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Construction of Wavelet Frames Using Lifting Like Schemes

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Abstract

The unitary extension principle and mixed extension principles are used to construct Parseval frames. Some desired properties like the order of vanishing moment of a frame is one of the common property in applications. The number of vanishing moment of a wavelet frame obtained through the extension principle depends on the way the polyphase or modulation matrix is constructed. The lifting like scheme has been presented to improve the number of vanishing moment of a given MRA based wavelet frame. A simple way of construction of wavelet frame using the polyphase components of the filter is also presented.

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1 Introduction

For practical purpose, a Hilbert space is required to have a nice basis like the orthonormal one. Fourier basis has been used in signal processing, image compression, etc. for a long time, however it has some disadvantages. In some applications wavelet bases have been proven to be better than the classical Fourier basis since they provide localizations. Frames, being linearly dependent spanning set, are generalizations of orthonormal basis and thus they provide redundancy. This is one of the reasons why fames are used in signal transmissions. Over the past few years wavelets and frames have made significant development in theory and applications . Like wavelets, frames have some desirable properties like vanishing moments, compactness of support, approximation order, orthogonality, symmetry etc. [4, 5, 7, 8]. A wavelet frame is a frame that is generated from affine system of functions. The order of vanishing moments is important in applications which are related to approximation order of wavelet frame decompositions [7]. It is important for applications to have symmetric/antisymmetric wavelet functions.

Multiresoltion analysis (MRA) based compactly supported wavelet frames constructed from the unitary extension principle (UEP), as in [7], are considered throughout this paper. The lifting [10] like schemes are given to increase the number of vanishing moments of wavelet frames. The resulting functions all have compact support. A few simple ways of generating a wavelet frame with some desired properties are presented. It is shown that the symmetric/antisymmetric wavelet frames can be constructed from the simple wavelets or wavelet frames. Finally a method to construct a polyphase matrix is provided from which the wavelet frame can be constructed.

2 Preliminaries

2.1 Frames

Let $\mathbb{H}$ be a separable Hilbert space. A sequence $\mathbb{X} = \{x_j\}_j$ in $\mathbb{H}$ called a Bessel sequence if there exists a constant $B > 0$ such that for all $f \in \mathbb{H}$

$$\sum_{j\in\mathbb{J}} | < f, x_j > |^2 \leq B||f||^2.$$

It is said to be a frame if there exists constants $0 < A \leq B$ such that for all $f \in \mathbb{H}$

$$A||f||^2 \leq \sum_{j\in\mathbb{J}} | < f, x_j > |^2 \leq B||f||^2.$$

A and B are called frame bounds. If$A = B$, then it is called a tight frame. Let $\mathbb{X} = \{x_j\}_j$ be a Bessel sequence on $\mathbb{H}$. The analysis operator $\Theta_{\mathbb{X}}$: $\mathbb{H} \to l^2(\mathbb{J})$ is defined by

$$\Theta_{\mathbb{X}} : f \to \{< f, x_j >\}_j;$$

and the synthesis operator is

$$\Theta^*_{\mathbb{X}} : \{c_j\} \to \sum_{j\in\mathbb{J}} c_j x_j.$$

Note that the analysis and synthesis operators are well defined and bounded because $\mathbb{X}$ is a Bessel sequence [8]. The synthesis operator is the Hilbert space adjoint of the analysis operator. The frame operator $S_{\mathbb{X}}f : \mathbb{H} \to \mathbb{H}$ is defined as

$$S_{\mathbb{X}}f = \Theta^*_{\mathbb{X}}\Theta_{\mathbb{X}}f :\to \sum_j < f, x_j > x_j.$$

Also note that $\mathbb{X}$ is a frame if and only if $S_{\mathbb{X}}$ is bounded and has bounded inverse [8]. In that case the reconstruction formula can be written as

$$f(x) = \sum_j < f, S_{\mathbb{X}}^{-1} x_j > x_j = \sum_j < f, x_j > S_{\mathbb{X}}^{-1} x_j.$$

$S_{\mathbb{X}}$ is preferred to be an identity. In that case $\mathbb{X}$ is called a normalized tight frame or Parseval frame. Let $\mathbb{X} = \{x_j\}, j \in \mathbb{J}$ and $\mathbb{Y} = \{y_j\}, j \in \mathbb{J}$ be two Bessel sequences in $\mathbb{H}$. The operator

$$\Theta_{\mathbb{Y}}^* \Theta_{\mathbb{X}} f :\rightarrow \sum_j < f, x_j > y_j,$$

is called a mixed dual gramian. If this operator is an identity, then the Bessel sequences $\mathbb{X}$ and $\mathbb{Y}$ are actually frames and are called dual frames [4]. The frames considered in this paper are generated from the affine system. They are all multiresolution analysis (MRA) based wavelet frames [7] where the scaling function and wavelets are all compactly supported . The MRA used in frame system is slightly relaxed from the usual MRA of wavelet ststem [5].

