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Convergence of Perturbed Iterative Algorithm for Generalized Quasi-Variational Inequalities

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Abstract

In this paper, we introduce and study a new class of generalized quasi-variational inequalities. By using the properties of the resolvent operator associated with a maximal monotone mapping in Hilbert space, we establish an existence theorem of solutions for the generalized quasi-variational inequality and suggest a perturbed iterative algorithm for finding approximate solutions which converges strongly to the unique solution of the generalized quasi-variational inequality.

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1 Introduction

Variational inequality theory has become a very effective and powerful tool for studying a wide range of problems arising in pure and applied sciences which include the work on differential equations, mechanics, contact problems in elasticity, control problems, general equilibrium problems in economics and transportation, and unilateral, obstacle, moving, and free boundary problems. The resolvent operator technique is an important technique to solve various variational inequalities. In recent years, the classical variational inequalities have been extended and generalized in many different directions, see [1]-[15]. It is known that the useful and important generalizations of these classical variational inequalities are quasi-variational inequalities and implicit variational inequalities, which have been introduced and studied by Ding [6], Ding and Tarafdar [10], [11], Harker and Yao [14] and others, respectively.

Inspired and motivated by the results in [6], [7], [8] and [15], in this paper, we introduce and study a new class of generalized quasi-variational inequalities. By applying the properties of the resolvent operator associated with a maximal monotone mapping in Hilbert spaces, we show that the quasi-variational inequality problem is equivalent to the fixed point problem. A perturbed iterative algorithm for finding approximate solutions which converges strongly to the exact solution of the generalized quasi-variational inequality are proposed and analyzed.

2 Preliminaries

Let H be a real Hilbert space with a norm $\|\cdot\|$ and an inner product $\langle\cdot,\cdot\rangle$. Let $N : H \times H \to H$, $T, A, g : H \to H$ be mappings and $\phi : H \times H \to R \cup \{+\infty\}$ be such that for each fixed point $y \in H$, $\phi(\cdot, y) : H \to R \cup \{+\infty\}$ is a proper convex lower semicontinuous functional on H and $g(H) \cap dom\, \partial\phi(\cdot, y) \neq \emptyset$. Then the problem of finding $x \in H$ such that $g(x) \in dom\, \partial\phi(\cdot, x)$ and

$$\langle N(Tx, Ax), y - g(x)\rangle \geq \phi(g(x), x) - \phi(y, x), \quad \forall y \in H \tag{2.1}$$

is called the *generalized quasi-variational inequality problem* (in short, GQVIP).

Special Cases:

(A) If $\phi(x, y) = \phi(x)$ for all $y \in H$, then the GQVIP reduces to the variational inequality problem considered by Hassouni and Moudafi in [15].

(B) If K is a given closed convex subset of H and $\phi = I_K$ is the indicator function of K,

$$I_K(x) = \begin{cases} 0, & \text{if } x \in K, \\ +\infty, & \text{otherwise,} \end{cases}$$

then the GQVIP reduces to the generalized strongly nonlinear variational inequality problem, that is, find $x \in H$ such that $g(x) \in K$ and

$$\langle N(Tx, Ax), y - g(x)\rangle \geq 0, \quad \forall y \in K.$$

(C) If $K : H \to 2^K$ is a set-valued mapping such that each $K(x)$ is a closed convex and for each fixed point $y \in H$, $\phi(\cdot, y) = I_{K(y)}(\cdot)$ is the indicator function of $K(y)$,

$$I_{K(y)}(x) = \begin{cases} 0, & \text{if } x \in K(y), \\ +\infty, & \text{otherwise,} \end{cases}$$

then the GQVIP reduces to the generalized strongly nonlinear quasi-variational inequality problem, that is, find $x \in H$ such that $g(x) \in K(x)$ and

$$\langle N(Tx, Ax), y - g(x)\rangle \geq 0, \quad \forall y \in K(x).$$

In brief, the GQVIP includes a lot of known variational inequalities in [1]-[15] as special cases. In order to prove our main results, we need the following concepts and results.

