \end{cases} \tag{8}$$

and

$$\widetilde{E}_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} \widetilde{E}_{l,q} = (x + \widetilde{E}_q)^n$$

with the usual convention about replacing $(\widetilde{E}_q)^n$ by $\widetilde{E}_{n,q}$. Thus, by (8), we get

$$q(\widetilde{E}_q + 1)^n + \widetilde{E}_{n,q} = \begin{cases} 2, & \text{if } n = 0, \\ 0, & \text{if } n > 0, \end{cases} \quad \text{(see [4])}. \tag{9}$$

For $f \in C(\mathbf{Z}_p)$, the p-adic analogue of Bernstein operator of order n for f is given by

$$\mathbf{B}_n(f|x) = \sum_{k=0}^{n} f(\frac{k}{n}) \binom{n}{k} x^k (1-x)^{n-k} = \sum_{k=0}^{n} f(\frac{k}{n}) B_{k,n}(x), \tag{10}$$

where $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{n!}$. For $n, k \in \mathbf{Z}_+$, the Bernstein polynomial of degree n is defined by

$$B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \text{ (see [6-8])}. \tag{11}$$

In this paper, we give the p-adic integral representation of Bernstein polynomials associated with q-Euler numbers and polynomials. From those integral representation, we derive some interesting identities on the q-Euler numbers and polynomials.

2 q-Euler numbers and Bernstein polynomials

From (1), (7), (8) and (9), we note that

$$\begin{aligned} \int_{\mathbf{Z_p}} q^x (1-x)^n d\mu_{-1}(x) &= (-1)^n \int_{\mathbf{Z_p}} q^x (x-1)^n d\mu_{-1}(x) \\ &= \frac{1}{q} \int_{\mathbf{Z_p}} q^{-x} (x+2)^n d\mu_{-1}(x). \end{aligned} \tag{12}$$

Hence, by (12), we have

$$q \int_{\mathbf{Z}_p} q^x (1-x)^n d\mu_{-1}(x) = q(-1)^n \widetilde{E}_{n,q}(-1) = E_{n,q^{-1}}(2). \tag{13}$$

Taking p-adic integral on $\mathbf{Z}_p$ in (11), we get

$$\begin{aligned} \int_{\mathbf{Z}_p} q^x B_{k,n}(x) d\mu_{-1}(x) &= \binom{n}{k} \int_{\mathbf{Z}_p} q^x x^k (1-x)^{n-k} d\mu_{-1}(x) \\ &= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \int_{\mathbf{Z}_p} x^{k+l} q^x d\mu_{-1}(x) \\ &= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \widetilde{E}_{k+l,q}. \end{aligned} \tag{14}$$

By (12), we can also see that

$$\begin{aligned}\int_{\mathbf{Z}_p} q^x B_{k,n}(x) d\mu_{-1}(x) &= \binom{n}{k} \int_{\mathbf{Z}_p} q^x x^k (1-x)^{n-k} d\mu_{-1}(x) \\ &= \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \int_{\mathbf{Z}_p} (1-x)^{n-l} q^x d\mu_{-1}(x) \\ &= \frac{1}{q} \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \widetilde{E}_{n-l,q^{-1}}(2). \end{aligned} \tag{15}$$

Therefore, by (15), we obtain the following proposition.

Theorem 2.1 *For $n \in \mathbf{Z}_+$, we have*

$$\int_{\mathbf{Z}_p} q^x B_{k,n}(x) d\mu_{-1}(x) = \frac{1}{q} \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \widetilde{E}_{n-l,q^{-1}}(2).$$