2.2 Notations

Let D be a $d \times d$ dilation matrix, a matrix with integer entries such that all of its eigenvalues are strictly greater than one in absolute value. Let the $m = |detD|$. Let $S = \{\mathbf{d}_0, \mathbf{d}_1, \ldots \mathbf{d}_{m-1}\}$ denote the representative of the cosets of $\mathbb{Z}^d / D\mathbb{Z}^d$ with $\mathbf{d}_0 = \mathbf{0}$. A scaling function φ is refinable (D-refinable) and satisfies

$$\varphi(\mathbf{x}) = \frac{1}{\sqrt{m}} \sum_{\mathbf{k} \in \mathbb{Z}^d} h_{\mathbf{k}} \varphi(D\mathbf{x} - \mathbf{k}). \tag{2.1}$$

The Fourier transform of $f \in L_1(\mathbb{R}^d)$ is given by

$$\widehat{f}(\boldsymbol{\xi}) = \int_{\mathbb{R}^d} f(\mathbf{t}) e^{-i \mathbf{t} \cdot \boldsymbol{\xi}} d\mathbf{t}$$

Taking the Fourier transform of 2.1, we get

$$\widehat{\varphi}(\boldsymbol{\xi}) = H^0(D^{-t}\boldsymbol{\xi}) \widehat{\varphi}(D^{-t}\boldsymbol{\xi})$$

where D^{-t} is the inverse transpose of D and $H^0(\boldsymbol{\xi})$ is called its filter or symbol. The construction of wavelet frame then relies on the construction of suitable $2\pi\mathbb{Z}^d$ periodic, bounded functions $H^l(\boldsymbol{\xi})$ $l = 1, 2, \ldots, r$ such that the affine system $\psi_{j,\mathbf{k}}^l := m^{1/2} \psi^l (D^j \cdot - \mathbf{k}),\ l = 1, 2, \ldots, r,\ j \in \mathbb{Z},\ \mathbf{k} \in \mathbb{Z}^d$ generated by the functions $\psi^l,\ l = 1, 2, \ldots, r$ is a frame for $L_2(\mathbb{R}^d)$ where

$$\widehat{\psi^l}(\boldsymbol{\xi}) = H^l(D^{-t}\boldsymbol{\xi}) \widehat{\varphi}(D^{-t}\boldsymbol{\xi}),\ \ l = 1, 2, \ldots, r.$$

We say that $\{\psi^1, \psi^2, \ldots, \psi^r\} \subset L_2(\mathbb{R}^d)$ generates a $D-$ wavelet frames for $L^2(\mathbb{R}^d)$ if the affine system is a frame for $L_2(\mathbb{R}^d)$. The pair $\{\psi^1, \psi^2, \ldots, \psi^r\}$ and $\{\tilde{\psi}^1, \tilde{\psi}^2, \ldots, \tilde{\psi}^r\}$ is said to generates a pair of dual $D-$ wavelet frames for $L^2(\mathbb{R}^d)$ if $\psi^l_{j,\mathbf{k}} := m^{1/2}\psi^l(D^j \cdot -\mathbf{k})$ and $\tilde{\psi}^l_{j,\mathbf{k}} := m^{1/2}\tilde{\psi}^l(D^j \cdot -\mathbf{k})$ are dual frames. In this case the following reconstruction formula holds for all $f \in L_2(\mathbb{R}^d)$,

$$f(\mathbf{x}) = \sum_{j=-\infty}^{\infty} \sum_{\mathbf{k}\in\mathbb{Z}^d} \sum_{l=1}^{r} < f(\mathbf{x}), \tilde{\psi}^l_{j,\mathbf{k}} > \psi^l_{j,\mathbf{k}}. \tag{2.2}$$

The approximation order of above representation depends on the vanishing moments of $\tilde{\psi}^l$, $l = 1, 2, \ldots, r$. The following multivariate notations are used throughout this paper. $\mathbf{z} := e^{-i\boldsymbol{\xi}} := e^{-i(\xi_1,\xi_2,\ldots,\xi_d)} = \left(e^{-i\xi_1}, e^{-i\xi_2}, \ldots, e^{-i\xi_d}\right)$. Given a complex valued vector $\mathbf{r} = (r_1, r_2, \ldots, r_d)$, an integer valued vector $\mathbf{s} = (s_1, s_2, \ldots, s_d)$ and an integer-valued matrix $M = (M_1, M_2, \ldots, M_d)$, where M_i is the i^{th} column of M. Then, the vector $\mathbf{r}^{\mathbf{s}}$ is defined to be $\mathbf{r}^{\mathbf{s}} := \prod_{i=1}^{d} r_i^{s_i}$, and $\mathbf{r}^M$ is a row vector defined as $\mathbf{r}^M := (\mathbf{r}^{M_1^t}, \mathbf{r}^{M_2^t}, \ldots, \mathbf{r}^{M_d^t})$. In particular, if $\mathbf{z} = (z_1, z_2, \ldots, z_d)$ and $\mathbf{k} = (k_1, k_2, \ldots, k_d)$, then $\mathbf{z}^{\mathbf{k}} = \prod_{i=1}^{d} z_i^{k_i}$, $z_j = e^{-i\xi_j}, j = 1, 2, \ldots, d$. Also the following is needed when polyphase component of the filters are used.

