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Some Properties for Subclass of Convex Functions with Respect to Symmetric Conjugate Points

Joyce Lim Chuen Shin and Aini Janteng

School of Science and Technology
Universiti Malaysia Sabah
Jalan UMS, 88400 Kota Kinabalu, Sabah, Malaysia
LCSJOYCE1104@yahoo.com.my, aini_jg@ums.edu.my

Abstract

This paper consider $C_{sc}(A, B)$ as a class of functions f which are analytic in an open unit disc $\mathcal{D} = \{z : |z| < 1\}$ and satisfying the condition $\frac{2(zf'(z))'}{(f(z)-\overline{f(-\bar{z})})'} \prec \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, $z \in \mathcal{D}$. We obtain some properties of functions $f \in C_{sc}(A, B)$ such as coefficient estimates, distortion theorem and integral operator.

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1 Introduction

Let $\mathcal{U}$ be the class of functions which are analytic in the open unit disc $\mathcal{D} = \{z : |z| < 1\}$ given by

$$w(z) = \sum_{k=1}^{\infty} b_k z^k$$

and satisfying the conditions

$$w(0) = 0, \ |w(z)| < 1, \ z \in \mathcal{D}.$$

Let $\mathcal{S}$ denote the class of functions f which are analytic and univalent in $\mathcal{D}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ z \in \mathcal{D}. \tag{1}$$

Also, let C_{sc} be the subclass of $\mathcal{S}$ consisting of functions given by (1) satisfying

$$\frac{2(zf'(z))'}{(f(z)-\overline{f(-\bar{z})})'} > 0, \quad z \in \mathcal{D}.$$

These functions are called convex with respect to symmetric conjugate points.

Further, let $f, g \in \mathcal{U}$. Then we say that f is subordinate to g, and we write $f \prec g$, if there exists a function $w \in \mathcal{U}$ such that $f(z) = g(w(z))$ for all $z \in \mathcal{D}$. Specially, if g is univalent in $\mathcal{D}$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathcal{D}) \subseteq g(\mathcal{D})$.

In terms of subordination, Goel and Mehrok in 1982 introduced a subclass of S_s^* denoted by $S_s^*(A,B)$. Let $S_s^*(A,B)$ denote the class of functions of the form (1) and satisfying the condition

$$\frac{2zf'(z)}{f(z)-f(-z)} \prec \frac{1+Az}{1+Bz}, \quad -1 \leq B < A \leq 1,\ z \in \mathcal{D}.$$

In this paper, let consider $C_{sc}(A,B)$ be the class of functions of the form (1) and satisfying the condition

$$\frac{2(zf'(z))'}{(f(z)-\overline{f(-\bar{z})})'} \prec \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1,\ z \in \mathcal{D}.$$

Obviously $C_{sc}(A,B)$ is a subclass of the class $C_{sc} = C_{sc}(1,-1)$.

By definition of subordination, it follows that $f \in C_{sc}(A,B)$ if and only if

$$\frac{2(zf'(z))'}{(f(z)-\overline{f(-\bar{z})})'} = \frac{1+Aw(z)}{1+Bw(z)} = P(z), \quad w \in \mathcal{U} \tag{2}$$

where

$$P(z) = 1 + \sum_{n=1}^{\infty} p_n z^n. \tag{3}$$

We study the class $C_{sc}(A,B)$ and obtain coefficient estimates, distortion theorem and integral operator.

2 Preliminary Result

We need the following preliminary lemmas, required for proving our result.

Lemma 2.1 *([2]) If $P(z)$ is given by (3) then*

$$|p_n| \leq (A-B). \tag{4}$$

Lemma 2.2 *([2]) Let $N(z)$ be analytic and $M(z)$ starlike in D and $N(0) = M(0) = 0$. Then*

$$\frac{\left|\left(\frac{N'(z)}{M'(z)} - 1\right)\right|}{\left|\left(A - B\frac{N'(z)}{M'(z)}\right)\right|} < 1$$

implies

$$\frac{\left|\left(\frac{N(z)}{M(z)} - 1\right)\right|}{\left|\left(A - B\frac{N(z)}{M(z)}\right)\right|} < 1, \quad z \in \mathcal{D}.$$

3 Main Result

We give the coefficient inequalities for the class $C_{sc}(A, B)$.