Definition 2.1. Let X be a Banach space with the dual space X^* and $\phi : X \to R \cup \{+\infty\}$ be a proper functional. ϕ is said to be *subdifferential* at a point $x \in X$ if there exists an $f^* \in X^*$ such that

$$\phi(y) - \phi(x) \geq \langle f^*, y - x\rangle, \quad \forall y \in X,$$

where f^* is called a *subgradient* of ϕ at x. The mapping $\partial\phi : X \to 2^{X^*}$ defined by

$$\partial\phi(x) = \{f^* \in X^* : \phi(y) - \phi(x) \geq \langle f^*, y - x\rangle, \ \forall y \in X\}, \quad x \in X$$

is said to be the *subdifferential* of ϕ.

Definition 2.2. Let H be a Hilbert space and $G : H \to 2^H$ be a maximal monotone mapping. For any fixed $\rho > 0$, the mapping $J_\rho^G : H \to H$ defined by

$$J_\rho^G(x) = (I + \rho G)^{-1}(x), \quad \forall x \in H$$

is said to be the *resolvent operator* of G, where I is the identity mapping on H.

Definition 2.3. Let $T : H \to H$ and $N : H \times H \to H$ be mappings. The mapping N is said to be

(1) β-*Lipschitz continuous* in the first argument if there exists a constant $\beta > 0$ such that

$$\|N(x, z) - N(y, z)\| \leq \beta\|x - y\|, \quad \forall x, y, z \in H;$$

(2) *α-strongly monotone* with the respect to T in the first argument, if there exists a constant $\alpha > 0$ such that

$$\langle N(Tx, z) - N(Ty, z), x - y \rangle \geq \alpha \|x - y\|^2, \quad \forall x, y, z \in H.$$

Similarly we can define the Lipschitz continuity of N with respect to the second argument.

Definition 2.4. A mapping $g : H \to H$ is said to be
(1) *γ-strongly monotone* if there exists a constant $\gamma > 0$ such that

$$\langle g(x) - g(y), x - y \rangle \geq \gamma \|x - y\|^2, \quad \forall x, y \in H;$$

(2) *σ-Lipschitz continuous* if there exists a constant $\sigma > 0$ such that

$$\|g(x) - g(y)\| \leq \sigma \|x - y\|, \quad \forall x, y \in H.$$

Lemma 2.1. *Let X be a real Hilbert space endowed with a strictly convex norm and $\phi : X \to R \cup \{+\infty\}$ be a proper convex lower semicontinuous functional. Then $\partial\phi : X \to 2^{X^*}$ is a maximal monotone mapping.*

Lemma 2.2. *Let $G : H \to 2^H$ be a maximal monotone mapping. Then the resolvent operator $J_\rho^G : H \to H$ is nonexpansive, that is,*

$$\left\| J_\rho^G(x) - J_\rho^G(y) \right\| \leq \|x - y\|, \quad \forall x, y \in H.$$

Lemma 2.3. ([16]) *Let $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$ and $\{c_n\}_{n\geq 0}$ be nonnegative sequences satisfying*

$$a_{n+1} \leq (1 - t_n)a_n + b_n t_n + c_n, \quad \forall n \geq 0,$$

where $0 \leq t_n \leq 1$, $\sum_{n=0}^\infty t_n = \infty$, $\lim_{n\to\infty} b_n = 0$ and $\sum_{n=0}^\infty c_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4. *The GQVIP has a solution $x^* \in H$ if and only if x^* satisfies the relation:*

$$g(x) = J_\rho^{\partial\phi(\cdot,x)}\big(g(x) - \rho N(Tx, Ax)\big), \quad \forall x \in H, \tag{2.2}$$

where $\rho > 0$ is a constant, $J_\rho^{\partial\phi(\cdot,x)} = (I + \rho\partial\phi(\cdot, x))^{-1}$ is the resolvent operator of $\partial\phi(\cdot, x)$ and I is the identity mapping on H.