By (8) and Theorem 2.1, we get

$$\begin{aligned} &\int_{\mathbf{Z}_p} q^x B_{k,n}(x) d\mu_q(x) \\ &= \begin{cases} \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^{n+l} (2 + q\widetilde{E}_{n-l,q^{-1}}) + 2 - \frac{2q^2}{1+q}, & \text{if } n = k, \\ \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} (2 + q\widetilde{E}_{n-l,q^{-1}}), & \text{if } n > k, \end{cases} \\ &= \begin{cases} q \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^{n+l} \widetilde{E}_{n-l,q^{-1}} + 2\delta_{1,n} + 2 - \frac{2q^2}{1+q}, & \text{if } n = k, \\ \binom{n}{k} q \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \widetilde{E}_{n-l,q^{-1}} + 2(-1)^k \binom{n}{k} \delta_{0,k}, & \text{if } n > k, \end{cases} \end{aligned} \tag{16}$$

where $\delta_{n,k}$ is the Kronecker symbol.

By (14) and (16), we obtain the following theorem.

Theorem 2.2 *For $n, k \in \mathbf{Z}_+$, we have*

$$\begin{aligned} &\sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \widetilde{E}_{k+l,q} \\ &= \begin{cases} q \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^{n+l} \widetilde{E}_{n-l,q^{-1}} + 2\delta_{1,n} + 2 - \frac{2q^2}{1+q}, & \textit{if } n = k, \\ q \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \widetilde{E}_{n-l,q^{-1}} + 2\delta_{0,k}(-1)^k, & \textit{if } n > k. \end{cases} \end{aligned}$$

Let $m, n, k \in \mathbf{Z}_+$, we have

$$\begin{aligned}
&\int_{\mathbf{Z}_p} q^x B_{k,n}(x) B_{k,m}(x) d\mu_{-1}(x) \\
=& \binom{n}{k}\binom{m}{k} \int_{\mathbf{Z}_p} q^x x^{2k}(1-x)^{n+m-2k} d\mu_{-1}(x) \\
=& \binom{n}{k}\binom{m}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} \int_{\mathbf{Z}_p} q^x (1-x)^{n+m-l} d\mu_{-1}(x) \\
=& \binom{n}{k}\binom{m}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} \left(\frac{1}{q} \widetilde{E}_{n+m-l,q^{-1}}(2)\right) \\
=& \begin{cases} \sum_{l=0}^{2k-1} \binom{2k}{l} (-1)^{l+2k} \left(\frac{1}{q} \widetilde{E}_{n+m-l,q^{-1}}(2)\right) + \left(2 - \frac{2q^2}{[2]_q}\right), & \text{if } n+m=2k, \\ \binom{n}{k}\binom{m}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} (2 + q\widetilde{E}_{n+m-l,q^{-1}}), & \text{if } n+m>2k, \end{cases} \\
=& \begin{cases} \sum_{l=0}^{2k-1} \binom{2k}{l} (-1)^{l+2k} \left(2 + q\widetilde{E}_{n+m-l,q^{-1}}\right) + \left(2 - \frac{2q^2}{[2]_q}\right), & \text{if } n+m=2k, \\ q\binom{n}{k}\binom{m}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} \widetilde{E}_{n+m-l,q^{-1}} + 2\delta_{0,2k}(-1)^{2k}, & \text{if } n+m>2k, \end{cases} \\
=& \begin{cases} q\sum_{l=0}^{2k-1} \binom{2k}{l} (-1)^{l+2k} \widetilde{E}_{n+m-l,q^{-1}} + \left(2 - \frac{2q^2}{[2]_q}\right), & \text{if } n+m=2k, \\ q\binom{n}{k}\binom{m}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} \widetilde{E}_{n+m-l,q^{-1}} + 2\delta_{0,2k}(-1)^{2k}, & \text{if } n+m>2k. \end{cases} \qquad (17)
\end{aligned}$$

Let $m, n, k \in \mathbf{Z}_+$. Then we easily see that

$$\begin{aligned}
&\int_{\mathbf{Z}_p} q^x B_{k,n}(x) B_{k,m}(x) d\mu_{-1}(x) \\
&= \binom{n}{k}\binom{m}{k} \int_{\mathbf{Z}_p} q^x x^{2k}(1-x)^{n+m-2k} d\mu_{-1}(x) \qquad (18) \\
&= \binom{n}{k}\binom{m}{k} \sum_{l=0}^{n+m-2k} \binom{n+m-2k}{l} (-1)^l \widetilde{E}_{l+2k,q}.
\end{aligned}$$

By (17) and (18), we obtain the following theorem.

