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Existence Results for Nonlinear Problems in $\mathbb{R}^N$ Involving the p-Laplacian

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Abstract

We establish an existence theorem for a problem of the type

$$\begin{cases} -\Delta_p u + \lambda(x)|u|^{p-2}u = f(x,u) + g(x,u) \quad \text{in} \quad \mathbb{R}^N \\ \\ \lim\limits_{|x|\to+\infty} u(x) = 0 \end{cases}$$

where $f, g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are Carthèodory functions, $\lambda \in L^\infty(\Omega)$, with $\operatorname{ess\,inf}_\Omega \lambda > 0$, and Δ_p is the p-Laplacian operator with $p > N$. This result extends to the case of $\mathbb{R}^N$ a previous result related to a Neumann problem in bounded domains.

1 Introduction

Let $p > 1$ and let $\Delta_p := \operatorname{div}(|\nabla(\cdot)|^{p-2}\nabla(\cdot))$ be the well known p-Laplacian operator.

Let $\Omega \subset \mathbb{R}^N$ be a nonempty bounded open set with regular boundary $\partial\Omega$ whose outward unit normal is denoted by ν, let $f, g : \mathbb{R} \to \mathbb{R}$ be two continuous functions and, finally, let $\alpha, \beta \in L^1(\Omega)$ and $\lambda \in L^\infty(\Omega)$ with $\operatorname{ess\,inf}_\Omega \lambda > 0$.

By using a variational argument based on a general existence theorem of critical points established by Ricceri (see [4]), in [2] the authors proved an existence result for the following Neumann problem

$$\begin{cases} -\Delta_p u + \lambda(x)|u|^{p-2}u = \alpha(x)f(u) + \beta(x)g(u) & \text{in} \quad \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega \end{cases} \qquad (P_\Omega)$$

by imposing, besides a sublinear growth condition on $\int_0^\xi g(t)dt$, that $p > N$ and that, for some interval $[-r, r]$ $(r > 0)$, the maximum of $F(\xi) = \int_0^\xi f(t)dt$ in $[-r, r]$ is attained at a point $\xi_0 \in [-r, r]$ suitably near to 0. In the proof of this result the authors exploit the fact that, because of the condition $p > N$, the space $W^{1,p}(\Omega)$, where the (weak) solutions are looked for, is compactly embedded in $C^0(\overline{\Omega})$ and the fact that the constant functions belong to $W^{1,p}(\Omega)$. This, jointly to the aforementioned the condition on f, allows to prove that the functional

$$u \in W^{1,p}(\Omega) \to \int_\Omega \left(\alpha(x) \int_0^{u(x)} f(t)dt \right) dx$$

attains its global minimum in a certain ball just at a constant function belonging to the interior of this ball. This circumstance makes the result of Ricceri (quoted above) successfully applicable in order to find solutions in $W^{1,p}(\Omega)$ for problem (P_Ω).

The aim of the present paper is to study the same problem (where we consider a more general right hand-side) in $\mathbb{R}^N$. We continue to keep the condition $p > N$ in order to have the embedding of $W^{1,p}(\mathbb{R}^N)$ in $C_0^0(\mathbb{R}^N)$ (the space of the continuous functions $u : \mathbb{R}^N \to \mathbb{R}$ such that $\lim_{|x|\to\infty} u(x) = 0$), equipped with the sup-norm $\|u\|_\infty = \sup_{\mathbb{R}^N} |u|$) which, however, is not compact in this case. Furthermore, we stress out that, unlike the bounded domain case, the constant functions do not belong to $W^{1,p}(\mathbb{R}^N)$. Nevertheless, we will show that, keeping substantially the same assumptions of the bounded domain case studied in [2], it is possible to establish the existence of a weak solution in $W^{1,p}(\mathbb{R}^N)$.

Our prove is again based on the results of [4].

2 The Result

We start introducing some notations and basic definitions and results. From now on we always assume $p > N$. Under this assumption, as already mentioned in the introduction, it is known that the space $W^{1,p}(\mathbb{R}^N)$ is continuously embedded in the space

$$C_0^0(\mathbb{R}^N) := \{u \in C^0(\mathbb{R}^N) : \lim_{|x|\to\infty} u(x) = 0\},$$

while, for a bounded domain $\Omega \subset \mathbb{R}^N$, the compact embedding of $W^{1,p}(\Omega)$ in $C^0(\overline{\Omega})$ holds (these basic results can be found in [3] for instance).

