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Improper Partial Bilateral Generating Functions for Some Orthogonal Polynomials

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Abstract

In this paper we used the group theoretic method to obtain three new classes of proper and improper partial bilateral generating functions from given class of proper and improper partial bilateral generating functions. First is a class of generating function involving Hermite and Bessel polynomials. Second is involving Bessel and Jacobi polynomials. Whereas, third is involving Hermite and Jacobi polynomials.

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1 Introduction

In a theoretical connection with the unification of generating functions has great importance in the study of special functions. With the steps forward in this directions has been made by some researchers [3, 4, 6, 15]. Also, the special functions has great deal with applications in pure and applied mathematics. They appears in different frameworks. They are often used in combinatorial analysis [13], and even in statistics [9]. In his study Mukherjee [11] extend the idea of bilateral generating function involving Jacobi polynomials derived by Chongdar [7] and it has been well presented by group-theoretic method. Also, he has been proved the existence of quasi bilinear generating function implies the existence of a more general generating function. One may refer an interesting article of Mujumdar [10] on bilateral generating functions. In their paper [1], Alam and Chongdar obtained some results on bilateral and trilateral generating functions of modified Laguerre polynomials. Furthermore, they made some comments on the results of Laguerre polynomials obtained by Das and Chatterjea [8]. Further, Banerji and Mohsen [2] established a result on generating relation involving modified Bessel polynomials.

In this paper we used the group theoretic method to obtain the new classes of generating functions from a given class of generating function. First is the class of generating function obtained from the given class of improper bilateral generating function involving Hermite and Bessel polynomials. Second is the class of generating function obtained from the given class of improper partial bilateral generating function involving Bessel and Jacobi polynomials. Whereas, third is obtained from the given class of improper bilateral generating function of Hermite and Jacobi polynomials.

2 Preliminaries

In this section we give the brief introduction to Hermite, Bessel and Jacobi polynomials. Also, we define the recently used terms proper and improper partial bilateral generating functions.

The Hermit and Jacobi polynomials, as introduced by Rainville [12], are defined as:

$$H_n(x) = \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k n!(2x)^{n-2k}}{k!(n-2k)!},$$

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!} {}_2F_1\left(-n, 1+\alpha+\beta+n; 1+\alpha; \frac{1-x}{2}\right).$$

The Bessel polynomials, as introduced by Srivastava and Manocha [16], are defined as:

$$Y_n(x,\alpha,\beta) = {}_2F_0\left(-n, \alpha+n-1; -; -\frac{x}{\beta}\right).$$

We consider the following partial differential operators (cf. [5, 11, 14]):

$$R_1 = 2xz - z\frac{\partial}{\partial x}, \tag{1}$$

$$R_2 = y^2 t^{-1} s\frac{\partial}{\partial y} + ys\frac{\partial}{\partial t} + yt^{-1}s^2\frac{\partial}{\partial s} + t^{-1}s(\beta - y) \tag{2}$$

and

$$R_3 = (1-u^2)g^{-1}h\frac{\partial}{\partial u} - h(u-1)\frac{\partial}{\partial g} - (1+u)g^{-1}h^2\frac{\partial}{\partial h} - (1+\gamma)(1+u)g^{-1}h. \tag{3}$$

So that

$$R_1\left[H_{m+n}(x)z^{m+n}\right] = H_{m+n+1}(x)z^{m+n+1}, \tag{4}$$

$$R_2\left[Y_{k+n}(y,\alpha,\beta)t^{\alpha}s^{k+n}\right] = \beta Y_{k+n+1}(y,\alpha-1,\beta)t^{\alpha-1}s^{k+n+1}, \tag{5}$$

$$R_3\left[P_{m+n}^{(\gamma,\mu)}(u)g^{\mu}h^{m+n}\right] = -2(m+n+1)P_{m+n+1}^{(\gamma,\mu-1)}(u)g^{\mu-1}h^{m+n+1}. \tag{6}$$

Consequently we have

$$\exp(wR_1)f(x,z) = \exp(2wxz - w^2z^2)f(x-wz,z), \tag{7}$$

$$\begin{aligned}&\exp(\xi R_2)f(y,t,s)\\ &= (1-\xi y t^{-1}s)\exp(\beta\xi t^{-1}s)f\left(\tfrac{y}{1-\xi yt^{-1}s}, \tfrac{t}{1-\xi yt^{-1}s}, \tfrac{s}{1-\xi yt^{-1}s}\right)\end{aligned} \tag{8}$$

and

$$\exp(\eta R_3)f(u,g,h) = \left(\frac{g}{g+2\eta h}\right)^{\gamma+1} f\left(\frac{ug+2\eta h}{g+2\eta h}, \frac{g(g+2\eta h)}{g+2\eta h}, \frac{gh}{g+2\eta h}\right). \tag{9}$$

Now we define the recently used terms proper partial bilateral generating relation and improper partial bilateral generating relation for two classical polynomials introduced by Sarkar (cf. [14]).