$$\begin{aligned}
\mathbf{z}^M &= (\mathbf{z}^{M_1^t}, \mathbf{z}^{M_2^t}, \ldots, \mathbf{z}^{M_d^t}) \\
&= \left((e^{-i\boldsymbol{\xi}})^{M_1^t}, (e^{-i\boldsymbol{\xi}})^{M_2^t}, \ldots, (e^{-i\boldsymbol{\xi}})^{M_d^t}\right) \\
&= \left(e^{-iM_1^t\cdot\boldsymbol{\xi}}, e^{-iM_2^t\cdot\boldsymbol{\xi}}, \ldots, e^{-iM_d^t\cdot\boldsymbol{\xi}}\right) \\
&= e^{-i\left(M_1^t\cdot\boldsymbol{\xi}, M_2^t\cdot\boldsymbol{\xi}, \ldots, M_d^t\cdot\boldsymbol{\xi}\right)} \\
&= e^{-iM^t\boldsymbol{\xi}}.
\end{aligned}$$

If M is a complex matrix, we take M^*, the complex conjugate transpose instead of M^t. By $f(\mathbf{z})$, we actually mean $f(\boldsymbol{\xi})$. All filters or polyphase components are functions of $\boldsymbol{\xi}$, however, sometimes they are written as functions of $\mathbf{z}$. In this notation $f(\mathbf{z}^*) = f(-\boldsymbol{\xi})$.

Definition 2.1 The *symbol* or a filter of the scaling function and wavelet functions are defined by

$$H^l(\boldsymbol{\xi}) = \frac{1}{\sqrt{m}} \sum_{\mathbf{k}} h^l_{\mathbf{k}}\, e^{-i\,\mathbf{k}\cdot\boldsymbol{\xi}}.$$

Equivalently,

$$H^l(\mathbf{z}) = \frac{1}{\sqrt{m}} \sum_{\mathbf{k}} h^l_{\mathbf{k}}\, \mathbf{z}^{\mathbf{k}}.$$

An arbitrary $\mathbf{k} \in \mathbb{Z}^d$ can be written uniquely as

$$\mathbf{k} = D\mathbf{j} + \mathbf{d},$$

where $\mathbf{d} \in S, \mathbf{j} \in \mathbb{Z}^d$. Let $h^{l,k}_{\mathbf{j}} = h^l_{D\mathbf{j}+\mathbf{d}_k}$.

Definition 2.2 The *polyphase symbols* or components are given by

$$H^{l,k}(\boldsymbol{\xi}) = \sum_{\mathbf{j}} h^l_{D\mathbf{j}+\mathbf{d}_k} e^{-i\,\mathbf{j}\cdot\boldsymbol{\xi}} = \sum_{\mathbf{j}} h^{l,k}_{\mathbf{j}} e^{-i\,\mathbf{j}\cdot\boldsymbol{\xi}}.$$

For $l = 1, 2, \ldots, r$, the symbols can be expressed in terms of polyphase components as

$$H^l(\mathbf{z}) = \frac{1}{\sqrt{m}} \sum_{j=0}^{m-1} \mathbf{z}^{\mathbf{d}_j} H^{l,j}(\mathbf{z}^D). \tag{2.3}$$

Equivalently,

$$H^l(\boldsymbol{\xi}) = \frac{1}{\sqrt{m}} \sum_{j=0}^{m-1} e^{-i\,\mathbf{d}_j\cdot\boldsymbol{\xi}}\, H^{l,j}(D^t\boldsymbol{\xi}). \tag{2.4}$$

Let $\mathbf{d}'_i = 2\pi D^{-t}\mathbf{d}_i$, $i = 0, 1, \ldots m-1$.

Definition 2.3 The *modulation matrix* is

$$M(\boldsymbol{\xi}) = \begin{bmatrix} H^0(\boldsymbol{\xi}) & H^0(\boldsymbol{\xi}+\mathbf{d'}_1) & \cdots & H^0(\boldsymbol{\xi}+\mathbf{d'}_{m-1}) \\ H^1(\boldsymbol{\xi}) & H^1(\boldsymbol{\xi}+\mathbf{d'}_1) & \cdots & H^1(\boldsymbol{\xi}+\mathbf{d'}_{m-1}) \\ H^2(\boldsymbol{\xi}) & H^2(\boldsymbol{\xi}+\mathbf{d'}_1) & \cdots & H^2(\boldsymbol{\xi}+\mathbf{d'}_{m-1}) \\ \vdots & \vdots & \cdots & \vdots \\ H^r(\boldsymbol{\xi}) & H^r(\boldsymbol{\xi}+\mathbf{d'}_1) & \cdots & H^r(\boldsymbol{\xi}+\mathbf{d'}_{m-1}) \end{bmatrix},$$

Definition 2.4 The *polyphase matrix* is

$$P(\boldsymbol{\xi}) = \begin{bmatrix} H^{0,0}(\boldsymbol{\xi}) & H^{0,1}(\boldsymbol{\xi}) & \cdots & H^{0,m-1}(\boldsymbol{\xi}) \\ H^{1,0}(\boldsymbol{\xi}) & H^{1,1}(\boldsymbol{\xi}) & \cdots & H^{1,m-1}(\boldsymbol{\xi}) \\ H^{2,0}(\boldsymbol{\xi}) & H^{2,1}(\boldsymbol{\xi}) & \cdots & H^{2,m-1}(\boldsymbol{\xi}) \\ \vdots & \vdots & \cdots & \vdots \\ H^{r,0}(\boldsymbol{\xi}) & H^{r,1}(\boldsymbol{\xi}) & \cdots & H^{r,m-1}(\boldsymbol{\xi}) \end{bmatrix}.$$