Theorem 2.3 *Let $m, n, k \in \mathbf{Z}_+$. Then we have that*

$$\sum_{l=0}^{n+m-2k} \binom{n+m-2k}{l} (-1)^l \widetilde{E}_{l+2k,q}$$

$$= \begin{cases} q\sum_{l=0}^{n+m-1}\binom{n+m}{l}(-1)^{l+n+m}\widetilde{E}_{n+m-l,q^{-1}} + \left(2-\frac{2q^2}{[2]_q}\right), & \text{if } n+m=2k, \\ q\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{l+2k}\widetilde{E}_{n+m-l,q^{-1}} + 2\delta_{0,2k}(-1)^{2k}, & \text{if } n+m>2k. \end{cases}$$

For $s \in \mathbf{N}$, let $k, n_1, n_2, \ldots, n_s \in \mathbf{Z}_+$. Then we get

$$\begin{aligned} & \int_{\mathbf{Z}_p} q^x \left(\prod_{i=1}^{s} B_{k,n_i}(x)\right) d\mu_{-1}(x) \\ = & \binom{n_1}{k}\cdots\binom{n_s}{k}\int_{\mathbf{Z}_p} q^x x^{sk}(1-x)^{n_1+\cdots+n_s-sk} d\mu_{-1}(x) \\ = & \binom{n_1}{k}\cdots\binom{n_s}{k}\sum_{l=0}^{sk}\binom{sk}{l}(-1)^{l+sk}\int_{\mathbf{Z}_p} q^x (1-x)^{n_1+\cdots+n_s-l} d\mu_{-1}(x) \\ = & \binom{n_1}{k}\cdots\binom{n_s}{k}\sum_{l=0}^{sk}\binom{sk}{l}(-1)^{l+sk}\left(\frac{1}{q}\widetilde{E}_{n_1+\cdots+n_s-l,q^{-1}}(2)\right) \\ = & \begin{cases} \sum_{l=0}^{sk-1}\binom{sk}{l}(-1)^{l+sk}\left(\frac{1}{q}\widetilde{E}_{n_1+\cdots+n_s-l,q^{-1}}(2)\right) + \left(2-\frac{2q^2}{[2]_q}\right), & \text{if } n_1+\cdots+n_s=sk, \\ \binom{n_1}{k}\cdots\binom{n_s}{k}\sum_{l=0}^{sk}\binom{sk}{l}(-1)^{l+sk}\left(2+q\widetilde{E}_{n_1+\cdots+n_s-l,q^{-1}}\right), & \text{if } n_1+\cdots+n_s>sk, \end{cases} \\ = & \begin{cases} \sum_{l=0}^{sk-1}\binom{sk}{l}(-1)^{l+sk}\left(2+q\widetilde{E}_{n_1+\cdots+n_s-l,q^{-1}}\right) + \left(2-\frac{2q^2}{[2]_q}\right), & \text{if } n_1+\cdots+n_s=sk, \\ q\binom{n_1}{k}\cdots\binom{n_s}{k}\sum_{l=0}^{sk}\binom{sk}{l}(-1)^{l+sk}\widetilde{E}_{n_1+\cdots+n_s-l,q^{-1}} + 2\delta_{0,sk}(-1)^{sk}, & \text{if } n_1+\cdots+n_s>sk. \end{cases} \end{aligned} \tag{19}$$

By a simple calculation, we easily see that

$$\begin{aligned} & \int_{\mathbf{Z}_p} q^x \left(\prod_{i=1}^{s} B_{k,n_i}(x)\right) d\mu_{-1}(x) \\ & = \left(\prod_{i=1}^{s}\binom{n_i}{k}\right)\sum_{l=0}^{n_1+\cdots+n_s-sk}\binom{n_1+\cdots+n_s-sk}{l}(-1)^l\widetilde{E}_{l+sk,q}. \end{aligned} \tag{20}$$

By comparing the coefficients on the both sides of (19) and (20), we obtain the following theorem.

