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Constrained Observability for Parabolic Systems

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Abstract

The objective of this paper is to develop the question of the constrained observability for distributed parabolic system evolving in spatial domain Ω. It consists in the reconstruction of the initial state and must be between two prescribed functions in a subregion ω of Ω. We give definitions and some properties of this kind of regional observability and we describe two reconstruction approaches where the first is based on subdifferential techniques and the second uses the lagrangian multiplier method. This last approach leads to an algorithm which is performed by example and simulation.

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1 Introduction

For distributed parameter systems, many works have been devoted to the observation problem ([1] and [2]). It has often been studied and the objective is the reconstruct the state in the whole system evolution domain Ω. The notion of sensors and actuators introduced in 1980 by El Jai and Pritchard, allows a better description of measurements and actions ([4] and [7]). The notion of regional observability is recently introduced and studied, and concerns the state observation in a subregion ω of Ω. One of the most important reasons for studying this kind of problem is that the observation error is smaller than in the general case. Various motivating examples of regional observability are given in ([3], [5]).
Here we are interested to approach the initial state and the reconstructed state must be between two prescribed function given only on a subregion ω of the geometric domain where the system is considered ([6]). There are many reasons for introducing this kind of notion : Firstly, the mathematical model of real system is obtained either from the measurements, or from approximation techniques and is very often affected by perturbations. Consequently the solution of such a system is only approximately known. Secondly the observation error is smaller than in general observation case and the reconstructed state is to be between two bounds. This problem is encountered in various real problems where the reconstructed state is required only to be between two functions. This is the case for example of a biological reactor in which the concentration regulation of a substrate at the bottom of the reactor is expected to be between two levels.
The paper is organized as follows: First we provide results on regional observability for distributed parameter system of parabolic type and we give definitions related to regional constrained observability of parabolic systems. In the next section we give characterization of this notion using a sub-differential tools. Section 4 deals with a reconstruction method based on Lagrangian multiplier approach which gives a practice algorithm. The Last section is devoted to compute the obtained algorithm with numerical example and simulations.

2 Problem statement

For an open regular and bounded domain Ω of $I\!R^n$(n=1,2,3). For $T > 0$, let $Q = \Omega \times]0, T[$ and $\Sigma = \partial\Omega \times]0, T[$. We consider the system described by

$$\begin{cases} \dfrac{\partial y(x,t)}{\partial t} = \mathcal{A}y(x,t) & Q \\ y(x,0) = y_0(x) & \Omega \\ y(\xi,t) = 0 & \Sigma \end{cases} \tag{1}$$

where $\mathcal{A}$ is a second-order linear differential operator with compact resolvent and generates a strongly continuous semi-group $(S(t))_{t\geq 0}$ in the state space $L^2(\Omega)$. We suppose that $y_0 \in L^2(\Omega)$, then the system (1) admits a unique solution $y(.)$ in $L^2(\Omega)$. The measurements are given by the output function given by

$$z(t) = Cy(t), \quad t \in [0, T] \tag{2}$$

where $C \in \mathcal{L}(L^2(\Omega), I\!R^q)$, q depends on the number of the considered sensors. The observation space is $\mathcal{O} = L^2(0, T, I\!R^q)$.
Since the system (1) is autonomous, the output can be expressed by

$$z(t) = CS(t)y_0 = (Ky_0)(t), \quad t \in]0, T[$$

where the operator K is given by

$$\begin{array}{rcl} K : L^2(\Omega) & \longrightarrow & \mathcal{O} \\ z & \longrightarrow & CS(.)z \end{array}$$

is linear bounded with the adjoint K^* given by

$$\begin{array}{rcl} K^* : \mathcal{O} & \longrightarrow & L^2(\Omega) \\ z^* & \longrightarrow & \displaystyle\int_0^T S^*(t)C^*z(t)dt \end{array}$$