A matrix $U(\boldsymbol{\xi})$ is said to a paraunitary if $U^*(\boldsymbol{\xi})U(\boldsymbol{\xi}) = I$ on the d dimensional torus. The following matrix is always a paraunitary matrix. The matrix $P(z) = (I_d - vv^t + vv^t z)$ is always paraunitary, i.e. $P^*P = I_d$ on the unit circle, where v is a unit column vector in $\mathbb{R}^d$. In fact,

$$P(\mathbf{z}) = M \prod_{j}^{d} \prod_{k} (I_d - v_j^k v_j^{k^t} + v_j^k v_j^{k^t} z_j), \tag{2.5}$$

where M is an $d \times d$ orthogonal matrix, $\mathbf{z} = (z_1, z_2, \ldots, z_d)$ and v_j^k are unit column vectors in $\mathbb{R}^d$ is always a paraunitary matrix. Symmetric paraunitary matrices have been used in [11] to obtain the symmetric/antisymmetric wavelet frames.

2.3 Vanishing Moments

The order of vanishing moments is one of the important properties of wavelets and wavelet frames. They provide higher degree of smoothness and approximation orders [?]. In case is $r = m - 1$ the vanishing moments can be described using the sum-rules or zero conditions, see [7] and citations therein. The polyphase criterion for the vanishing moments for wavelet frames are given in [7], which seems easy to check if the new wavelets are constructed through the polyphase matrices like. The polyphase criterion for vanishing moment of order $\boldsymbol{\alpha}$ holds if and only if $D^{\boldsymbol{\beta}}(H^l(M^{-t}\boldsymbol{\xi}))|_{\boldsymbol{\xi}=\mathbf{0}} = 0$, for $l = 1, 2, \ldots, r$, for all $\boldsymbol{\beta} \in \mathbb{Z}_+^d, \boldsymbol{\beta} \leq \boldsymbol{\alpha}$, with the notations given in [7].

2.4 Extension Principles

Extension principles [5, 1] are powerful tools in creating frames. We are interested in compactly supported (MRA) based wavelet frame as in [7]. The function satisfying the following properties is called a scaling function and provides such an MRA.

Definition 2.5 *Let $\varphi \in L^2(\mathbb{R}^d)$ satisfies*

(1) $\hat{\varphi}$ is continuous at the origin and $\hat{\varphi}(0) = 1$,

(2) there exists an $K > 0$ such that $\sum_{\mathbf{k}\in\mathbb{Z}^d} |\hat{\varphi}(\boldsymbol{\xi} + D\mathbf{k}\pi)|^2 \leq K$, for a.e. $\boldsymbol{\xi} \in \mathbb{R}^d$

(3) $\hat{\varphi}(\boldsymbol{\xi}) = H^0(D^{-t}\boldsymbol{\xi})\hat{\varphi}(D^{-t}\boldsymbol{\xi})$, where $H^0(\boldsymbol{\xi})$ is a $2\pi\mathbb{Z}^d$ periodic bounded function.

Here $H^0(\boldsymbol{\xi})$ is filter or symbol of the scaling function. The compact support of the scaling function implies that $H^0(\boldsymbol{\xi})$ is a trigonometric polynomial, also called a Laurent polynomial. The condition (1) is required in all of constructions. The condition (2) is satisfied if the integer shifts of φ form a Bessel system. This can be relaxed since the support of φ is compact. The condition (3) is simply refinability 2.1 of φ.

In frame theory one constructs $r(r \geq m-1)$ wavelets such that that affine system generated by them is a frame. For simplicity we will denote the scaling function by ψ^0 and wavelets by $\psi^1, \ldots, \psi^r$ and their symbols by $H^0, H^1, \ldots, H^r$ respectively. Their dual counterparts are denoted by $\widetilde{\psi}^0, \widetilde{\psi}^1, \ldots, \widetilde{\psi}^r$. The corresponding filters are denoted by $\widetilde{H}^l(\boldsymbol{\xi}), l = 0, 1, \ldots, r$. It turns out that the filters H^0 and $\widetilde{H}^0$ are continuous at the origin and $H^0(0) = \tilde{H}^0(0) = 1$ too. The unitary extension principle is about constructing $H^l(\boldsymbol{\xi}), l = 1, 2, \ldots, r$ such that the affine system $\psi^l_{j,\mathbf{k}}$, $l = 1, 2, \ldots, r$ is a tight frame, and the mixed extension principle is about constructing $H^l(\boldsymbol{\xi})$, and $\tilde{H}^l(\boldsymbol{\xi}), l = 1, 2, \ldots, r$ such that the affine systems $\psi^l_{j,\mathbf{k}}$ and $\widetilde{\psi}^l_{j,\mathbf{k}}$ form a pair of dual wavelet frame, where

$$\widehat{\psi}^l(\boldsymbol{\xi}) = H^l(D^{-t}\boldsymbol{\xi})\widehat{\psi}^0(D^{-t}\boldsymbol{\xi}), \text{ and } \widehat{\widetilde{\psi}}^l(\boldsymbol{\xi}) = \widetilde{H}^l(D^{-t}\boldsymbol{\xi})\widehat{\psi}^0(D^{-t}\boldsymbol{\xi}),$$

$l = 1, 2, \ldots, r$. So in this setting, as in [7], we have the following version of unitary extension principle.