Proof. It is easy to show that the GQVIP has a solution $x^* \in H$ if and only if

$$\begin{aligned}
&\langle N(Tx^*, Ax^*), y - g(x^*) \rangle \geq \phi(g(x^*), x^*) - \phi(y, x^*), \quad \forall y \in H \\
\iff \quad & -N(Tx^*, Ax^*) \in \partial\phi(\cdot, x^*)(g(x^*)) \\
\iff \quad & g(x^*) - \rho N(Tx^*, Ax^*) \in g(x^*) + \rho\partial\phi(\cdot, x^*)(g(x^*)) \\
\iff \quad & g(x^*) = J_\rho^{\partial\phi(\cdot,x^*)}(g(x^*) - \rho N(Tx^*, Ax^*)),
\end{aligned}$$

where $\rho > 0$ is a constant. This completes the proof. □

Based on Lemma 2.4, we suggest the following perturbed iterative algorithm for the GQVIP.

Algorithm 2.1. *Let $N : H \times H \to H$, T, A, $g : H \to H$ be mappings and $\phi_n : H \times H \to R \cup \{+\infty\}$ be functional for each $n \geq 0$. For any given $x_0 \in H$, $u_0 \in H$, $v_0 \in H$, $w_0 \in H$, compute the sequence $\{x_n\}_{n\geq 0}$ by the following iterative scheme*

$$z_n = \alpha_n'' x_n + \beta_n'' \big[x_n - g(x_n) + J_\rho^{\partial\phi_n(\cdot,x_n)}\big(g(x_n) - \rho N(Tx_n, Ax_n)\big)\big] + \gamma_n'' w_n,$$
$$y_n = \alpha_n' x_n + \beta_n' \big[z_n - g(z_n) + J_\rho^{\partial\phi_n(\cdot,z_n)}\big(g(z_n) - \rho N(Tz_n, Az_n)\big)\big] + \gamma_n' v_n,$$
$$x_{n+1} = \alpha_n x_n + \beta_n \big[y_n - g(y_n) + J_\rho^{\partial\phi_n(\cdot,y_n)}\big(g(y_n) - \rho N(Ty_n, Ay_n)\big)\big] + \gamma_n u_n$$

for all $n \geq 0$, where $\{u_n\}_{n\geq 0}$, $\{v_n\}_{n\geq 0}$, $\{w_n\}_{n\geq 0}$ are bounded sequences in H and $\{\alpha_n\}_{n\geq 0}$, $\{\beta_n\}_{n\geq 0}$, $\{\gamma_n\}_{n\geq 0}$, $\{\alpha_n'\}_{n\geq 0}$, $\{\beta_n'\}_{n\geq 0}$, $\{\gamma_n'\}_{n\geq 0}$, $\{\alpha_n''\}_{n\geq 0}$, $\{\beta_n''\}_{n\geq 0}$ and $\{\gamma_n''\}_{n\geq 0}$ are sequences in $[0, 1]$ satisfying suitable conditions and $\rho > 0$ is a constant.

3 Main results

In this section, we establish an existence theorem of solutions for the GQVIP and the convergence of the iterative sequence generalized by Algorithm 2.1.

Theorem 3.1. *Let H be a Hilbert space, T, $A : H \to H$ be Lipschitz continuous with constants ξ, ζ, respectively, and $N : H \times H \to H$ be α-strongly monotone in the first argument with respect to T and Lipschitz continuous in the first and second arguments with constants β and γ, respectively. Let $g : H \to H$ be λ-strongly monotone and σ-Lipschitz continuous and $\phi : H \times H \to R \cup \{+\infty\}$ be such that for each fixed $y \in H$, $\phi(\cdot, y)$ is a proper convex lower semicontinuous functional on H, $g(H) \cap dom\, \partial\phi(\cdot, y) \neq \emptyset$ and*

$$\big\|J_\rho^{\partial\phi(\cdot,x)}(z) - J_\rho^{\partial\phi(\cdot,y)}(z)\big\| \leq \mu\|x - y\|, \quad \forall x, y, z \in H,$$

where μ is a positive constant. Suppose that $K = \mu + 2\sqrt{1 - 2\lambda + \sigma^2}$ and there exists a constant $\rho > 0$ satisfying

$$\rho\gamma\zeta < 1 - K \tag{3.1}$$

and one of the following conditions:

$$\beta^2\xi^2 - \gamma^2\zeta^2 > 0, \quad \alpha > \gamma\zeta(1 - K) + \sqrt{(\beta^2\xi^2 - \gamma^2\zeta^2)K(2 - K)},$$
$$\left|\rho - \frac{\alpha + \gamma\zeta(K - 1)}{\beta^2\xi^2 - \gamma^2\zeta^2}\right| < \frac{\sqrt{(\alpha + \gamma\zeta(K - 1))^2 - (\beta^2\xi^2 - \gamma^2\zeta^2)K(2 - K)}}{\beta^2\xi^2 - \gamma^2\zeta^2}; \tag{3.2}$$

$$\begin{aligned}&\beta^2\xi^2-\gamma^2\zeta^2<0,\\&\left|\rho-\frac{\alpha+\gamma\zeta(K-1)}{\beta^2\xi^2-\gamma^2\zeta^2}\right|\\&\quad>\frac{\sqrt{(\alpha+\gamma\zeta(K-1))^2-(\beta^2\xi^2-\gamma^2\zeta^2)K(2-K)}}{\gamma^2\zeta^2-\beta^2\xi^2}.\end{aligned}\tag{3.3}$$

Then the GQVIP has a unique solution $x^*\in H$.

Proof. By Lemma 2.4, it is sufficient to prove that there exists $x^*\in H$ satisfying (2.2). Define a mapping $F:H\to H$ by

$$F(x)=x-g(x)+J_\rho^{\partial\phi(\cdot,x)}\big(g(x)-\rho N(Tx,Ax)\big),\quad \forall x\in H.$$

Since g is λ-strongly monotone and σ-Lipschitz continuous, it follows that

$$\begin{aligned}&\|x-y-(g(x)-g(y))\|^2\\&=\|x-y\|^2-2\langle x-y,g(x)-g(y)\rangle+\|g(x)-g(y)\|^2\\&\le(1-2\lambda+\sigma^2)\|x-y\|^2\end{aligned}\tag{3.4}$$

for all $x,y\in H$. In view of the assumptions of N, T and A, we get that

$$\begin{aligned}&\|x-y-\rho(N(Tx,Ax)-N(Ty,Ax)))\|^2\\&=\|x-y\|^2-2\rho\langle x-y,N(Tx,Ax)-N(Ty,Ax)\rangle\\&\quad+\rho^2\|N(Tx,Ax)-N(Ty,Ax)\|^2\\&\le(1-2\rho\alpha+\rho^2\beta^2\xi^2)\|x-y\|^2\end{aligned}\tag{3.5}$$

and

$$\|N(Ty,Ax)-N(Ty,Ay)\|\le\gamma\zeta\|x-y\|\tag{3.6}$$

for all $x,y\in H$. By Lemmas 2.1 and 2.2, we deduce that for any $x,y\in H$

$$\begin{aligned}&\|F(x)-F(y)\|\\&\le\|x-y-(g(x)-g(y))\|\\&\quad+\left\|J_\rho^{\partial\phi(\cdot,x)}\big(g(x)-\rho N(Tx,Ax)\big)-J_\rho^{\partial\phi(\cdot,x)}\big(g(y)-\rho N(Ty,Ay)\big)\right\|\\&\quad+\left\|J_\rho^{\partial\phi(\cdot,x)}\big(g(y)-\rho N(Ty,Ay)\big)-J_\rho^{\partial\phi(\cdot,y)}\big(g(y)-\rho N(Ty,Ay)\big)\right\|\\&\le2\|x-y-(g(x)-g(y))\|+\|x-y-\rho(N(Tx,Ax)-N(Ty,Ax))\|\\&\quad+\rho\|N(Ty,Ax)-N(Ty,Ay)\|+\mu\|x-y\|.\end{aligned}\tag{3.7}$$

Substituting (3.4)-(3.6) into (3.7), we have

$$\|F(x)-F(y)\|\le\theta\|x-y\|,\quad\forall x,y\in H,$$

where

$$\theta=K+\sqrt{1-2\rho\alpha+\rho^2\beta^2\xi^2}+\rho\gamma\zeta.\tag{3.8}$$

It is easy to verify that (3.1) and one of (3.2) and (3.3) mean $\theta < 1$. Hence F is a contraction mapping and it has a unique fixed point $x^* \in H$. It follows from Lemma 2.4 that x^* is a unique solution of the GQVIP. This completes the proof. □