Let $\lambda \in L^\infty(\mathbb{R}^N)$ with $\operatorname{ess\,inf}_{\mathbb{R}^N} \lambda > 0$.

In what follows, the space $W^{1,p}(\mathbb{R}^N)$ will be equipped with the following norm

$$\|\cdot\| := \left(\int_{\mathbb{R}^N} (|\nabla(\cdot)|^p + \lambda(x)|\cdot|^p)dx\right)^{\frac{1}{p}}$$

which is equivalent to the usual one. As it is known, the function

$$u \in W^{1,p}(\mathbb{R}^N) \to \frac{1}{p}\|u\|^p$$

is weakly lower semicontinuous and Gâteaux differentiable in $W^{1,p}(\mathbb{R}^N)$ and its Gâteaux derivative at $u \in W^{1,p}(\mathbb{R}^N)$ is the following linear operator

$$v \in W^{1,p}(\mathbb{R}^N) \to \int_\Omega \left(|\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x) + \lambda(x)|u(x)|^{p-2}u(x)v(x)\right) dx.$$

We denote by C_λ the best embedding constant of $W^{1,p}(\mathbb{R}^N)$ in $C_0^0(\mathbb{R}^N)$, that is

$$C_\lambda = \sup_{u\in W^{1,p}(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|_\infty}{\|u\|}. \tag{1}$$

Let us now introduce the following summability condition for a Carathèodory function $h : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$

$H)$ $\quad \sup_{|t|\leq s} |h(\cdot, t)| \in L^1(\mathbb{R}^N)$ for all $s > 0$.

We denote by $\mathcal{M}$ the set of all Carathèodory functions $h : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfying condition $H)$.

Moreover, for every $R \geq 0$, let us denote by B_R the closed ball in $\mathbb{R}^N$ centered at 0 with radius R if $R > 0$, the empty set if $R = 0$.

Finally, for every $h \in \mathcal{M}$, we define the following functionals

$$\begin{aligned} J_h(u) &= \int_{\mathbb{R}^N} \left(\int_0^{u(x)} h(x,t)dt\right) dx. \\ \Psi_h(u) &= \tfrac{1}{p}\|u\|^p - J_h(u) \end{aligned} \qquad u \in W^{1,p}(\mathbb{R}^N). \tag{2}$$

Lemma 1 *Let $h \in \mathcal{M}$. Then, the functional J_h is sequentially weakly continuous and Gâteaux differentiable in $W^{1,p}(\mathbb{R}^N)$.*

Proof By Corollary 41.9 of [5] it is suffice to show that J_h is Gâteaux differentiable in $W^{1,p}(\mathbb{R}^N)$ and that $J_h' : W^{1,p}(\mathbb{R}^N) \to (W^{1,p}(\mathbb{R}^N))'$ is compact. Let $u, v \in W^{1,p}(\mathbb{R}^N)$ and let $\tau \in]0,1]$. One has

$$\frac{J_h(u+\tau v) - J_h(u)}{\tau} = \int_{\mathbb{R}^N} \left(\frac{1}{\tau} \int_{u(x)}^{u(x)+\tau v(x)} h(x,t)dt \right) dx.$$

Because of the embedding of $W^{1,p}(\mathbb{R}^N)$ in $C_0^0(\mathbb{R}^N)$ we can find a constant $s_0 > 0$ such that $|u(x)| + \tau|v(x)| \le s_0$ for all $x \in \mathbb{R}^N$ and $\tau \in [0,1]$. Then we have

$$\left| \frac{1}{\tau} \int_{u(x)}^{u(x)+\tau v(x)} h(x,t)dt \right| \le \frac{1}{\tau} \left| \int_{u(x)}^{u(x)+\tau v(x)} |h(x,t)|dt \right| \le$$
$$\frac{1}{\tau} \left| \int_{u(x)}^{u(x)+\tau v(x)} \sup_{|\sigma| \le s_0} |h(x,\sigma)|dt \right| = \sup_{|\sigma| \le s_0} |h(x,\sigma)||v(x)|$$

for all $x \in \mathbb{R}^N$, where $\sup_{|\sigma|\le s_0} |h(\cdot,\sigma)||v(\cdot)| \in L^1(\mathbb{R}^N)$.