Theorem 2.1 *Unitary Extension Principle: Let ψ^0 be a scaling function and let H^0 be its filter. Let $H^1, H^2, \ldots, H^r$ be $2\pi\mathbb{Z}^d$ periodic bounded functions. Let $\psi^l(\boldsymbol{\xi}) = H^l(D^{-t}\boldsymbol{\xi})\hat{\psi}^0(D^{-t}\boldsymbol{\xi})$. If the polyphase matrix P satisfies $P^*P = I_m$, then the system $\psi^l_{j,\mathbf{k}}$, $l = 1, 2, \cdots, r$ is a tight wavelet frames for $L^2(\mathbb{R}^d)$.*

The detailed extension principles are given in [5, 1]. Under some mild conditions see [7] and citations therein, the following theorem, mixed extension principles, provides a pair of dual wavelet frames.

Theorem 2.2 *Mixed Extension Principle: Let ψ^0 and $\widetilde{\psi}^0$ be two scaling functions, and let H^0 and $\widetilde{H}^0$ be their corresponding filters. Let $H^1, H^2, \ldots, H^r$ and $\widetilde{H}^1, \widetilde{H}^2, \ldots, \widetilde{H}^r$ be $2\pi\mathbb{Z}^d$ periodic bounded functions. If their corresponding polyphase matrices P and $\widetilde{P}$ satisfy $\widetilde{P}^*P = I_m$, then the system $\psi^l_{j,\mathbf{k}}$, and $\widetilde{\psi}^l_{j,\mathbf{k}}$, $l = 1, 2, \ldots, r$ form dual wavelet frames for $L^2(\mathbb{R}^d)$, where*

$$\widehat{\psi}^l(\boldsymbol{\xi}) = H^l(D^{-t}\boldsymbol{\xi})\widehat{\psi}^0(D^{-t}\boldsymbol{\xi}) \text{ and } \widehat{\widetilde{\psi}}^l(\boldsymbol{\xi}) = \widetilde{H}^l(D^{-t}\boldsymbol{\xi})\widehat{\widetilde{\psi}^0}(D^{-t}\boldsymbol{\xi}).$$

We note that the condition of $\tilde{P}^*P = I_m$ is equivalent to $\tilde{M}^*M = I_m$, see [6, 3]. Thus creating wavelet frames from a scaling function is just a matrix completion problem where the first row of the polyphase matrix is given.

3 Main Results

In the construction of wavelet frames, the extension principle [5, 1], uses the completion of modulation matrix of the filters, however it can be polyphase matrix instead [6, 7]. In modulation matrix, the first column contains all the information, the other columns are obtained from a shift applied to the entries of the first column, however this is not case with the polyphase matrix. As far as the extension principles are concerned, they are equivalent whether modulation matrices are used or the polyphase matrices are used [6]. The polyphase matrix has been used to study a number of properties of wavelet frames in the literatures [9, 7, 6, 2]. A few reasons of using the polyphase matrix here are the following. (i) Skopina in [7] has characterized the number of vanishing moments of wavelet frames in terms of polyphase components. (ii) The use of polyphase matrix is preferred in implementations for computational efficiency. (iii) The polyphase components are not structured [2] and (iv) Goyal and Kovačević, have used polyphase matrices to study the robustness to erasures of a frame in [9].

Theorem 3.1 can be used to raise the vanishing moment of a MRA based wavelet frame from a given system of a wavelet frame with given vanishing moment. A lifting like schene [10] has been presented (theorem 3.1) to generate a pair of dual wavelet frames when a wavelet frame, constructed from the UEP, is given. This leads to a pair of dual wavelet frames (synthesis and analysis). The process can be repeated on dual side too. Theorem 3.2 can be used to generate a wavelet frame from a given system of wavelet with the given vanishing moments and theorem 3.3 uses a suitably constructed polyphase matrix from which the wavelet frames can be constructed.

3.1 Raising vanishing moments through lifting

In a wavelet system the number of vanishing moment for the wavelet is same as the number of approximation order for the scaling function. This is however not the case in frame theory. The vanishing moment of the resulting wavelets not only depends on the scaling function but also on the way the matrix was completed. This scheme is what is called a lifting scheme in wavelet theory. It creates a pair of dual wavelet frames from a wavelet frame. It increases the length of the filters but the support remains compact. Let $P(\mathbf{z})$ be the polyphase matrix of a MRA based frame system obtained from the unitary extension principle. Let $L(\mathbf{z})$ be the following matrix.

$$L(\mathbf{z}) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ L_1(\mathbf{z}) & 1 & 0 & \cdots & 0 \\ L_2(\mathbf{z}) & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ L_r(\mathbf{z}) & 0 & 0 & \cdots & 1 \end{bmatrix},$$

where $L_l(\mathbf{z}),\ l = 1, 2, \ldots r$ are Laurant polynomials. A new polyphase matrix for a new wavelet frame can be constructed as $P^{new}(\mathbf{z}) := L(\mathbf{z})P(\mathbf{z})$. On the dual side, let $\tilde{P}^{new}(\mathbf{z}) := \tilde{L}(\mathbf{z})P(\mathbf{z})$ where

$$\tilde{L}(\mathbf{z}) = \begin{bmatrix} 1 & -L_1(\mathbf{z^{-1}}) & -L_2(\mathbf{z^{-1}}) & \cdots & -L_r(\mathbf{z^{-1}}) \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Lemma 3.1 *The above constructed matrices satisfy* $\tilde{P}^{new*}P^{new} = I$.

proof: Since $\tilde{P}^{new*}P^{new} = P^*\tilde{L}^*LP = P^*IP = P^*P = I$. The middle terms is an identity because of their constructions. The lemma immediately provides the following theorem.