Theorem 3.1 *Let $f \in C_{sc}(A, B)$, then for $n \geq 1$,*

$$|a_{2n}| \leq \frac{(A - B)}{(2n)n!2^n} \prod_{j=1}^{n-1} (A - B + 2j), \tag{5}$$

and

$$|a_{2n+1}| \leq \frac{(A - B)}{(2n+1)n!2^n} \prod_{j=1}^{n-1} (A - B + 2j). \tag{6}$$

Proof.

For (2) and (3), we have

$$1 + 2^2a_2z + 3^2a_3z^2 + ... + (2n)^2a_{2n}z^{2n-1} + (2n+1)^2a_{2n+1}z^{2n} + ...$$

$$= (1 + 3a_3z^2 + 5a_5z^4 + ... + (2n-1)a_{2n-1}z^{2n-2} + (2n+1)a_{2n+1}z^{2n} + ...) \bullet (1 + p_1z + p_2z^2 + ... + p_{2n}z^{2n} + p_{2n+1}z^{2n+1} + ...)$$

Equating the coefficients of like powers of z, we have

$$2^2a_2 = p_1, \quad 3(2)a_3 = p_2 \tag{7}$$

$$4^2a_4 = p_3 + 3a_3p_1, \quad 5(4)a_5 = p_4 + 3a_3p_2 \tag{8}$$

$$(2n)^2a_{2n} = p_{2n-1} + 3a_3p_{2n-3} + 5a_5p_{2n-5} + ... + (2n-1)a_{2n-1}p_1 \tag{9}$$

$$(2n)(2n+1)a_{2n+1} = p_{2n} + 3a_3p_{2n-2} + 5a_5p_{2n-4} + ... + (2n-1)a_{2n-1}p_2. \tag{10}$$

Easily using Lemma 2.1 and (7), we get

$$|a_2| \leq \frac{(A - B)}{2(2)}, \quad |a_3| \leq \frac{(A - B)}{3(2)}. \tag{11}$$

Again by applying (11) and followed by Lemma 2.1, we get from (8)

$$|a_4| \le \frac{(A-B)(A-B+2)}{4(4)2}, \quad |a_5| \le \frac{(A-B)(A-B+2)}{5(4)(2)}.$$

It follows that (5) and (6) hold for n=1,2. We now prove (5) using induction. Equation (9) in conjuction with Lemma 2.1 yield

$$|a_{2n}| \le \frac{(A-B)}{(2n)^2} \left[1 + \sum_{k=1}^{n-1}(2k+1)|a_{2k+1}|.\right] \tag{12}$$

We assume that (5) holds for k=3,4,...,(n-1). Then from (12), we obtain

$$|a_{2n}| \le \frac{A-B}{(2n)^2} \left[1 + \sum_{k=1}^{n-1} \frac{A-B}{k!2^k} \prod_{j=1}^{k-1}(A-B+2j)\right]. \tag{13}$$

In order to complete the proof, it is sufficient to show that

$$\frac{A-B}{(2m)^2} \left[1 + \sum_{k=1}^{m-1} \frac{A-B}{k!2^k} \prod_{j=1}^{k-1}(A-B+2j)\right]$$

$$= \frac{A-B}{(2m)m!2^m} \prod_{j=1}^{m-1}(A-B+2j), \quad (m = 3, 4, ..., n). \tag{14}$$

(14) is valid for $m = 3$.

Let us suppose that (14) is true for all m, $3 < m \le (n-1)$. Then from (13)

$$\frac{A-B}{(2n)^2} \left[1 + \sum_{k=1}^{n-1} \frac{A-B}{k!2^k} \prod_{j=1}^{k-1}(A-B+2j)\right]$$

$$= \left(\frac{n-1}{n}\right)^2 \left(\frac{A-B}{(2(n-1))^2} \left(1 + \sum_{k=1}^{n-2} \frac{A-B}{k!2^k} \prod_{j=1}^{k-1}(A-B+2j)\right)\right)$$

$$+ \frac{A-B}{(2n)^2} \frac{A-B}{(n-1)!2^{n-1}} \prod_{j=1}^{n-2}(A-B+2j)$$

$$= \left(\frac{n-1}{n}\right)^2 \frac{A-B}{2(n-1)(n-1)!2^{n-1}} \prod_{j=1}^{n-2}(A-B+2j)$$

$$+ \frac{A-B}{(2n)^2} \frac{A-B}{(n-1)!2^{n-1}} \prod_{j=1}^{n-2}(A-B+2j)$$

$$= \frac{A-B}{2n^2(n-1)!2^{n-1}} \prod_{j=1}^{n-2}(A-B+2j)\frac{(A-B+2(n-1))}{2}$$

$$= \frac{A-B}{(2n)n!2^n} \prod_{j=1}^{n-1}(A-B+2j)$$

Thus, (14) holds for $m = n$ and hence (5) follows. Similarly, we can prove (6).