Thus, noticing that

$$\frac{1}{\tau} \int_{u(x)}^{u(x)+\tau v(x)} h(x,t)dt \to h(x,u(x))v(x)$$

for all $x \in \mathbb{R}^N$, we infer, by the Dominate Convergence Theorem, that

$$\lim_{\tau \to 0^+} \frac{J_h(u+\tau v) - J_h(u)}{\tau} = \int_{\mathbb{R}^N} h(x,u(x))v(x)dx = J_h'(u)(v).$$

Let us now prove that J_h' is compact. To this end, we consider a bounded sequence u_n in $W^{1,p}(\mathbb{R}^N)$ and we prove that there exists a subsequence u_{n_k} such that $J_h'(u_{n_k})$ converges. Since $W^{1,p}(\mathbb{R}^N)$ is a reflexive space, up to a subsequence, we can suppose u_n weakly convergent to some $u \in W^{1,p}(\mathbb{R}^N)$. Let us to show that $J_h'(u_n)$ strongly converges to $J_h'(u)$. Fix a sequence $\{\varepsilon_k\}$ of positive real numbers converging to zero. By the embedding of $W^{1,p}(\mathbb{R}^N)$ in $C_0^0(\mathbb{R}^N)$ and the assumption on h, we can find $s_1 > 0$ and, for every $k \in \mathbb{N}$, $M_k > 0$ such that

$$\sup_{x \in \mathbb{R}^N} \left(\sup_{n \in \mathbb{N}} |u_n(x)| + |u(x)| \right) \le s_1$$

and

$$\int_{\mathbb{R}^N \setminus B_{M_k}} \sup_{|t| \le s_1} |h(x,t)|dx < \frac{\varepsilon_k}{4C_\lambda},$$

where C_λ denotes the best embedding constant of $W^{1,p}(\mathbb{R}^N)$ in $C_0^0(\mathbb{R}^N)$ introduced in (1). Using the compact embedding of $W^{1,p}(B_{M_k})$ in $C^0(B_{M_k})$, we infer that (up to a subsequence) $u_{n|B_{M_k}} \to u_{|B_{M_k}}$ uniformly in B_{M_k}. Thus, taking into account that $\sup_{n\in\mathbb{N}}|h(x,u_n(x))| \le \sup_{|t|\le s_1}|h(x,t)|$ with $\sup_{|t|\le s_1}|h(\cdot,t)| \in L^1(B_{M_k})$, by the Dominate Convergence Theorem, there exists a strictly increasing sequence $\{n_k\}$ of positive integers such that

$$\int_{B_{M_k}} |h(x,u_{n_k}(x)) - h(x,u(x))|dx < \frac{\varepsilon_k}{2C_\lambda} \tag{3}$$

for all $k \in \mathbb{N}$.

At this point, by (3) and the choice of M_k we have

$$\begin{aligned}
&|J_h'(u_{n_k})(v) - J_h'(u)(v)| \le \\
&\int_{B_{M_k}} |h(x,u_{n_k}(x)) - h(x,u(x))||v(x)|dx + 2\int_{\mathbb{R}^N\setminus B_{M_k}} \sup_{|t|\le s_1}|h(x,t)||v(x)|dx \\
&\le C_\lambda\left(\int_{B_{M_k}} |h(x,u_{n_k}(x)) - h(x,u(x))|dx + 2\int_{\mathbb{R}^N\setminus B_{M_k}} \sup_{|t|\le s_1}|h(x,t)|dx\right) < \\
&\varepsilon_k,
\end{aligned}$$

for all $v \in W^{1,p}(\mathbb{R}^N)$ with $\|v\| \le 1$ and $k \in \mathbb{N}$. Therefore,

$$\|J_h'(u_{n_k}) - J_h'(u)\|_{(W^{1,p}(\mathbb{R}^N))'} = \sup_{\|v\|\le 1}|J_h'(u_{n_k})(v) - J_h'(u)(v)| < \varepsilon_k$$

for all $k \in \mathbb{N}$. This concludes the proof. □

Remark 1 Note that, thanks to Lemma 1, if $h \in \mathcal{M}$, we have that the functional Ψ_h is sequentially weakly lower semicontinuous and Gâteaux differentiable in $W^{1,p}(\mathbb{R}^N)$.