Let ω be a subregion of Ω with positive of Lesbegue measure. Let χ_ω be the restriction function defined by

$$\begin{array}{rcl} \chi_\omega : L^2(\Omega) & \longrightarrow & L^2(\omega) \\ y & \longmapsto & \chi_\omega y = y_{|\omega} \end{array} \tag{3}$$

with the adjoint χ_ω^* given by

$$\chi_\omega^* y = \left\{ \begin{array}{lll} y & \text{in} & \omega \\ 0 & \text{in} & \Omega \setminus \omega \end{array} \right.$$

We recall that a sensor is conventionally defined by a couple (D, f), where $D \subset \bar{\Omega}$ is the geometric support of the sensor and f is the spatial distribution of the information on the support D ([7]).
In the case of a pointwise sensor (internal or boundary) D={b} and $f = \delta(b - \cdot)$, where δ is the Dirac mass concentrated in b, and the sensor is then denoted by (b, δ_b). For definitions and properties of strategic and regionally strategic sensors we refer to ([5] and [7]).
We also recall that the system (1) together with the output (2) is said to be exactly (respectively weakly) regionally observable in ω if $Im\chi_\omega K^* = L^2(\omega)$ (respectively $KerK\chi_\omega^* = \{0\}$). For more details, we refer the reader to ([5]).
Let $\alpha(.)$ and $\beta(.)$ be two functions defined in $L^2(\omega)$ such that $\alpha(.) \leq \beta(.)$ a.e. in ω. In the sequel we set

$$[\alpha(.), \beta(.)] = \big\{ y \in L^2(\omega) \ \mid \ \alpha(.) \leq y(.) \leq \beta(.) \text{ a.e. in } \omega \big\}$$

Definition 2.1 *The system (1) together with the output (2) is said to be exactly* $[\alpha(.), \beta(.)]$*-observable in* ω *if*

$$(Im\chi_{\omega}K^{*}) \cap [\alpha(.), \beta(.)] \neq \emptyset.$$

Definition 2.2 *The system (1) together with the output (2) is said to be weakly* $[\alpha(.), \beta(.)]$*-observable in* ω *if*

$$(\overline{Im\chi_{\omega}K^{*}}) \cap [\alpha(.), \beta(.)] \neq \emptyset.$$

Definition 2.3 *The sensor* (D, f) *is said to be* $[\alpha(.), \beta(.)]$*-strategic in* ω *if the observed system is weakly* $[\alpha(.), \beta(.)]$*-observable in* ω*.*

Remark 2.4

1. *If the system (1) together with the output (2) is exactly* $[\alpha(.), \beta(.)]$*-observable in* ω *then it is weakly* $[\alpha(.), \beta(.)]$*-observable in* ω*.*

2. *If the system (1) together with the output (2) is exactly (resp. weakly)* $[\alpha(.), \beta(.)]$*-observable in* ω_1 *then it is exactly (resp. weakly)* $[\alpha(.), \beta(.)]$*-observable in any subregion* $\omega_2 \subset \omega_1$*.*

3. *If the system (1) together with the output (2) is exactly regionally observable in* ω *then it is exactly* $[\alpha(.), \beta(.)]$*-observable in* ω*. Indeed, if the system (1) together with the output (2) is exactly regionally observable in* ω *then* $Im\chi_{\omega}K^{*} = L^2(\omega)$ *which gives*

$$(Im\chi_{\omega}K^{*}) \cap [\alpha(.), \beta(.)] \neq \emptyset$$

that is to say that the system (1) together with the output (2) is exactly $[\alpha(.), \beta(.)]$*-observable in* ω*.*

There exist systems which are not weakly observable but are weakly $[\alpha(.)\beta(.)]$-observable.