Theorem 3.1 *The above constructed polyphase matrices* $P^{new}(\mathbf{z})$ *and* $\tilde{P}^{new}(\mathbf{z})$ *satisfy the conditions of mixed extension principle. Thus the affine system obtained from these filters are dual frames.*

Proof: The new polyphase matrices satisfy all the criterion of the mixed extension principle because of lemma 3.1, hence the result.

It is clear that the resulting functions are all compactly supported. Suitable choice of the Laurant polynmials L_l improve the vanishing moments and the required condition $\hat{\phi}^{new}(0) = 1$.

Note: The the scaling function on the primal side doesn't change only the wavelet functions change, however on the dual side the scaling function changes but the wavelets filters change. This changes wavelets as well since $\tilde{\psi}^l(\boldsymbol{\xi}) = \tilde{H}^l(D^{-t}\boldsymbol{\xi})\hat{\psi}^0(D^{-t}\boldsymbol{\xi})$.

The new filters of wavelets $H_l^{new},\ l = 1, 2, \ldots, r$ in the primal side are given by

$$H_l^{new}(\boldsymbol{\xi}) = H^l(\boldsymbol{\xi}) + L_l(D^t\boldsymbol{\xi})\left(H^{0,0}(D^t\boldsymbol{\xi}) + H^{0,1}(D^t\boldsymbol{\xi}) + \cdots + H^{0,m-1}(D^t\boldsymbol{\xi})\right).$$

The dual scaling function changes to

$$\tilde{H}_0^{new}(\boldsymbol{\xi}) = \tilde{H}^0(\boldsymbol{\xi}) - L_1^*(D^t\boldsymbol{\xi})\tilde{H}^1(\boldsymbol{\xi}) - L_2^*(D^t\boldsymbol{\xi})\tilde{H}^2(\boldsymbol{\xi}) - \cdots - L_r^*(D^t\boldsymbol{\xi})\tilde{H}^r(\boldsymbol{\xi}).$$

Example: 1 The following one dimensional examples are taken from [7] where a given scaling function can give rise to more than one wavelet frame system depending on how the polyphase matrix is completed.

$$P(\xi) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \quad P'(\xi) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}, \quad \tilde{P}'(\xi) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

The polyphase matrix $P(\xi)$ generates wavelet system whose filters are given by $H^{(1)}(\xi) = H^{(2)}(\xi) = \frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}e^{-i\xi}$. It satisfies the vanishing moment VM_0 property [7] as $H^1(0) = H^2(0) = 0$. The wavelet filters obtained from $P'(\boldsymbol{\xi})$ and $P'(\boldsymbol{\xi})$ generate a pair of dual wavelet frames that do not satisfy VM_0 property. However this can be lifted to get desired vanishing moments. Let $L_1(\xi) = ae^{-i\xi}$ and $L_2(\xi) = be^{-i\xi}$ and let

$$L(\xi) = \begin{pmatrix} 1 & 0 & 0 \\ L_1(\xi) & 1 & 0 \\ L_2(\xi) & 0 & 1 \end{pmatrix}.$$

Thus the new polyphase matrix is obtained as:

$$P^{new}(\xi) = L(\xi)P'(\xi).$$

However this creates new scaling function on the dual side as

$$\widetilde{P}^{new}(\xi) = \tilde{L}(\xi)\widetilde{P}'(\xi),$$

where

$$\widetilde{L}(\xi) = \begin{pmatrix} 1 & -L_1(-\xi) & -L_2(-\xi) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We note that $L(z^*) = L(-\xi)$. Thus the wavelet filters on the primal side are given by the following.

$$H^1(\xi) = \frac{1}{2}(1 + ae^{-2i\xi} + ae^{-3i\xi}),$$

$$H^2(\xi) = \frac{1}{2}(e^{-i\xi} + be^{-2i\xi} + be^{-3i\xi}).$$

Also taking $a = -\dfrac{1}{2}$ and $b = -\dfrac{1}{2}$ they satisfy the VM_0 property. The new scaling function on the dual side, obtained from the first row of $\tilde{L}(\xi)\tilde{P}(\xi)$, is

$$\widetilde{H}^0(\xi) = \frac{1}{2}\left(1 - (a-b)e^{2i\xi} + e^{-i\xi} + (a-b)e^{i\xi}\right).$$

Note: Since the resulting functions are all compactly supported, they satisfy all the conditions of unitary extension principle since $\widetilde{H}^0(0) = 1$.