Lemma 2 *Let $h \in \mathcal{M}$. Assume that*

$$\limsup_{|\xi|\to+\infty} \frac{\int_0^\xi h(x,t)dt}{|\xi|^p} < \frac{\operatorname{ess\,inf}_{\mathbb{R}^N}\lambda}{p}. \tag{4}$$

Then, the functional Ψ_h is bounded below and

$$\lim_{\substack{\|u\|\to+\infty \\ u\in W^{1,p}(\mathbb{R}^N)}} \Psi_h(u) = +\infty$$

Proof By (4) we can fix $\varepsilon > 0$ with $\varepsilon < \dfrac{\operatorname{ess\,inf}_{\mathbb{R}_N}\lambda}{p} - \limsup\limits_{|\xi|\to+\infty}\dfrac{\int_0^\xi h(x,t)dt}{|\xi|^p}$ and $\eta > 0$ such that $\dfrac{\int_0^\xi h(x,t)dt}{|\xi|^p} < \dfrac{\operatorname{ess\,inf}_{\mathbb{R}_N}\lambda}{p} - \varepsilon$ for every $\xi \in \mathbb{R}\setminus[-\eta,\eta]$. Now, let $u \in W^{1,p}(\mathbb{R}^N)$. We have the following estimate

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(\int_0^{u(x)} h(x,t)dt \right) dx = \\
&\int_{|u(x)| \leq \eta} \left(\int_0^{u(x)} h(x,t)dt \right) dx + \int_{|u(x)| > \eta} \left(\int_0^{u(x)} h(x,t)dt \right) dx \leq \\
&\int_{\mathbb{R}^N} \left(\int_0^{\eta} \sup_{|t| \leq \eta} |h(x,t)|dt \right) dx + \int_{\mathbb{R}^N} \left(\frac{\operatorname{ess\,inf}_{\mathbb{R}_N} \lambda}{p} - \varepsilon \right) |u(x)|dx \leq \\
&c_\eta + \left(\frac{\operatorname{ess\,inf}_{\mathbb{R}_N} \lambda}{p} - \varepsilon \right) \int_{\mathbb{R}^N} \frac{\lambda(x)}{\operatorname{ess\,inf}_{\mathbb{R}_\mathrm{N}} \lambda} |u(x)|^p dx \leq \\
&c_\eta + \left(\frac{1}{p} - \frac{\varepsilon}{\operatorname{ess\,inf}_{\mathbb{R}_N} \lambda} \right) \|u\|^p,
\end{aligned}$$

where $c_\eta = \int_{\mathbb{R}^N} \left(\int_0^\eta \sup_{|t| \leq \eta} |h(x,t)|dt \right) dx$. Consequently,

$$\frac{1}{p}\|u\|^p - \int_{\mathbb{R}^N} \left(\int_0^{u(x)} h(x,t)dt \right) dx \geq \frac{\varepsilon}{\operatorname{ess\,inf}_{\mathbb{R}_N} \lambda} \|u\|^p - c_\eta$$

from which the thesis easily follows. □

In view of Lemma 2, for every $h \in \mathcal{M}$ satisfying (4) and every $s > \inf_{W^{1,p}(\mathbb{R}^N)} \Psi_h(u)$ we can define the following (non-empty) set

$$A(s,h) \stackrel{def}{=} \{\sigma > 0 : \Psi_h^{-1}(]-\infty, s[) \subseteq \mathcal{B}_{\frac{\sigma}{C_\lambda}}(0)\}$$

where, for $r > 0$, $\mathcal{B}_r(0)$ denotes the closed ball in $W^{1,p}(\mathbb{R}^N)$ centered at 0 with radius r and C_λ is the constant defined in (1).

For every fixed nonnegative $\alpha \in L^1(\mathbb{R}^N)$ and $\tau > 0$, we also define the following set

$$\Lambda(\tau,\alpha) \stackrel{def}{=} \{h \in \mathcal{M} : \sup_{|\xi| \leq \tau} \left| \int_0^\xi h(x,t)dt \right| \leq \alpha(x) \text{ for almost every } x \in \mathbb{R}^N\}$$

We are now in position to state our main result.

Let $f, g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be two Carathèodory functions and let us consider the following problem in $\mathbb{R}^N$

$$\begin{cases} -\Delta_p u + \lambda(x)|u|^{p-2}u = f(x,u) + g(x,u) & \text{in} \quad \mathbb{R}^N \\ \lim_{|x| \to +\infty} u(x) = 0 \end{cases} \qquad (P)$$

As it is known, a *weak solution* of problem (P) is any $u \in W^{1,p}(\mathbb{R}^N)$ satisfying the equation

$$\int_{\mathbb{R}^N} |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x) + \int_{\mathbb{R}^N} \left[\lambda(x)|u(x)|^{p-2}u(x) - f(x,u(x)) - g(x,u(x))\right] v(x)dx = 0$$

for all $v \in W^{1,p}(\mathbb{R}^N)$. Therefore, the weak solutions of problem (P) are exactly the critical points of the functional $\Psi_g - J_f$.