Example 2.5 *Consider the following one-dimensional system*

$$\begin{cases} \dfrac{\partial y(x,t)}{\partial t} = \dfrac{\partial^2 y(x,t)}{\partial x^2} &]0,1[\times]0,T[\\ y(0,t) = y(1,t) = 0 &]0,T[\\ y(0) = y_0 &]0,1[\end{cases} \tag{4}$$

augmented with the output function

$$z(t) = y(b,t), \quad b \in]0,1[\cap \mathbb{Q} \tag{5}$$

The operator $\dfrac{\partial^2}{\partial x^2}$ *has a complete set of eigenfunctions* (φ_i) *in* $L^2(\Omega)$ *associated to the eigenvalues* λ_i *given by*

$$\varphi_i(x) = \sqrt{2}\sin(i\pi x) \text{ and } \lambda_i = -i^2\pi^2$$

We have the lemma

Lemma 2.6 *The system (4) together with the output (5) is not weakly observable in]0,1[but it is weakly* $[\alpha(.), \beta(.)]$*-observable in]0,1[.*

Proof.
Let us consider $I = \{i \in I\!N^* \mid C\varphi_i = \varphi_i(b) = 0\}$ *and* $J = I^c$*, then we have easily* $\ker K(t) = \overline{\{\varphi_i\}}_{i\in I}$.
Since $I \neq \emptyset$*, we have* $\ker K(t) \neq \{0\}$*, so the system (4) augmented with (5) is not weakly observable in]0,1[but we can show that it is weakly* $[\alpha(.), \beta(.)]$*-observable in]0,1[, indeed,*
Let $i_0 \in J$ *such that* $\varphi_{i_0}(b) \neq 0$*, then*

$$\begin{aligned} K(t)\varphi_{i_0} &= \sum_{j=1}^{\infty} \exp(\lambda_j t)\langle\varphi_{i_0}, \varphi_j\rangle\varphi_j(b) \\ &= \exp(\lambda_{i_0} t)\varphi_{i_0}(b) \neq 0, \forall t \in]0, T[\end{aligned}$$

which shows that φ_{i_0} *is weakly observable in]0,1[.*
For $\alpha(x) = -|\varphi_{i_0}(x)| - 1\langle\varphi_{i_0}(x)$ *and* $\beta(x) = |\varphi_{i_0}(x)| + 1\rangle\varphi_{i_0}(x), \forall x \in]0, 1[$
then the system (4) together with the output (5) is weakly $[\alpha(.), \beta(.)]$*-observable in]0,1[.*

Propisition 2.7 *The system (1) together with the output (2) is exactly* $[\alpha(.), \beta(.)]$*-observable in* ω *if and only if*

$$(\ker \chi_\omega + ImK^*) \cap [\alpha(.), \beta(.)] \neq \emptyset$$

Proof.

- Suppose that $(\ker \chi_\omega + ImK^*) \cap [\alpha(.), \beta(.)] \neq \emptyset$ then, there exists $z \in [\alpha(.), \beta(.)]$ such that $z \in \ker \chi_\omega + ImK^*$ so $z = z_1 + z_2$, where $\chi_\omega z_1 = 0$ and $z_2 = K^*\theta$ with $\theta \in \mathcal{O}$, then $\chi_\omega z = \chi_\omega z_1 + \chi_\omega z_2 = \chi_\omega z_2 = \chi_\omega K^*\theta$ and $\chi_\omega z \in Im(\chi_\omega K^*)$, thus

$$(Im\chi_\omega K^*) \cap [\alpha(.), \beta(.)] \neq \emptyset$$

  which shows that the system (1) together with the output (2) is exactly $[\alpha(.), \beta(.)]$-observable in ω.

- Suppose that the system (1) together with the output (2) is exactly $[\alpha(.), \beta(.)]$-observable in ω, which is equivalent to

$$(Im\chi_\omega K^*) \cap [\alpha(.), \beta(.)] \neq \emptyset$$

then there exists $z \in [\alpha(.), \beta(.)]$ and $\theta \in L^2(0, T; I\!R^q)$ such that $\chi_\omega z = \chi_\omega K^*\theta$ which gives $\chi_\omega(z - K^*\theta) = 0$. Let $y_1 = z - K^*\theta$ and $y_2 = K^*\theta$, then $z = y_1 + y_2$ with $y_1 \in \ker \chi_\omega$ and $y_2 \in ImK^*$ which shows that $z \in \ker \chi_\omega + ImK^*$ and therefore

$$(\ker \chi_\omega + ImK^*) \cap [\alpha(.), \beta(.)] \neq \emptyset$$

Propisition 2.8 *The system (1) together with the output (2) is weakly $[\alpha(.), \beta(.)]$-observable in ω if and only if*

$$\left(\ker \chi_\omega + \overline{ImK^*}\right) \cap [\alpha(.), \beta(.)] \neq \emptyset$$

Proof.

