Int. Journal of Math. Analysis, Vol. 5, 2011, no. 48, 2421 - 2425

An Analytic Study of the Cauchy Problems

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Abstract

In this paper, homotopy perturbation method is applied to solve the Cauchy problems. The results show that the homotopy perturbation method is of high accuracy, more convenient and efficient for solving the Cauchy problems.

Mathematics Subject Classification: 47G99

Keywords: Homotopy perturbation method, initial condition, Cauchy problem

1 Introduction

In this paper we propose homotopy perturbation method [1-4] to solve Cauchy problems. In this method the solution is considered as the summation of an infinite series which usually converges rapidly to the exact solutions. Using homotopy technique in topology, a homotopy is constructed with an embedding parameter $p \in [0, 1]$ which is considered as a small parameter. To illustrate the basic concepts of homotopy perturbation method, consider the following non-linear functional equation:

$$A\left(u\right) = f\left(r\right), \quad r \in \Omega, \tag{1}$$

With the following boundary conditions:

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$$B\left(u, {}^{\partial u}\!/_{\partial n}\right) = 0, \quad r \in \Gamma.$$

Where A is a functional operator, B is a boundary operator, $f(r)$ is a known analytic function, and Γ is the boundary of the domain Ω. Generally speaking, the operator A can be decomposed into two parts L and N, where L is a linear and N is a non-linear operator. Therefore Eq.(1) can be rewritten as the following:

$$L(u) + N(u) - f(r) = 0. \tag{2}$$

We construct a homotopy $U(r,p) : \Omega \times [0,1] \to R$, which satisfies:

$$H(U,p) = (1-p)\left[L(U) - L(u_0)\right] + p\left[A(U) - f(r)\right] = 0, \quad p \in [0,1], \quad r \in \Omega.$$

Or

$$H(U,p) = L(U) - L(u_0) + pL(u_0) + p\left[N(U) - f(r)\right] = 0, \quad p \in [0,1], \quad r \in \Omega.,$$

Where u_0 is an initial approximation to the solution of Eq.(1). In this method, homotopy perturbation parameter p is used to expand the solution, as a power series, say;

$$U = U_0 + pU_1 + p^2U_2 + \cdots,$$

And setting $p = 1$ results in the approximate solution of Eq.(1) as:

$$u = \lim_{p \to 1} U = U_0 + U_1 + U_2 \cdots,$$

In this paper, we use the method to discuss the first-order partial differential equation in the form [5]:

$$\begin{array}{l} \frac{\partial u(x,t)}{\partial t} + a(x,t)\frac{\partial u(x,t)}{\partial x} = \varphi(x), \qquad x \in R, \ \ t > 0. \\ u(x,0) = \psi(x), \ \ x \in R. \end{array} \tag{3}$$

For solving Eq.(3), by homotopy perturbation method, we construct the following homotopy:

$$\frac{\partial U}{\partial t} - \frac{\partial u_0}{\partial t} + p\left(a(x,t)\frac{\partial U}{\partial x} - \varphi(x) + \frac{\partial u_0}{\partial t}\right) = 0, \tag{4}$$

Suppose that the solution of Eq.(4) to be in the following form

$$U = U_0 + pU_1 + p^2U_2 + \ldots \tag{5}$$

Substituting Eq.(5) into Eq.(4), and equating the coefficients of the terms with the identical powers of p,

$$\begin{array}{l} p^0 : \frac{\partial U_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, \\ p^1 : \frac{\partial U_1}{\partial t} + \frac{\partial u_0}{\partial t} - \varphi(x) + a(x,t)\frac{\partial U_0}{\partial x} = 0, \\ p^2 : \frac{\partial U_2}{\partial t} + a(x,t)\frac{\partial U_1}{\partial x} = 0, \\ p^3 : \frac{\partial U_3}{\partial t} + a(x,t)\frac{\partial U_1}{\partial x} = 0, \\ \vdots \\ p^j : \frac{\partial U_j}{\partial t} + a(x,t)\frac{\partial U_{j-1}}{\partial x} = 0, \\ \vdots \end{array}$$

For simplicity we take

$$U_0(x,t) = u_0(x,t) = u(x,0) = \psi(x)$$

Having this assumption we get the following iterative equation

$$U_1 = \int_0^t \left(-a(x,t)\frac{\partial U_0}{\partial x} + \varphi(x) - \frac{\partial u_0}{\partial t} \right) dt,$$

$$U_j = \int_0^t \left(-a(x,t)\frac{\partial U_{j-1}}{\partial x} \right) dt, \qquad j = 2, 3, ...$$

Therefore, the solutions of Eq.(3) can be obtained, by setting $p = 1$

$$u = \lim_{p \to 1} U = U_0 + U_1 + U_2 + U_3 + ...$$

2 Numerical examples

To illustrate the method and to show the ability of the method three examples are presented.