Theorem 1 *Assume $g \in \mathcal{M}$ satisfying condition* (4) *of Lemma 2. Let $s > 0$ and $r \geq A(s,g)$. Moreover, let $\alpha \in L^1(\mathbb{R}^N)$ be nonnegative. Then, there exists $r_0 \in]0,r]$ such that for every $\xi_0 \in]-r_0, r_0[$ and every $f \in \Lambda(r,\alpha)$ satisfying*

$$I_{\xi_0}) \int_0^{\xi_0} f(x,t)dt = \sup_{|\xi|\leq r} \int_0^{\xi} f(x,t)dt \text{ for almost every } x \in \mathbb{R}^N,$$

problem (P) admits at least one weak solution $u \in W^{1,p}(\mathbb{R}^N)$ satisfying $\Psi_g(u) < s$.

Proof At first, we fix $R \geq 0$ such that

$$2\int_{\mathbb{R}^N\setminus B_R} \alpha(x)dx < s. \tag{5}$$

After that, for every $\xi \in \mathbb{R}$, let us define

$$u_\xi(x) = \begin{cases} \xi & \text{if} \quad x \in B_R \\ \xi e^{-(|x|-R)} & \text{if} \quad x \in \mathbb{R}^N \setminus B_R. \end{cases}$$

Then, $u_\xi \in W^{1,p}(\mathbb{R}^N)$ and

$$\|u_\xi\|^p = |\xi|^p \int_{\mathbb{R}^N\setminus B_R} (1+\lambda(x))e^{-p(|x|-R)}dx + |\xi|^p \int_{B_R} \lambda(x)dx.$$

Hence, taking (5) into account and noticing that $\int_{\mathbb{R}^N} \left(\int_0^{u_\xi(x)} g(x,t)\right) dx \to 0$ as $|\xi| \to 0^+$, an easy computation show that there exists $r_0 > 0$ such that

$$s > \Psi(u_\xi) + 2\int_{\mathbb{R}^N\setminus B_R} \alpha(x)dx \tag{6}$$

for all $\xi \in]-r_0, r_0[$.

Note that, since α is nonnegative, by (6) we have $\Psi(u_{r_0}) < s$ and so, taking into account that $r \geq A(s,g)$, one has $r_0 = \|u_{r_0}\|_\infty \leq C_\lambda \|u_{r_0}\| \leq r$.

At this point, let $\xi_0 \in [-r_0, r_0]$ and let $f \in \Lambda(s, \alpha)$ satisfying condition I_{ξ_0}).

If we show that

$$\rho \stackrel{def}{=} \inf_{u \in \Psi_g^{-1}(]-\infty, s[)} \frac{\sup_{\mathcal{B}_{\frac{r}{C_\lambda}}(0)} J_f - J_f(u)}{s - \Psi_g(u)} < 1 \tag{7}$$

then, in view of Lemma 1 and Lemma 2, we can apply Theorem 2.1 of [4], to the functionals Ψ_g and $-J_f$. Thus, the functional $\Psi_g - J_f$ attains its global minimum on the weak closure of $\Psi_g^{-1}(]-\infty, s[)$ at a point u belonging to $\Psi_g^{-1}(]-\infty, s[)$. Therefore, u is a critical point of $\Psi_g - J_f$ and so u is a weak solution for problem (P) as well.

Thus, to complete the proof, we only have to show that inequality (7) holds.