Definition 2.1 *Proper partial bilateral generating function for two classical polynomials is the relation*

$$G(x,z,w) = \sum_{n=0}^{\infty} a_n w^n p_{m+n}^{(\alpha)}(x) q_{m+n}^{(\beta)}(z). \tag{10}$$

Where the coefficients a_n's are quite arbitrary, $p_{m+n}^{(\alpha)}$ and $q_{m+n}^{(\beta)}$ are any two classical polynomials of order $(m+n)$ with the parameters α and β respectively. (α and β are complex numbers, m is integer.)

Definition 2.2 *Improper partial bilateral generating function for two classical polynomials is the relation*

$$G(x,z,w) = \sum_{n=0}^{\infty} a_n w^n p_{m+n}^{(\alpha)}(x) q_{k+n}^{(\beta)}(z). \tag{11}$$

Where the coefficients a_n's are quite arbitrary, $p_{m+n}^{(\alpha)}$ and $q_{m+k}^{(\beta)}$ are any two classical polynomials of order $(m+n)$ and $(k+n)$ respectively with the parameters α and β. (α and β are complex numbers and m, k are integers.)

In the following section we have obtained new classes of improper bilateral generating functions for orthogonal polynomials.

3 Improper Partial Bilateral Generating Functions

In this section we have obtained new classes of improper partial bilateral generating functions for the orthogonal polynomials such as Hermite, Bessel and Jacobi polynomials. First is improper partial bilateral generating function involving the Hermite and Bessel polynomials. It is given in the form of theorem 3.1. Second is involving Bessel and Jacobi polynomials given in the form of theorem 3.2, whereas the generating function involving the Hermite and Jacobi polynomials is in the form of theorem 3.3.

Theorem 3.1 *If there exist the following class of improper partial bilateral generating function for the Hermite and Bessel polynomials by means of the relation*

$$G(x,y,w)=\sum_{n=0}^{\infty}a_n w^n H_{m+n}(x)Y_{k+n}(y,\alpha,\beta), \tag{12}$$

where a_n's are arbitrary, then the following relation is hold:

$$\begin{aligned}(1-\xi y)^{1-\alpha-k}&\exp\left(2wx-w^2+\beta\xi\right)f\left(x-w,\tfrac{y}{1-\xi y},\tfrac{w\xi}{1-\xi y}\right)\\ &=\sum_{n,p,q=0}^{\infty}a_n\frac{w^{n+p}\xi^{n+q}}{p!q!}\beta^q H_{m+n+p}(x)Y_{k+n+q}(y,\alpha-q,\beta),\end{aligned} \tag{13}$$

where $|\xi y|<1$. Equation (13) represents a new class of improper partial bilateral generating function for Hermite and Bessel polynomials.

Theorem 3.2 *If there exist the following class of improper partial bilateral generating functions for the Bessel and Jacobi polynomials by means of the relation*

$$G(y,u,\xi)=\sum_{n=0}^{\infty}a_n\xi^n Y_{k+n}(y,\alpha,\beta)P_{m+n}^{(\gamma,\mu)}(u), \tag{14}$$

where a_n's are arbitrary, then the following relation is hold:

$$\begin{aligned}(1-\xi y)^{1-\alpha-k}&\exp\left(\beta\xi\right)(1+2\eta)^{-(1+\gamma+m)}G\left(\tfrac{y}{1-\xi y},\tfrac{u+2\eta}{1+2\eta},\tfrac{\xi\eta}{(1-\xi y)(1+2\eta)}\right)\\ =\sum_{n,p,q=0}^{\infty}&a_n\frac{\xi^{n+p}\eta^{n+q}}{p!q!}(-2)^q\beta^p(m+n+1)_q Y_{k+n+p}(y,\alpha-p,\beta)P_{m+n+q}^{(\gamma,\mu-q)}(u),\end{aligned} \tag{15}$$

where $|\xi y|<1$. Equation (15) represents a new class of improper partial bilateral generating function for Bessel and Jacobi polynomials.