Theorem 3.2. *Let H, N, T, A, g and ϕ be as in Theorem 3.1 and for each $n \geq 0$, $\phi_n : H \to R \cup \{+\infty\}$ satisfy that for any fixed $y \in H$, $\phi_n(\cdot, y)$ is a proper convex lower semicontinuous functional on H and $g(H) \cap dom\, \partial\phi_n(\cdot, y) \neq \emptyset$. Assume that*

$$\lim_{n\to\infty} \left\| J_\rho^{\partial\phi_n(\cdot,y)}(z) - J_\rho^{\partial\phi(\cdot,y)}(z) \right\| = 0, \quad \forall y, z \in H$$

and

$$\left\| J_\rho^{\partial\phi_n(\cdot,x)}(z) - J_\rho^{\partial\phi_n(\cdot,y)}(z) \right\| \leq \mu \|x - y\|, \quad \forall n \geq 0,\, x, y, z \in H.$$

Suppose that $\{u_n\}_{n\geq 0}$, $\{v_n\}_{n\geq 0}$ and $\{w_n\}_{n\geq 0}$ are any bounded sequences in H and $\{\alpha_n\}_{n\geq 0}$, $\{\beta_n\}_{n\geq 0}$, $\{\gamma_n\}_{n\geq 0}$, $\{\alpha'_n\}_{n\geq 0}$, $\{\beta'_n\}_{n\geq 0}$, $\{\gamma'_n\}_{n\geq 0}$, $\{\alpha''_n\}_{n\geq 0}$, $\{\beta''_n\}_{n\geq 0}$ and $\{\gamma''_n\}_{n\geq 0}$ are sequences in $[0, 1]$ satisfying the following conditions:

$$\begin{aligned} &\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1, \\ &\gamma_n = a_n\beta_n, \quad \forall n \geq 0, \end{aligned} \tag{3.9}$$

$$\lim_{n\to\infty} \gamma'_n = \lim_{n\to\infty} \gamma''_n = \lim_{n\to\infty} a_n = 0, \tag{3.10}$$

$$\sum_{n=0}^{\infty} \beta_n = \infty. \tag{3.11}$$

If there exists a positive constant ρ satisfying (3.1) *and one of* (3.2) *and* (3.3), *then the GQVIP has a unique solution $x^* \in H$ and the sequence $\{x_n\}_{n\geq 0}$ defined by Algorithm 2.1 converges strongly to x^*.*

Proof. It follows from Theorem 3.1 that the GQVIP has a unique solution $x^* \in H$ and for all $n \geq 0$

$$\begin{aligned} x^* &= \alpha''_n x^* + \beta''_n \big[x^* - g(x^*) + J_\rho^{\partial\phi(\cdot,x^*)}\big(g(x^*) - \rho N(Tx^*, Ax^*)\big)\big] + \gamma''_n x^*, \\ &= \alpha'_n x^* + \beta'_n \big[x^* - g(x^*) + J_\rho^{\partial\phi(\cdot,x^*)}\big(g(x^*) - \rho N(Tx^*, Ax^*)\big)\big] + \gamma'_n x^*, \\ &= \alpha_n x^* + \beta_n \big[x^* - g(x^*) + J_\rho^{\partial\phi(\cdot,x^*)}\big(g(x^*) - \rho N(Tx^*, Ax^*)\big)\big] + \gamma_n x^*. \end{aligned}$$

In light of Algorithm 2.1 and the proof of Theorem 3.1, we conclude easily that

$$\|z_n - x^* - (g(z_n) - g(x^*))\| \leq \sqrt{1 - 2\lambda + \sigma^2}\|z_n - x^*\|$$

and

$$\begin{aligned} &\|z_n - x^* - \rho(N(Tz_n, Az_n) - N(Tx^*, Ax^*))\| \\ &\leq \big(\sqrt{1 - 2\rho\alpha + \rho^2\beta^2\xi^2} + \rho\gamma\zeta\big)\|z_n - x^*\| \end{aligned}$$