- We assume that $\left(\ker \chi_\omega + \overline{ImK^*}\right) \cap [\alpha(.), \beta(.)] \neq \emptyset$ so, there exists $z_1 \in \ker \chi_\omega$ and $z_2 \in \overline{ImK^*}$ such that $z = z_1 + z_2 \in [\alpha(.), \beta(.)]$, which implies that $\chi_\omega z = \chi_\omega z_2 = \chi_\omega(\lim_{n\to+\infty} K^*\theta_n) = \lim_{n\to+\infty} \chi_\omega K^*\theta_n$ with θ_n is a sequence of elements of $\mathcal{O}$ and $\chi_\omega z \in \overline{Im(\chi_\omega K^*)}$ then

$$\left(\overline{Im\chi_\omega K^*}\right) \cap [\alpha(.), \beta(.)] \neq \emptyset$$

  finally the system (1) together with the output (2) is weakly $[\alpha(.), \beta(.)]$-observable in ω.

- Suppose that the system (1) together with the output (2) is weakly $[\alpha(.), \beta(.)]$-observable in ω, which is equivalent to

$$(\overline{Im\chi_\omega K^*}) \cap [\alpha(.), \beta(.)] \neq \emptyset$$

  then there exists $z \in [\alpha(.), \beta(.)]$ and θ_n a sequence of elements of $\mathcal{O}$ such that $\chi_\omega z = \lim_{n\to+\infty} \chi_\omega K^*\theta_n$ which gives $\chi_\omega(z - \lim_{n\to+\infty} K^*\theta_n) = 0$. We set $y_1 = z - \lim_{n\to+\infty} K^*\theta_n$ and $y_2 = \lim_{n\to+\infty} K^*\theta_n$, then $z = y_1 + y_2$ with $y_1 \in \ker \chi_\omega$ and $y_2 \in \overline{ImK^*}$. This shows that $z \in \ker \chi_\omega + \overline{ImK^*}$ and finally

$$(\ker \chi_\omega + \overline{ImK^*}) \cap [\alpha(.), \beta(.)] \neq \emptyset$$

3 Subdifferential approach

Here we consider the problem of regional reconstruction of the initial state from measurements taken in a finite time interval and the purpose is to use subdifferential approach ([8]) to solve the problem of constrained observability. Without loss of generality we consider the distributed parameter system represented by the autonomous state equation

$$\begin{cases} \dfrac{\partial y(x,t)}{\partial t} = \mathcal{A}y(x,t) & Q \\ y(x,0) = y_0(x) & \Omega \\ y(\xi,t) = 0 & \Sigma \end{cases} \tag{6}$$

augmented with the output function

$$z(t) = CS(t)y_0 \tag{7}$$

The exact $[\alpha(.),\beta(.)]$-observability in ω turns up to minimize the reconstruction error given by

$$\begin{cases} \min ||Ky - z||^2_{\mathcal{O}} \\ y \in Y \end{cases} \tag{8}$$

where $Y = \{y \in L^2(\Omega) \mid \chi_\omega y \in [\alpha(.),\beta(.)]\}$
In this section we propose to solve the problem (8) using the subdifferential approach ([8]).
Let us denote by

- $\Gamma_0(L^2(\Omega))$ the set of functions $f : L^2(\Omega) \longmapsto \tilde{I\!R} =]-\infty,+\infty]$ proper, lower semi-continuous (l.s.c.) and convex in $L^2(\Omega)$.
- For $f \in \Gamma_0(L^2(\Omega))$
  $dom(f) = \{y \in L^2(\Omega) \mid f(y) < +\infty\}$ and f^* the polar function of f, given by

  $$f^*(y^*) = \sup_{y \in dom(f)}\{\langle y^*, y\rangle - f(y)\}, \forall y^* \in L^2(\Omega)$$