Theorem 3.3 *If there exist the following class of improper partial bilateral generating functions for the Hermite and Jacobi polynomials by means of the relation*

$$G(x,u,w)=\sum_{n=0}^{\infty} a_n w^n H_{m+n}(x) P_{k+n}^{(\gamma,\mu)}(u), \tag{16}$$

where a_n's are arbitrary, then the following relation is hold:

$$\begin{aligned}(1+2\eta)^{-(1+\gamma+k)} &\exp\left(2wx-w^2\right) f\left(x-w, \tfrac{u+2\eta}{1+2\eta}, \tfrac{w\eta}{1+2\eta}\right)\\ =\sum_{n,p,q=0}^{\infty} a_n &\frac{w^{n+p}\eta^{n+q}}{p!q!}(-2)^q (k+n+1)_q H_{m+n+p}(x) P_{k+n+q}^{(\gamma,\mu-q)}(u).\end{aligned} \tag{17}$$

Equation (17) represents a new class of improper partial bilateral generating function for Hermite and Jacobi polynomials.

Proof of the Theorem 3.1.
Let

$$G(x,y,w)=\sum_{n=0}^{\infty} a_n w^n H_{m+n}(x) Y_{k+n}(y,\alpha,\beta), \tag{18}$$

on multiplying both sides of above equation by $z^m t^\alpha s^k$, which gives us

$$z^m t^\alpha s^k G(x,y,w)=\sum_{n=0}^{\infty} a_n w^n H_{m+n}(x) z^m Y_{k+n}(y,\alpha,\beta) t^\alpha s^k. \tag{19}$$

Now replacing w by $w\xi zs$ in (19), we get

$$z^m t^\alpha s^k G(x,y,w\xi zs)=\sum_{n=0}^{\infty} a_n (w\xi)^n H_{m+n}(x) z^{m+n} Y_{k+n}(y,\alpha,\beta) t^\alpha s^{k+n}. \tag{20}$$

Applying the results in (4), (5), (7) and (8) to equation (20) we obtain

$$\begin{aligned}&(1-\xi y t^{-1}s)^{1-\alpha-k}\exp\left(2wxz-w^2z^2+\beta\xi t^{-1}s\right) z^m t^\alpha s^k\\ &\times f\left(x-w, \tfrac{y}{1-\xi y t^{-1}s}, \tfrac{w\xi zs}{1-\xi y t^{-1}s}\right)\\ &=\sum_{n,p,q=0}^{\infty} a_n \frac{w^{n+p}\xi^{n+q}}{p!q!}\beta^q H_{m+n+p}(x) Y_{k+n+q}(y,\alpha-q,\beta) z^{m+n+p} t^{\alpha-q} s^{k+n+q}.\end{aligned} \tag{21}$$

Finally, when putting $z=t=s=1$ in the above equation (21), we arrive at the result (13).

Remark 1 *The proofs of theorems 3.2 and 3.3 are similar to that of theorem 3.1.*

In the following section we have obtained the new class of proper partial bilateral generating functions.

4 Particular Cases:

It may be of special interest to point out that, for $k = m$, the above theorems (3.1, 3.2 and 3.3) become a nice class of generating functions forms proper bilateral generating functions. We state these results in the form of following corollaries:

Corollary 4.1 *If there exist the following class of (proper) partial bilateral generating function for the Hermite and Bessel polynomials by means of the relation*

$$G(x, y, w) = \sum_{n=0}^{\infty} a_n w^n H_{m+n}(x) Y_{m+n}(y, \alpha, \beta), \tag{22}$$

where a_n's are arbitrary, then the following general class of generating function is hold:

$$\begin{aligned}&(1-\xi y)^{1-\alpha-m} \exp\left(2wx - w^2 + \beta\xi\right) f\left(x - w, \tfrac{y}{1-\xi y}, \tfrac{w\xi}{1-\xi y}\right) \\ &= \sum_{n,p,q=0}^{\infty} a_n \frac{w^{n+p}\xi^{n+q}}{p!q!} \beta^q H_{m+n+p}(x) Y_{m+n+q}(y, \alpha - q, \beta),\end{aligned} \tag{23}$$

where $|\xi y| < 1$.