Example1. Consider the transport equation [6]:

$$\begin{array}{l} \frac{\partial u(x,t)}{\partial t} + a\frac{\partial u(x,t)}{\partial x} = 0, \qquad x \in R, \ \ t > 0. \\ u(x,0) = x^2, \ \ x \in R. \end{array} \tag{6}$$

A homotopy can be readily constructed as follows:

$$\frac{\partial U(x,t)}{\partial t} - \frac{\partial u_0(x,t)}{\partial t} + p\left(a\frac{\partial U(x,t)}{\partial x} \right) = 0 \tag{7}$$

Substituting Eq.(5) into Eq.(7), and equating the terms with identical powers of p, we have

$$
\begin{aligned}
&p^0 \;:\; U_0\left(x,t\right) = u\left(x,0\right) = x^2, \\
&p^1 \;:\; U_1\left(x,t\right) = -\textstyle\int_0^t \left(a\frac{\partial U_0}{\partial x}\right)\, dt = -2atx, \\
&p^2 \;:\; U_2\left(x,t\right) = -\textstyle\int_0^t \left(a\frac{\partial U_1}{\partial x}\right)\, dt \;= a^2t^2\,, \\
&p^3 \;:\; U_3\left(x,t\right) = -\textstyle\int_0^t \left(a\frac{\partial U_2}{\partial x}\right)\, dt = 0\,, \\
&\vdots \\
&p^j \;:\; U_j\left(x,t\right) = -\textstyle\int_0^t \left(a\frac{\partial U_{j-1}}{\partial x}\right)\, dt \;= 0\,,
\end{aligned}
$$

And so on. Therefore, an exact solution of Eq.(6) can be obtained, by setting $p = 1$.

$$
u\left(x,t\right) = \lim_{p\to 1} U\left(x,t\right) = \sum_{i=0}^{\infty} U_i\left(x,t\right) = x^2 - 2atx + a^2t^2
$$

Example 2. Consider the nonlinear Cauchy problem [6]:

$$
\begin{aligned}
&\tfrac{\partial u(x,t)}{\partial t} + x\tfrac{\partial u(x,t)}{\partial x} = 0, \qquad x \in R,\;\; t > 0. \\
&u\left(x,0\right) = x^2,\;\; x \in R.
\end{aligned} \tag{8}
$$

According to Eqs.(4) and (5) we have

$$
\begin{aligned}
&p^0 \;:\; U_0\left(x,t\right) = u\left(x,0\right) = x^2, \\
&p^1 \;:\; U_1\left(x,t\right) = -\textstyle\int_0^t \left(x\frac{\partial U_0}{\partial x}\right)\, dt = -2tx^2, \\
&p^2 \;:\; U_2\left(x,t\right) = -\textstyle\int_0^t \left(x\frac{\partial U_1}{\partial x}\right)\, dt \;= 2t^2x^2\,, \\
&p^3 \;:\; U_3\left(x,t\right) = -\textstyle\int_0^t \left(x\frac{\partial U_2}{\partial x}\right)\, dt = -\tfrac{4}{3}\,t^3x^2, \\
&\vdots \\
&p^j \;:\; U_j\left(x,t\right) = -\textstyle\int_0^t \left(x\frac{\partial U_{j-1}}{\partial x}\right)\, dt \;= (-1)^j\; \tfrac{(2t)^j}{j!}x^2,
\end{aligned}
$$

And so on. Therefore, an exact solution of Eq.(8) can be obtained, by setting $p = 1$.

$$
u\left(x,t\right) = \lim_{p\to 1} U\left(x,t\right) = \sum_{i=0}^{\infty} U_i\left(x,t\right) = x^2 \sum_{i=0}^{\infty} \left(-1\right)^i \frac{\left(2t\right)^i}{i!} = x^2e^{-2t}.
$$

Example 3. Consider the following non-homogeneous Cauchy problem [6]:

$$
\begin{aligned}
&\tfrac{\partial u(x,t)}{\partial t} + \tfrac{\partial u(x,t)}{\partial x} = x, \qquad x \in R,\;\; t > 0. \\
&u\left(x,0\right) = e^x,\;\; x \in R.
\end{aligned} \tag{9}
$$

According to Eqs.(4) and (5) we have

$$
\begin{aligned}
&p^0 \;:\; U_0\,(x,t) = u\,(x,0) = e^x, \\
&p^1 \;:\; U_1\,(x,t) = \textstyle\int_0^t \left(-\frac{\partial U_0}{\partial x} + x\right)\, dt = -te^x + tx, \\
&p^2 \;:\; U_2\,(x,t) = -\textstyle\int_0^t \left(\frac{\partial U_1}{\partial x}\right)\, dt \;=\frac{t^2}{2}e^x - \frac{t^2}{2}\,, \\
&p^3 \;:\; U_3\,(x,t) = -\textstyle\int_0^t \left(\frac{\partial U_2}{\partial x}\right)\, dt = -\,\frac{t^3}{6}e^x, \\
&\vdots \\
&p^j \;:\; U_j\,(x,t) = -\textstyle\int_0^t \left(\frac{\partial U_{j-1}}{\partial x}\right)\, dt \;= (-1)^j\; \frac{t^j}{j!}e^x,
\end{aligned}
$$

And so on. Therefore, an exact solution of Eq.(8) can be obtained, by setting $p = 1$.

$$
u\,(x,t) = \lim_{p\to 1} U\,(x,t) = \sum_{i=0}^{\infty} U_i\,(x,t) = t\left(x - \frac{t}{2}\right) + e^{x-t}.
$$

3 Conclusion

In this paper the homotopy perturbation method is used to solve the Cauchy problems. We described the method, used it on three test problems, by this technique we obtain the exact solution. In addition, this technique is algorithmic and it is easy to implementation by symbolic computation software, such as Maple and Matlab. The obtained solution shows that the method is vary convenient and effective to solve wide classes of problems.

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Received: September, 2011