Theorem 2.4 *For $s \in \mathbf{N}$, let $k, n_1, n_2, \ldots, n_s \in \mathbf{Z}_+$. Then we have*

$$\sum_{l=0}^{n_1+\cdots+n_s-sk}\binom{n_1+\cdots+n_s-sk}{l}(-1)^l\widetilde{E}_{l+sk,q}$$

$$= \begin{cases} q \sum_{l=0}^{n_1+\cdots+n_s-1} \binom{n_1+\cdots+n_s}{l} (-1)^{n_1+\cdots+n_s+l} & \widetilde{E}_{n_1+\cdots+n_s-l,q^{-1}} + \left(2 - \frac{2q^2}{[2]_q}\right), \\ & \text{if } \ n_1+\cdots+n_s = sk, \\ \qquad q \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{l+sk} \widetilde{E}_{n_1+\cdots+n_s-l,q^{-1}} & +2\delta_{0,sk}(-1)^{sk}, \\ & \text{if } \ n_1+\cdots+n_s > sk. \end{cases}$$

References

[1] A. Bayad, Modular properties of elliptic Bernoulli and Euler functions, *Adv. Stud. Contemp. Math.,* **20** (2010), 389–401.

[2] M. Can, M. Cenkci, V. Kurt, Y. Simsek, Twisted Dedekind type sums associated with Barnes' type multiple Frobenius-Euler l-functions, *Adv. Stud. Contemp. Math.,* **18** (2009), no. 2, 135-160.

[3] C.-P. Chen, L. Lin, An inequality for the generalized Euler constant function, *Adv. Stud. Contemp. Math.,* **17** (2008), 105–107.

[4] T. Kim, J. Choi, Y. H. Kim, A note on the q-analogues of Euler numbers and polymials, *Honam Math.,* Accepted.

[5] T. Kim, A note on q-Bernstein polynomials, *Russ. J. Math. Phys.,* **18** (2011), 73-82.

[6] T. Kim, Some identities on the q-Euler polynomials of higher order and q-Stirling numbers by the fermionic p-adic integral on $\mathbf{Z}_p$, *Russ. J. Math. Phys.,* **16** (2009), 484-491.

[7] L. C. Jang, W.-J. Kim, Y. Simsek, A study on the p-adic integral representation on $\mathbf{Z}_p$ associated with Bernstein and Bernoulli polynomials, *Adv. Difference Equ.,* **2010** (2010), Article ID 163217, 6 pages.

[8] S.-H. Rim, S.-J. Lee, E.-J. Moon, J.-H. Jin, On the q-Genocchi numbers and polynomials associated with q-Zeta function, *Proc. Jangjeon Math. Soc.,* **12** (2009), 261-267.

[9] C.S. Ryoo, A note on the weighted q-Euler numbers and polynomials, *Adv. Stud. Contemp. Math.,* **21** (2011), 47-54.

[10] C. S. Ryoo, On the generalized Barnes type multiple q-Euler polynomials twisted by ramified roots of unity, *Proc. Jangjeon Math. Soc.,* **13** (2010), 255-263.

[11] Y. Simsek, Generating functions of the twisted Bernoulli numbers and polynomials associated with their interpolation functions, *Adv. Stud. Contemp. Math.*, **16** (2008), 251–278.

[12] H. Ozden, I. N. Cangul, Y. Simsek, Remarks on q-Bernoulli numbers associated with Daehee numbers, *Adv. Stud. Contemp. Math.,* **18** (2009), 41–48.

Received: September, 2011