We again observe that by (6) and taking into account that α is nonnegative, we have $s > \Psi_g(u_{\xi_0})$. It follows

$$\rho < \frac{\sup_{\mathcal{B}_{\frac{r}{C_\lambda}}(0)} J_f - J_f(u_{\xi_0})}{s - \Psi_g(u_{\xi_0})}. \tag{8}$$

Moreover, using the embedding of $W^{1,p}(\mathbb{R}^N)$ in $C_0^0(\mathbb{R}^N)$ and condition I_{ξ_0}, we get

$$\sup_{\mathcal{B}_{\frac{r}{C_\lambda}}(0)} J_f - J_f(u_{\xi_0}) \leq \int_{\mathbb{R}^N} \sup_{|\xi| \leq r} \left(\int_0^\xi f(x,t)dt \right) dx - J_f(u_{\xi_0}) \leq$$
$$\int_{B_R} \left(\sup_{|\xi| \leq r} \int_0^\xi f(x,t)dt - \int_0^{\xi_0} f(x,t)dt \right) dx +$$
$$\int_{\mathbb{R}^N \setminus B_R} \left(\sup_{|\xi| \leq r} \int_0^\xi f(x,t)dt - \int_0^{\xi_0 e^{-(|x|-R)}} f(x,t)dt \right) dx \leq$$
$$2 \int_{\mathbb{R}^N \setminus B_R} \left| \sup_{|\xi| \leq r} \int_0^\xi f(x,t)dt \right| dx \leq 2 \int_{\mathbb{R}^N \setminus B_R} \alpha(x) dx$$

Then, (7) easily follows from (6) and (8). □

Remark 2 As we have said, Theorem 1 is the version of Theorem 2.1 of [2] to the case in which the domain is $\mathbb{R}^N$. Our assumptions are the same of Theorem 2.1 of [2] with the further condition that the nonlinearity f must belong to the set $\Lambda(r, \alpha)$ for a fixed nonnegative $\alpha \in L^1(\mathbb{R}^N)$. This condition needs in order to the constant r_0 be independent of f (as in Theorem 2.1 of [2]). Actually, by the proof of Theorem 1, we see that the condition $f \in \Lambda(r, \alpha)$ can be weakened in the sense that , fixed $R_0, r > 0$ and $\alpha \in L^1(\mathbb{R}^N)$, it is suffice to assume that $\sup_{|t| \leq r} \left| \int_0^\xi f(x,t)dt \right| \leq \alpha(x)$ for almost every $x \in \mathbb{R}^N \setminus B_{R_0}$.

Remark 3 Since the exact evaluation of the constant C_λ in (1) is not known in general, the number $A(s, h)$ could be hard to calculate. Nevertheless, it is easy to check that in the definition of the number $A(s, h)$, the constant C_λ can be replaced by any constant greater than C_λ.

We now state a Corollary of Theorem 1 for the ordinary case where we also provide a computation of the number r_0. To get this further information we need the exact value of C_λ or an upper estimate of this latter (Remark 3). We first find the exact embedding constant C_λ in the case $N = 1$ and $\lambda \equiv 1$ which we denote by C_1. Then, it is easy to see that, for any λ, one has $C_\lambda \leq \frac{1}{\min\{1, \operatorname{essinf}_{\mathbb{R}^N} \lambda\}} C_1$

Lemma 3 *If $N = 1$ and $\lambda \equiv 1$, then $C_1 = 2^{-\frac{1}{p}} (p-1)^{\frac{p-1}{p^2}}$.*

Proof The function $u_0(t) = e^{-(p-1)^{-\frac{1}{p}}|t|}$ belongs to $W^{1,p}(\mathbb{R})$ and, by elementary calculation, we have

$$C_1 \geq \frac{\|u_0\|_\infty}{\|u_0\|} = 2^{-\frac{1}{p}} (p-1)^{\frac{p-1}{p^2}}. \tag{9}$$

Now, let a, b be positive numbers and put $\rho = \frac{a}{b}$. Then, noticing that

$$\inf_{\xi > 0} (\xi + \xi^{1-p}) = p(p-1)^{\frac{1}{p}-1},$$

we have

$$\frac{a^{p-1}b}{a^p + b^p} = \frac{1}{t + t^{1-p}} \leq \frac{1}{p}(p-1)^{1-\frac{1}{p}}$$

from which

$$a^{p-1}b \leq \frac{1}{p}(p-1)^{1-\frac{1}{p}}(a^p + b^p). \tag{10}$$

Of course, the previous inequality trivially holds if a or b is 0. At this point, let $u \in W^{1,p}(\mathbb{R}) \setminus \{0\}$ and fix $t \in \mathbb{R}$. Using (10), we get

$$|u(t)|^p = p \int_{-\infty}^{t} |u(s)|^{p-2} u(s) u'(s) ds \leq p \int_{-\infty}^{t} |u(s)|^{p-1} |u'(s)| ds \leq$$
$$(p-1)^{1-\frac{1}{p}} \int_{-\infty}^{t} (|u(s)|^p + |u(s)'|^p) ds$$

