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Note on Reverses of the Triangle Inequality

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Abstract

In this paper we establish the recently results of reverses of the triangle inequality. Some of these results are given and analyzed. Thus, we obtain some similar forms of reverses of the triangle inequalities.

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1. Introduction

First of all, let's remind on the classical triangle inequality (see [4])
Let $z_1, \dots, z_2$ be complex numbers. Then the following inequality holds

$$\left|\sum_{i=1}^{n} z_i\right| \leq \sum_{i=1}^{n} |z_i| \tag{1.1}$$

There also exists integral analogues of the triangle inequality (1.1).

Let f be a continuous complex (real) function of real variable x on a segment $[a,b]$. Then the following inequality holds

$$\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx \tag{1.2}$$

which is continuous version of the triangle inequality (1.1).

Also, let's remind on classical reversed triangle inequality (1.1) (see also [4]).

If $0<\theta<\frac{\pi}{2}$ and $a-\theta<\arg z_i<a+\theta,\ i=1,\ldots,n$, then following inequality holds

$$\left|\sum_{i=1}^n z_i\right| \ge \cos\theta \sum_{i=1}^n |z_i| \tag{1.3}$$

There exist the continuous version of a reverse inequality (1.2).

If f is complex-valued integrable function defined on [*a,b*] and $-\theta<\arg f(x)<\theta$ for $0<\theta<\frac{\pi}{2}$, then

$$\left|\int_a^b f(x)dx\right| \ge \cos\theta \int_a^b |f(x)|dx \tag{1.4}$$

There are a lot of results on reverses of the triangle inequalities in a recent literature. Some of these results ore given and analyzed (see [2] and [4]).

In [4] we cal also find more general abstract triangle and reversed triangle inequality, i.e. generalization of triangle inequality with positive linear functionals.

2. Reverses of the triangle inequality

Let E nonempty set, L linear class of function $g: E\to R$ which the following properties

(1) $f,g\in L \Rightarrow (af+bg)\in L$ for $a,b\in R$,

(2) $1\in L$, i. e. if $f(t)\equiv 1\ (t\in L)$, then $f\in L$

Let's consider positive linear functional $A: L\to R$, i. e. which satisfy conditions:

(I) $A(af+bg)=aA(f)+bA(g)$, for $f,g\in L$, and $a,b\in R$,

(II) If $f\in L$, and $f(t)\ge 0$ on E, then $A(f)\ge 0$.

Further, let's consider the class

$$\overline{L}=\{f: E\to C \mid \operatorname{Re} f\in L,\ \operatorname{Im} f\in L\}$$

Let A be a positive linear functional and let the functional $\overline{A}:\overline{L}\to C$ is defined by

$$\overline{A}(f) = A(\operatorname{Re} f) + i\,A(\operatorname{Im} f) = \operatorname{Re}(\overline{A}(f)) + i\ \operatorname{Im}(\overline{A}(f))$$

Then, $\overline{A}$ is complex linear functional on $\overline{L}$, i. e,

$$\overline{A}(af + bg) = a\overline{A}(f) + b\overline{A}(g), \text{ for } f, g \in \overline{L},\ a, b \in C.$$

The following results is obtained in [4] :

Theorem 1. If $f \in \overline{L}$ and $|f| \in L$, then

$$\left|\overline{A}(f)\right| \le A(|f|)$$

Proof. For arbitrary $\theta \in R$, we have

$$\operatorname{Re}(e^{i\theta}\overline{A}(f)) = \operatorname{Re}(\overline{A}(e^{i\theta} f)) = A(\operatorname{Re}(e^{i\theta} f)) \le A(|f|)$$

Suppose that $\overline{A}(f) = r\,e^{it}$. Set in the previous consideration $\theta = -t$ we get

$$\left|\overline{A}(f)\right| = r = \operatorname{Re}(e^{-it}\overline{A}(f)) \le A(|f|)$$

Then, we'll state abstract reversed triangle inequality:

Theorem 2. If $f \in \overline{L}$, and $a - \theta < \arg f(x) < a + \theta,\ 0 < \theta < \dfrac{\pi}{2}$, then

$$\left|\overline{A}(f)\right| \ge (\cos\theta) A(|f|).$$

Proof. We have

$$\begin{aligned}
\left|\overline{A}(f)\right| &= \left|e^{-ia}\ \overline{A}(f)\right| \ge \operatorname{Re}(e^{-ia}\ \overline{A}\,(f)) = \\
&= A(\operatorname{Re}(e^{-ia}\ f)) = A(|f| \operatorname{Re}(e^{i(\arg f - a)})) = \\
&= A(|f| \cos(\arg f - a)) \ge \\
&\ge A(\cos\theta\, |f|) = (\cos\theta)\, A(|f|).
\end{aligned}$$