  For $y^0 \in dom(f)$ the set

  $$\partial f(y^0) = \left\{y^* \in L^2(\Omega) \mid f(y) \geq f(y^0) + \langle y^*, y - y^0\rangle, \forall y \in L^2(\Omega)\right\}$$

  denotes the subdifferential of f at y^0, then we have the following property

  $$y_1 \in \partial f(y^*) \text{ if and only if } f(y^*) + f^*(y_1) = \langle y^*, y_1\rangle$$

- For D a nonempty subset of $L^2(\Omega)$

  $$\Psi_D(y) = \begin{cases} 0 & \text{if } y \in D \\ +\infty & \text{otherwise} \end{cases}$$

  denotes the indictor functional of D.

With these notations the problem (8) is equivalent to the problem

$$\begin{cases} \inf(\|Ky - z\|^2_{\mathcal{O}} + \Psi_Y(y)) \\ y \in L^2(\Omega) \end{cases} \tag{9}$$

The solution of this problem may be characterized by the following result.

Propisition 3.1 *If the system (6) together with the output (7) is exactly $[\alpha(.), \beta(.)]$-observable in ω, then y^* is a solution of (9) if and only if*

$$y^* \in Y \quad and \quad \Psi^*_Y(-2K^*(Ky^* - z)) = -2\|Ky^*\|^2_{\mathcal{O}} + 2\langle K^*z, y^*\rangle$$

Proof.
y^* is a solution of (9) if and only if $0 \in \partial(f + \Psi_Y)(y^*)$, with $f(y^*) = \|Ky^* - z\|^2_{\mathcal{O}}$ but $f \in \Gamma_0(L^2(\Omega))$, and since Y is closed, convex and nonempty, then $\Psi_Y \in \Gamma_0(L^2(\Omega))$. Moreover under the hypothesis of the exact $[\alpha(.), \beta(.)]$-observability in ω, we have $Dom(f) \cap Dom(\Psi_Y) \neq \emptyset$, but f is continuous then

$$\partial(f + \Psi_Y)(y^*) = \partial f(y^*) + \partial\Psi_Y(y^*)$$

it follows that y^* is a solution of (9) if and only if $0 \in \partial f(y^*) + \partial\Psi_Y(y^*)$.
Moreover f is Frechet-differentiable, then

$$\partial f(y^*) = \{\nabla f(y^*)\} = \{2K^*(Ky^* - z)\}$$

and y^* is a solution of (9) if and only if $-2K^*(Ky^* - z) \in \partial\Psi_Y(y^*)$
which is equivalent to

$$y^* \in Y,\ \Psi_Y(y^*) + \Psi^*_Y(-2K^*(Ky^* - z)) = \langle y^*, -2K^*Ky^* + 2K^*z\rangle$$

and gives

$$y^* \in Y,\ \Psi^*_Y(-2K^*(Ky^* - z)) = -2\|Ky^*\|^2_{\mathcal{O}} + 2\langle K^*z, y^*\rangle.$$

4 Lagrangian multiplier approach

In this section we purpose to solve the problem (8) using the Lagrangian multiplier method([9]). An algorithm is also given for the numerical construction of the estimated initial state which can be illustrated by numerical example and simulations which lead to some conjectures. From the definition of the exact $[\alpha(.), \beta(.)]$-observability in ω, there exists at least $\theta \in \mathcal{O}$ that allows to observe a state $y \in [\alpha(.), \beta(.)]$ in ω of the form $K^*\theta$ that is

$$\exists\theta \in \mathcal{O} \ \mid\ \chi_\omega K^*\theta \in [\alpha(.), \beta(.)]$$