3.2 Frames from Modulation matrix

Authors in [4, 3] have used matrices like (2.5) to construct a pair of orthogonal wavelet frames. Their method can be used to obtain symmetric/antysymmetric filter coefficients for the wavelet frames. Let ψ^0, ψ^1 be a pair of scaling function and wavelet with filters $H^0(\boldsymbol{\xi})$ and $H^1(\boldsymbol{\xi})$. Let $M(\boldsymbol{\xi})$ be the modulation matrix. Let $A = \{a_{i,j}\}$ be any $r \times r$ paraunitary matrix with $2\pi/m$ periodic trigonometric polynomial entries. Let A_i be the i^{th} columns of A and let

$$M^{new}(\boldsymbol{\xi}) = \; A_i(\boldsymbol{\xi})M(\boldsymbol{\xi}) = \begin{pmatrix} a_{1,i}(\boldsymbol{\xi}) \\ a_{2,i}(\boldsymbol{\xi}) \\ \vdots \\ a_{r,i}(\boldsymbol{\xi}) \end{pmatrix} M(\boldsymbol{\xi}) \; .$$

Let $H^l(\boldsymbol{\xi}) = a_{l,i}(\boldsymbol{\xi})H^1(\boldsymbol{\xi}),\;\; l = 1, 2, \ldots, r$. So the following theorem follows.

Theorem 3.2 *The filters obtained above satisfy all the conditions of unitary extension principle. So the affine systems generated by ψ^l, $l = 1, 2, \cdots r$, provides tight wavelet frames for $L_2(\mathbb{R})$, where*

$$\widehat{\psi}^l(\boldsymbol{\xi}) = H^l(D^{-t}\boldsymbol{\xi})\widehat{\psi}^0(D^{-t}\boldsymbol{\xi}),\;\; l = 1, 2, \ldots, r.$$

Proof: Since $M^{new*}(\boldsymbol{\xi})M^{new}(\boldsymbol{\xi}) = I$, it satisfies all conditions of UEP. The order of vanishing moment of wavelet frames obtained above equals the vanishing moment of the used wavelet system.

Example 2: Let us consider a well known Haar wavelet system in dimension one. The modulation matrix is given by

$$M(z) = \begin{pmatrix} \dfrac{1+z}{2} & \dfrac{1-z}{2} \\ \dfrac{1-z}{2} & \dfrac{1+z}{2} \end{pmatrix},$$

where $z = e^{-i\xi}$. Taking the first column of an arbitrary paraunitary matrix obtained by (2.5) as

$$A(z) = \begin{pmatrix} \frac{5}{6} + \frac{z^2}{6} \\ -\frac{1}{3} + \frac{z^2}{3} \\ -\frac{1}{6} + \frac{z^2}{6} \end{pmatrix}.$$

Then the filters for the wavelets are given by

$$H^1(z) = \left(\frac{5}{6} + \frac{z^2}{6}\right)\frac{1-z}{2} = \frac{1}{12}\left(5 - 5z + z^2 - z^3\right),$$

$$H^2(z) = \left(-\frac{1}{3} + \frac{z^2}{3}\right)\frac{1-z}{2} = \frac{1}{6}\left(-1 + z - z^2 - z^3\right),$$

$$H^3(z) = \left(-\frac{1}{6} + \frac{z^2}{6}\right)\frac{1-z}{2} = \frac{1}{12}\left(-1 + z - z^2 - z^3\right).$$

This scheme can be carried to higher dimensions in a very straight forward way. One can take the symmetric paraunitary matrix as in [11].

Example 3: Using the following symmetric paraunitary matrix from [11], with same $M(z)$ as above,

$$A(z) = \begin{pmatrix} \frac{z^{-1}+z}{2} \\ -\frac{z^{-1}-z}{2} \end{pmatrix}.$$

the wavelet filters come out to be

$$H^1(z) = \left(\frac{z^{-1}+z}{2}\right)\frac{1-z}{2} \quad = \quad \frac{1}{4}(z^{-1} - 1 + z - z^2),$$

$$H^2(z) = \left(-\frac{z^{-1}-z}{2}\right)\frac{1-z}{2} \quad = \quad \frac{1}{4}(-z^{-1} + 1 + z - z^2).$$

Notice that the first filter is anti-symmetric and the second one is symmetric.

3.3 Frames from Polyphase Matrix

Let P denote the polyphase matrix of lowpass highpass filters of an MRA based wavelet system. We assume that the entries of P are trigonometric polynomials.

Let $A = \{a_{i,j}\}$ be any $r \times r$ paraunitary matrix. Let P_1 be following matrix.

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}.$$

Let $P^{new} = P_1 P$. Thus

$$P^{new}(\boldsymbol{\xi}) = \begin{pmatrix} H^{0,0}(\boldsymbol{\xi}) & H^{0,1}(\boldsymbol{\xi}) & \cdots & H^{0,m-1}(\boldsymbol{\xi}) \\ \sum_{i=1}^{r} a_{1,i}(\boldsymbol{\xi})H^{i,0}(\boldsymbol{\xi}) & \sum_{i=1}^{r} a_{1,i}(\boldsymbol{\xi})H^{i,1}(\boldsymbol{\xi}) & \cdots & \sum_{i=1}^{r} a_{1,i}(\boldsymbol{\xi})H^{i,m-1}(\boldsymbol{\xi}) \\ \sum_{i=1}^{r} a_{2,i}(\boldsymbol{\xi})H^{i,0}(\boldsymbol{\xi}) & \sum_{i=1}^{r} a_{2,i}(\boldsymbol{\xi})H^{i,1}(\boldsymbol{\xi}) & \cdots & \sum_{i=1}^{r} a_{2,i}(\boldsymbol{\xi})H^{i,m-1}(\boldsymbol{\xi}) \\ \vdots & \vdots & \cdots & \\ \sum_{i=1}^{r} a_{r,i}(\boldsymbol{\xi})H^{i,0}(\boldsymbol{\xi}) & \sum_{i=1}^{r} a_{r,i}(\boldsymbol{\xi})H^{i,1}(\boldsymbol{\xi}) & \cdots & \sum_{i=1}^{r} a_{r,i}(\boldsymbol{\xi})H^{i,m-1}(\boldsymbol{\xi}) \end{pmatrix}.$$