for all $n \geq 0$. In view of the assumptions of ϕ, ϕ_n, Algorithm 2.1 and Lemmas 2.1 and 2.2, we arrive at

$$\begin{aligned}
&\|z_n - x^*\| \\
&= \big\|\alpha_n'' x_n + \beta_n''\big[x_n - g(x_n) + J_\rho^{\partial\phi_n(\cdot,x_n)}\big(g(x_n) - \rho N(Tx_n, Ax_n)\big)\big] + \gamma_n'' w_n \\
&\quad - \big\{\alpha_n'' x^* + \beta_n''\big[x^* - g(x^*) + J_\rho^{\partial\phi(\cdot,x^*)}\big(g(x^*) - \rho N(Tx^*, Ax^*)\big)\big] + \gamma_n'' x^*\big\}\big\| \\
&\leq \alpha_n''\|x_n - x^*\| + \beta_n''\|x_n - x^* - (g(x_n) - g(x^*))\| \\
&\quad + \beta_n''\big\|J_\rho^{\partial\phi_n(\cdot,x_n)}\big(g(x_n) - \rho N(Tx_n, Ax_n)\big) \\
&\quad\quad - J_\rho^{\partial\phi_n(\cdot,x^*)}\big(g(x_n) - \rho N(Tx_n, Ax_n)\big)\big\| \\
&\quad + \beta_n''\big\|J_\rho^{\partial\phi_n(\cdot,x^*)}\big(g(x_n) - \rho N(Tx_n, Ax_n)\big) \\
&\quad\quad - J_\rho^{\partial\phi_n(\cdot,x^*)}\big(g(x^*) - \rho N(Tx^*, Ax^*)\big)\big\| \\
&\quad + \beta_n''\big\|J_\rho^{\partial\phi_n(\cdot,x^*)}\big(g(x^*) - \rho N(Tx^*, Ax^*)\big) \\
&\quad\quad - J_\rho^{\partial\phi(\cdot,x^*)}\big(g(x^*) - \rho N(Tx^*, x^*)\big)\big\| + \gamma_n''\|w_n - x^*\| \\
&\leq \alpha_n''\|x_n - x^*\| + \beta_n''\theta\|x_n - x^*\| + \beta_n'' M_n + M\gamma_n'',
\end{aligned}$$

where

$$M = \sup\{\|w_n - x^*\|,\ \|v_n - x^*\|,\ \|u_n - x^*\| : n \geq 0\}$$

and

$$M_n = \big\|J_\rho^{\partial\phi_n(\cdot,x^*)}\big(g(x^*) - \rho N(Tx^*, Ax^*)\big) - J_\rho^{\partial\phi(\cdot,x^*)}\big(g(x^*) - \rho N(Tx^*, Ax^*)\big)\big\|$$

for all $n \geq 0$ and θ is defined by (3.7). Similarly, we deduce that

$$\|y_n - x^*\| \leq \alpha_n'\|x_n - x^*\| + \beta_n'\theta\|z_n - x^*\| + \beta_n' M_n + M\gamma_n'$$

and

$$\|x_{n+1} - x^*\| \leq \alpha_n\|x_n - x^*\| + \beta_n\theta\|y_n - x^*\| + \beta_n M_n + M\gamma_n$$

for all $n \geq 0$. From the above inequalities, we get that

$$\begin{aligned}
&\|x_{n+1} - x^*\| \\
&\leq \big[\alpha_n + \theta\beta_n\big(\alpha_n' + \theta\beta_n'\big(\alpha_n'' + \theta\beta_n''\big)\big)\big]\|x_n - x^*\| \\
&\quad + M_n\beta_n\big(1 + \theta\beta_n' + \theta^2\beta_n'\beta_n''\big) + M\theta\beta_n\big(\gamma_n' + \theta\beta_n'\gamma_n''\big) + M\gamma_n \\
&\leq (1 - (1-\theta)\beta_n)\|x_n - x^*\| + \big[3M_n + M\big(\gamma_n' + \gamma_n'' + a_n\big)\big]\beta_n
\end{aligned}$$

for all $n \geq 0$. It follows from Lemma 2.3 and (3.8)-(3.10) that $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

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