Of course, we can also obtain in the case when

$$a < \arg f(t) \le a + \theta, \text{ for } 0 \le \theta \le \pi$$

that

$$\left|\overline{A}(f)\right| \ge \left(\cos\frac{\theta}{2}\right) A(|f|).$$

Now, by introducing two parameters, we obtain the following result :

Theorem 3. If $f \in \overline{L}$ and $|f| \in L$. Let $a + \phi_1 < \arg f(\alpha) < a + \phi_2$ and $0 < \varphi_2 - \varphi_1 < \pi$. Then the following inequality hold:

$$\left|\overline{A}(f)\right| \geq \cos\frac{\phi_2-\phi_1}{2}\cdot A(|f|)$$

Proof. Since

$$\left(a+\frac{\phi_1+\phi_2}{2}\right)-\frac{\phi_2-\phi_1}{2} \leq \arg f(x) \leq \left(a+\frac{\phi_1+\phi_2}{2}\right)+\frac{\phi_2-\phi_1}{2}$$

The inequality obviously follows from result of Theorem 2.

On the other hand, in [3] S. S. Dragomir obtained the following result:

Theorem 4. Let $f:[a,b]\to C$ strongly measurable such that the Lebesgue integral $\int_a^b \|f(t)\|\,dt$ exist and it is finite. Let

$$0<\phi_1<\arg f(x)<\phi_2<\frac{\pi}{2}$$

for a. e. $t\in[a,b]$. Then the following inequality holds

$$\left|\int_a^b f(x)\,dx\right| \geq \sqrt{\sin^2\phi_1+\cos^2\phi_2}\cdot\int_a^b |f(x)|\,dx$$

with equality if and only if

$$\int_a^b f(x)\,dx = (\cos\phi_2 + i\sin\phi_1)\int_a^b |f(x)|\,dx$$

Now, if we apply Theorem 1, we can obtain similar result in the following form :

Theorem 5. Let $f:[a,b]\to C$ strongly measurable such that the Lebesgue integral $\int_a^b |f(t)|\,dt$ exist and it is finite. Let $a+\phi_1<\arg f(\alpha)<a+\phi_2$ and $0<\varphi_2-\varphi_1<\pi$. Then the following inequality holds

$$\left|\int_a^b f(x)\,dx\right| \geq \cos\frac{\phi_2-\phi_1}{2}\int_a^b |f(x)|\,dx$$

Proof. Now we are going to compare the results obtained in theorem 4 and theorem 5. Then, claim that the following inequality is valid :

$$\cos\frac{\phi_2-\phi_1}{2} \geq \sqrt{\sin^2\phi_1+\cos^2\phi_2}$$

for $0 < \phi_1 < \arg f(x) < \phi_2 < \frac{\pi}{2}$.

Let's fix $\alpha = \phi_2 - \phi_1$. Now, consider the function F defined as

$$F(\phi_1) = \sin^2 \phi_1 + \cos^2 (\phi_1 + \alpha) - \cos^2 \frac{\alpha}{2}$$

We are going to prove that, for each $\alpha \in \left(0, \frac{\pi}{2}\right)$, the function F is negative for each $\phi_1 \in \left(0, \frac{\pi}{2} - \alpha\right)$.

Indeed, we calculate the first derivative of function F :

$$F'(\phi_1) = \sin(2\phi_1) - \sin(2\phi_1 + 2\alpha).$$

Now, it is easy to find that the minimum of F is obtained for $\phi_1 = \frac{\pi}{4} - \frac{\alpha}{2}$. In both cases, (when ϕ_1 tends to 0 or to $\frac{\pi}{2} - \alpha$), the value of $F(\phi_1)$ tends to $\cos^2 \alpha - \cos^2 \frac{\alpha}{2}$ and this value is obviously negative for $\alpha \in \left(0, \frac{\pi}{2}\right)$. So, the function F is negative for each $\phi_1 \in \left(0, \frac{\pi}{2} - \alpha\right)$.

Hence, the result of Theorem 4 can be improved in the following way :

$$\left| \int_a^b f(x)\,dx \right| \geq \cos\frac{\phi_2 - \phi_1}{2} \int_a^b |f(x)|\,dx$$

for $c + \phi_1 < \arg f(x) < c + \phi_2$, $0 < \phi_2 - \phi_1 < \pi$ (a.e.).

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