Likewise, we also have

$$|u(t)|^p \leq (p-1)^{1-\frac{1}{p}} \int_{t}^{+\infty} (|u(s)|^p + |u(s)'|^p) ds.$$

Hence

$$2|u(t)|^p \le \int_{\mathbb{R}} (|u(s)|^p + |u(s)'|^p)ds = (p-1)^{1-\frac{1}{p}} \|u\|^p.$$

that is

$$\frac{|u(t)|^p}{\|u\|^p} \le 2(p-1)^{1-\frac{1}{p}}$$

Therefore, since t and u are arbitrarily chosen, one has

$$C_1 = \sup_{u \in W^{1,p}(\mathbb{R}) \setminus \{0\}} \frac{\|u\|_\infty}{\|u\|} \le 2^{-\frac{1}{p}}(p-1)^{\frac{p-1}{p^2}}.$$

From the previous inequality and (9) we have the conclusion.

Corollary 1 *Let p, M, q, r be positive numbers with $p > 1$ and set*

$$r_0 = \left[\sup_{R \ge 0} \frac{\frac{1}{p}\left(\frac{r}{C_1}\right)^p - \frac{4M}{q}(R+1)^{-q}}{2R + \frac{4}{p}} \right]^{\frac{1}{p}}.$$

Then, for every continuous function $h : \mathbb{R} \to \mathbb{R}$ satisfying

a) $\sup_{|t| \le r} |h(t)| \le M;$

b) $\int_0^{\xi_0} h(t)dt = \sup_{|\xi| \le r} \int_0^{\xi} h(t)dt$ *for some* $\xi_0 \in [-r_0, r_0]$,

the problem

$$\begin{cases} -(|u'|^{p-2}u')' + |u|^{p-2}u = \frac{h(u)}{(1+|x|)^q} & \text{in} \quad \mathbb{R} \\ \\ \lim_{|x| \to +\infty} u(x) = 0 \end{cases}$$

admits at least a weak solution $u \in W^{1,p}(\mathbb{R})$ satisfying $\int_{\mathbb{R}} [(u')^p + (u)^p]dx \le r^p$.

Proof We apply Theorem 1 with $N = 1$, $s = \frac{1}{p}\left(\frac{r}{C_1}\right)^p$, $\lambda \equiv 1$, $g \equiv 0$, $\alpha(x) = \frac{M}{(1+|x|)^{q+1}}$ and $f(x,t) = \frac{h(t)}{(1+|x|)^{q+1}}$ for all $x \in \mathbb{R}$. Since $g \equiv 0$, it follows $r = A(\frac{1}{p}(\frac{r}{C_1})^p, 0) = A(s, 0)$. Moreover, note that, thanks to condition a), one has $f \in \Lambda(r, \alpha)$. So, by the proof of Theorem 1, we only have to show that the number r_0 (which is positive as it is easy to check) is such that for all $\xi \in [-r_0, r_0]$ inequality (6) is fulfilled for some $R_0 \ge 0$. Note that, in our case, for a fixed $R \geq 0$, an easy computation show that (6) has the following statement

$$\frac{r^p}{p} > \left(4\int_R^{+\infty} e^{-p(t-R)}dt + 2R\right)|\xi|^p + 4M\int_R^{+\infty}\frac{1}{(1+t)^{q+1}} = \left(\frac{4}{p} + 2R\right)|\xi|^p + \frac{4M}{q(1+R)^q}$$

for every $\xi \in]-r_0, r_0[$. Then, the existence of R_0 follows from the fact that supremum involved in the definition of r_0 is actually attained at some $R_0 \geq 0$. □

References

[1] R. A. Adams, *Sobolev Spaces*, Academic Press, Orlando, 1975;

[2] G. Anello, G. Cordaro, *Existence of soltions of the Neumann problem for a class of equations involving the p-Laplacian via a variational principle of Ricceri*, Arch. Math. **79**(2002), 274-287;

[3] H. Brezis, *Analyse Fonctionelle - Théorie et Applications*, Masson, (1983);

[4] B. Ricceri, *A general variational principle and some of its applications*, J. Comput. Appl. Math. **113** (2000), 401-410.

[5] E. Zeidler, *Nonlinear functional analysis and its applications*, vol III, Springer -Verlag, (1985).

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