Corollary 4.2 *If there exist the following class of (proper) partial bilateral generating function for the Bessel and Jacobi polynomials by means of the relation*

$$G(y, u, \xi) = \sum_{n=0}^{\infty} a_n \xi^n Y_{m+n}(y, \alpha, \beta) P_{m+n}^{(\gamma,\mu)}(u), \tag{24}$$

where a_n's are arbitrary, then the following general class of generating function is hold:

$$\begin{aligned}&(1-\xi y)^{1-\alpha-m} \exp(\beta\xi)(1+2\eta)^{-(1+\gamma+m)} G\left(\tfrac{y}{1-\xi y}, \tfrac{u+2\eta}{1+2\eta}, \tfrac{\xi\eta}{(1-\xi y)(1+2\eta)}\right) \\ &= \sum_{n,p,q=0}^{\infty} a_n \frac{\xi^{n+p}\eta^{n+q}}{p!q!} (-2)^q \beta^p (m+n+1)_q Y_{m+n+p}(y, \alpha - p, \beta) P_{m+n+q}^{(\gamma,\mu-q)}(u),\end{aligned} \tag{25}$$

where $|\xi y| < 1$.

Corollary 4.3 *If there exist the following class of (proper) partial bilateral generating function for Hermite and Jacobi polynomials by means of the relation*

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n w^n H_{m+n}(x) P_{m+n}^{(\gamma,\mu)}(u), \tag{26}$$

where a_n's are arbitrary, then the following general class of generating function is hold:

$$\begin{aligned}&(1+2\eta)^{-(1+\gamma+m)} \exp\left(2wx - w^2\right) f\left(x-w, \tfrac{u+2\eta}{1+2\eta}, \tfrac{w\eta}{1+2\eta}\right)\\ = &\sum_{n,p,q=0}^{\infty} a_n \frac{w^{n+p}\eta^{n+q}}{p!q!}(-2)^q (m+n+1)_q H_{m+n+p}(x) P_{m+n+q}^{(\gamma,\mu-q)}(u).\end{aligned} \tag{27}$$

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References

[1] S. Alam and A. K. Chongdar, *On generating functions of modified Laguerre polynomials*, Rev. Real Academia de Ciencias, Zaragoza, **62**, (2007), pp. 91–98.

[2] P. K. Banerji and F. B. F. Mohsen, *On Generating Functions Involving Laguerre and Modified Bessel Polynomials*, Varāhmihir Journal of Mathematical Sciences, **Vol 2** No. 1, (2002), pp. 191–196.

[3] S. K. Chatterjea, *Unification of a class of bilateral generating relations for certain special functions*, Bull. Cal. Math. Soc., **67** (1975), pp. 115–127.

[4] S. K. Chatterjea, *An extension of a class of bilateral generating functions for certain special functions*, Bull. Inst. Math. Acad. Sinica **5(2)** (1977).

[5] A. K. Chongdar, *Some generating functions involving Laguerre polynomials*, Bull. Calcutta Math. Soc. **76**, (1984), pp. 262-269.

[6] A. K. Chongdar, *On the unification of a class of bilateral generating functions for certain special functions*, Tamkang Jour. Math., **18 (3)** (1987), pp. 53–59.

[7] A. K. Chongdar, *On Certain Bilateral Generating Functions*, Bull. Cal. Math.Soc.,

http://www.dmi.units.it/~rimut/volumi/24/07.pdf

[8] S. Das and S. K. Chatterjea, *On a partial differential operator for Laguerre polynomials*, Pure math. Manuscript, **4**, (1985), pp. 187–193.

[9] M. G. Kendall and A. Stuart, *The Advanced Theory of Statistics*, Griffin, London, (1958).

[10] A. B. Mujumdar, *On Bilateral Generating Functions III*, Journal of Indian Acad. math., **Vol 20** No.1, (1998), pp. 101-104.

[11] M. C. Mukherjee, *An Extension of bilateral generating function of certain special Function*, Real Academia de Ciencias, Zaragoza, **57**, (2002), pp. 143–146.

[12] E. D. Rainville, *Special functions*, Macmillan (1960), New York.

[13] J. Riordan, *An Introduction to Combinatorial Analysis*, Wiley Publications in Mathematical Statistics, John Wiley & Sons, New York, (1958).

[14] A. K. Sarkar, *On a partial bilateral and improper partial bilateral generating functions involving Some special functions*, FILMAT(Nis), **18**, (2004), pp. 41-49.

[15] J. P. Singhal and H. M. Srivastava, *A class of bilateral generating functions for certain classical polynomials*, Pacific J. Math. **42** (1972), pp. 355–362.

[16] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Halsted Press, John, Wiley and Sons(1984), New York.

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