So all state we will consider are of the form $K^*\theta$ such that $\theta \in \mathcal{O}$.
Then to solve the problem (8) and achieve our purpose it is sufficient to consider the following problem

$$\begin{cases} \min \|KK^*\theta - z\|^2_{\mathcal{O}} \\ \theta \in G = \{\hat{\theta} \in \mathcal{O} \mid \chi_\omega K^*\hat{\theta} \in [\alpha(.), \beta(.)]\} \end{cases} \tag{10}$$

Then we have the following result

Propisition 4.1 *If the system (1) together with the output (2) is exactly observable in Ω, then the solution of (10) is given by*

$$\theta^* = (KK^*KK^*)^{-1}KK^*z - \frac{1}{2}(KK^*KK^*)^{-1}K\chi^*_\omega\lambda^*$$

and the solution in ω of the problem (8) for all the states which are of the form $K^*\theta$ with $\theta \in \mathcal{O}$ is given by

$$y^* = R_\omega K^*z - \frac{1}{2}R_\omega\chi^*_\omega\lambda^*$$

where λ^* is the solution of

$$\begin{cases} \frac{1}{2}R_\omega\chi^*_\omega\lambda^* = -y^* + R_\omega K^*z \\ y^* = P_{[\alpha(.),\beta(.)]}(\rho\lambda^* + y^*) \end{cases} \tag{11}$$

while $P_{[\alpha(.),\beta(.)]} : L^2(\omega) \to [\alpha(.), \beta(.)]$, denotes the projection operator, $\rho > 0$ and $R_\omega = \chi_\omega K^*(KK^*KK^*)^{-1}K$

Proof.
If the system (1) together with the output (2) is exactly $[\alpha(.), \beta(.)]$-observable in ω then $G \neq \emptyset$ and the problem (10) has a solution. To address the constraint problem (10) we use a Lagrange multiplies and reduce the problem (10) to the following saddle point problem

$$\begin{cases} \min \|KK^*\theta - z\|^2_{\mathcal{O}}, \\ (\theta, y) \in W, \end{cases} \tag{12}$$

where

$$W = \{(\theta, y) \in \mathcal{O} \times [\alpha(.), \beta(.)] \mid \chi_\omega K^*\theta - y = 0\}$$

To the problem (12) we associate the Lagrangian L defined by
$L(\theta, y, \lambda) = \|KK^*\theta - z\|^2_{\mathcal{O}} + \langle\lambda, \chi_\omega K^*\theta - y\rangle$ for $(\theta, y, \lambda) \in \mathcal{O} \times [\alpha(.), \beta(.)] \times L^2(\omega)$.
Let us recall that $(\theta^*, y^*, \lambda^*)$ is a saddle point of L if

$$\max_{\lambda \in L^2(\omega)} L(\theta^*, y^*, \lambda) = L(\theta^*, y^*, \lambda^*) = \min_{\substack{\theta \in \mathcal{O} \\ y \in [\alpha(.), \beta(.)]}} L(\theta, y, \lambda^*)$$

- The set $\mathcal{O} \times [\alpha(.), \beta(.)]$ is nonempty, closed and convex, moreover the function $\lambda \to L(\theta, y, \lambda)$ is concave, upper semi-continuous and differentiable. The same $(\theta, y) \to L(\theta, y, \lambda)$ is convex, lower semi-continuous and differentiable. Morover,

$$\exists \lambda_0 \in L^2(\omega) \text{ such that } \lim_{\|(\theta,y)\| \to +\infty} L(\theta, y, \lambda_0) = +\infty$$

and

$$\exists (\theta_0, y_0) \in \mathcal{O} \times [\alpha(.), \beta(.)] \text{ such that } \lim_{\|\lambda\| \to +\infty} L(\theta_0, y_0, \lambda) = -\infty$$

This shows that L admits a saddle point.