Next, we give distortion bound and preserving integral operator for the class $C_{sc}(A, B)$.

Theorem 3.2 *Let $f \in C_{sc}(A, B)$, then for $|z| = r,\ 0 < r < 1$,*

$$\frac{1}{r}\int_0^r \frac{1}{(1+t^2)^2}\frac{1-At}{(1-Bt)}dt \le |f'(z)| \le \frac{1}{r}\int_0^r \frac{1}{(1-t^2)^2}\frac{1+At}{(1+Bt)}dt. \tag{15}$$

Proof.

Put $h(z) = \frac{f(z)-\overline{f(-\bar{z})}}{2}$. Then from (2), we obtain

$$|(zf'(z))'| = |h'(z)| \left|\frac{1+Aw(z)}{1+Bw(z)}\right|. \tag{16}$$

Since h is convex, it follows that (see [1])

$$\frac{r}{1+r} \le |h(z)| \le \frac{r}{1-r}.$$

Therefore by differentiating once, the following is obtained

$$\frac{1}{1+r^2} \le |h'(z)| \le \frac{1}{1-r^2}. \tag{17}$$

Furthermore, for $w \in \mathcal{U}$, it can also be easily established that

$$\frac{1-Ar}{1-Br} \le \left|\frac{1+Aw(z)}{1+Bw(z)}\right| \le \frac{1+Ar}{1+Br}. \tag{18}$$

Next, applying results (17) and (18) in (16), we obtain

$$\frac{1}{1+r^2}\frac{1-Ar}{1-Br} \le |(zf'(z))'| \le \frac{1}{1-r^2}\frac{1+Ar}{1+Br}. \tag{19}$$

Finally, setting $|z| = r$ and integrating (19) gives our result.

Theorem 3.3 *If $f \in C_{sc}(A, B)$ then $F \in C_{sc}(A, B)$, where*

$$F(z) = \frac{2}{z}\int_0^z f(t)\, dt.$$

Proof.

With the given F above, consider

$$\frac{2(zF'(z))'}{(F(z)-\overline{F(-\bar{z})})'} = \frac{z^2 f'(z) - zf(z) + \int_0^z f(t)dt}{\frac{1}{2}\left[z(f(z)-\overline{f(-\bar{z})}) - \int_0^z (f(t)-\overline{f(-\bar{t})})dt\right]}.$$

Suppose, we let $N(z)$ and $M(z)$ be the numerator and denominator functions respectively. It can be shown that

$$M(z) = \frac{1}{2}\left[z(f(z) - \overline{f(-\overline{z})}) - \int_0^z (f(t) - \overline{f(-\overline{t})})dt\right]$$

is convex. Hence, $M(z)$ is also starlike. Furthermore,

$$\frac{N'(z)}{M'(z)} = \frac{2(zf'(z))'}{(f(z) - \overline{f(-\overline{t})})'} \quad with \quad f \in C_{sc}(A, B).$$

Thus

$$\frac{N'(z)}{M'(z)} = \frac{1 + Aw(z)}{1 + Bw(z)}, \quad w \in \mathcal{U}.$$

This implies that

$$\frac{\left|\left(\frac{N'(z)}{M'(z)} - 1\right)\right|}{\left|\left(A - B\frac{N'(z)}{M'(z)}\right)\right|} < 1.$$

Hence, by Lemma 2.2, we have

$$\frac{\left|\left(\frac{N(z)}{M(z)} - 1\right)\right|}{\left|\left(A - B\frac{N(z)}{M(z)}\right)\right|} < 1, \quad z \in \mathcal{D}$$

or equivalently,

$$\frac{N(z)}{M(z)} = \frac{1 + Aw_1(z)}{1 + Bw_1(z)}, \quad w_1 \in \mathcal{U}.$$

Thus $F \in C_{sc}(A, B)$.

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