This provides a polyphase matrix for new wavelet frames. Using 2.3 or 2.4 the new filters H_l^{new}, $l = 1, 2, \ldots, r$ are given by,

$$\begin{aligned} H_l^{new}(\boldsymbol{\xi}) &= \frac{1}{\sqrt{m}} \sum_{j=0}^{m-1} [P^{new}(D^*\boldsymbol{\xi})]_{l,j}\, e^{-i\,\mathbf{d}_j\cdot\boldsymbol{\xi}} \\ &= \frac{1}{\sqrt{m}} \sum_{j=0}^{m-1} \sum_{i=1}^{r} a_{l,i}(D^*\boldsymbol{\xi}) H^{i,j}(D^*\boldsymbol{\xi})\, e^{-i\,\mathbf{d}_j\cdot\boldsymbol{\xi}} \\ &= \sum_{i=1}^{r} a_{l,i}(D^*\boldsymbol{\xi}) \left(\frac{1}{\sqrt{m}} \sum_{j=0}^{m-1} H^{i,j}(D^*\boldsymbol{\xi})\, e^{-i\,\mathbf{d}_j\cdot\boldsymbol{\xi}} \right) \\ &= \sum_{i=1}^{r} a_{l,i}(D^*\boldsymbol{\xi}) H^i(\boldsymbol{\xi}). \end{aligned} \tag{3.1}$$

So we have the following theorem:

Theorem 3.3 *Let $H_l^{new}(\boldsymbol{\xi})$ be as given as above and let*

$$\widehat{\psi}_l^{new}(\boldsymbol{\xi}) = H_l^{new}(D^{-t}\boldsymbol{\xi})\widehat{\psi}^0(D^{-t}\boldsymbol{\xi}),$$

for $l = 1, 2, \ldots, r$. Then the affine system generated by ψ_l^{new}, $l = 1, 2, \ldots, r$ forms a tight wavelet frames for $L_2(\mathbb{R}^d)$.

Proof: Since $P^{new}(\boldsymbol{\xi})P^{new}(\boldsymbol{\xi}) = P^* P_1^* P_1 P = I$, the unitary extension principle applies to the matrices $P^{new}(\boldsymbol{\xi})$, hence the theorem follows. The paraunitary matrix doesn't have to be $2\pi/m\mathbb{Z}^d$ periodic as required in theorem 3.2.
Note: The vanishing moments of the resulting frames are preserved from the vanishing moments of the wavelet, the resulting wavelet frame is symmetryc/antysymmetric

if paraunitary symmetric matrix is used and symetric/antysymmetric wavelet is used.

Example 4: Consider the following polyphase matrix for the frame taken from [7] where the polyphase matrix $P(\xi)$ generates wavelet system whose filters are given by $H^1(z) = H^2(z) = \frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}z$. Consider the following paraunitary matrix (2.5) with some chosen $\mathbf{v}$.

$$A(z_1, z_2) = \frac{1}{5}\begin{pmatrix} 4+z_1 & -2+2z_1 \\ -2+2z_1 & 1+4z_1 \end{pmatrix}.$$

Using the dilation matrix

$$D = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

$\mathbf{z}^D = \left((z_1, z_2)^{(1,1)}, (z_1, z_2)^{(1,-1)}\right) = (z_1 z_2, z_1 z_2^{-1})$. So

$$A(\mathbf{z}^D) = \frac{1}{5}\begin{pmatrix} 4+z_1z_2 & -2+2z_1z_2 \\ -2+2z_1z_2 & 1+4z_1z_2 \end{pmatrix}.$$

Therefore the new filters for the wavelet frames are given by

$$\begin{aligned} H_1^{new}(z_1, z_2) &= A(\mathbf{z}^D)_{[1,1]}H^1(z) + A(\mathbf{z}^D)_{[1,2]}H^2(z) \\ &= \frac{1}{10\sqrt{2}}\left(2 - 2z_1 + 3z_1z_2 - 3z_1^2z_2\right), \\ H_2^{new}(z_1, z_2) &= A(\mathbf{z}^D)_{[2,1]}H^1(z) + A(\mathbf{z}^D)_{[2,2]}H^2(z) \\ &= \frac{1}{10\sqrt{2}}\left(-1 + z_1 + 6z_1z_2 - 6z_1^2z_2\right), \end{aligned}$$

where the variable z in $H^1(z)$ and $H^2(z)$ has been replaced by z_1. These are 2-D wavelets frames. One can start from 2-D polyphase matrix as well.

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