- Assume that $(\theta^*, y^*, \lambda^*)$ is a saddle point of L and show that $y^* = \chi_\omega K^* \theta^*$ is the restriction in ω of the solution of (8) for all the states which are of the form $K^*\theta$ with $\theta \in \mathcal{O}$. We have

$$L(\theta^*, y^*, \lambda) \leq L(\theta^*, y^*, \lambda^*) \leq L(\theta, y, \lambda^*), \ \forall (\theta, y, \lambda) \in \mathcal{O} \times [\alpha(.), \beta(.)] \times L^2(\omega)$$

From the first inequality, we have $\forall \lambda \in L^2(\omega)$

$$\|KK^*\theta^* - z\|^2_{\mathcal{O}} + \langle \lambda, \chi_\omega K^*\theta^* - y^* \rangle \leq \|KK^*\theta^* - z\|^2_{\mathcal{O}} + \langle \lambda^*, \chi_\omega K^*\theta^* - y^* \rangle$$

So

$$\langle \lambda, \chi_\omega K^*\theta^* - y^* \rangle \leq \langle \lambda^*, \chi_\omega K^*\theta^* - y^* \rangle, \quad \forall \lambda \in L^2(\omega)$$

Which implies that $\chi_\omega K^*\theta^* = y^*$, hence $\chi_\omega K^*\theta^* \in [\alpha(.), \beta(.)]$.
From the second inequality it follows that

$$L(\theta^*, y^*, \lambda^*) \leq L(\theta, y, \lambda^*), \forall (\theta, y) \in \mathcal{O} \times [\alpha(.), \beta(.)]$$

This means that $\forall (\theta, y) \in \mathcal{O} \times [\alpha(.), \beta(.)]$

$$\|KK^*\theta^* - z\|^2_{\mathcal{O}} + \langle \lambda^*, \chi_\omega K^*\theta^* - y^* \rangle \leq \|KK^*\theta - z\|^2_{\mathcal{O}} + \langle \lambda^*, \chi_\omega K^*\theta - y \rangle$$

Since $y^* = \chi_\omega K^*\theta^*$ we have

$$\|KK^*\theta^* - z\|^2_{\mathcal{O}} \leq \|KK^*\theta - z\|^2_{\mathcal{O}} + \langle \lambda^*, \chi_\omega K^*\theta - y \rangle, \forall (\theta, y) \in \mathcal{O} \times [\alpha(.), \beta(.)]$$

Taking $y = \chi_\omega K^*\theta \in [\alpha(.), \beta(.)]$, we obtain

$$\|KK^*\theta^* - z\|^2_{\mathcal{O}} \leq \|KK^*\theta - z\|^2_{\mathcal{O}}$$

which implies that θ^* is a solution of (10), and so $y_0^* = K^*\theta^*$ whose the restriction $y^* = \chi_\omega K^*\theta^*$ is solution of (8) for all the states which are of the form $K^*\theta$ with $\theta \in \mathcal{O}$.

- $(\theta^*, y^*, \lambda^*)$ is a saddle point of L if the following assumptions hold

$$2\langle KK^*\theta^* - z, KK^*(\theta - \theta^*)\rangle + \langle \lambda^*, \chi_\omega K^*(\theta - \theta^*)\rangle = 0, \quad \forall\theta \in \mathcal{O} \tag{13}$$

$$-\langle \lambda^*, y - y^*\rangle \geq 0, \quad \forall y \in [\alpha(.), \beta(.)] \tag{14}$$

$$\langle \lambda - \lambda^*, \chi_\omega K^*\theta^* - y^*\rangle = 0, \quad \forall\lambda \in L^2(\omega) \tag{15}$$

From (13) we have
$2\langle KK^*\theta^* - z, KK^*(\theta - \theta^*)\rangle + \langle \lambda^*, \chi_\omega K^*(\theta - \theta^*)\rangle = 0, \forall\theta \in \mathcal{O}$
so
$\langle 2(KK^*)^*(KK^*\theta^* - z), (\theta - \theta^*)\rangle + \langle (\chi_\omega K^*)^*\lambda^*, (\theta - \theta^*)\rangle = 0, \forall\theta \in \mathcal{O}$
then $-2(KK^*)^*KK^*\theta^* + 2(KK^*)^*z = (\chi_\omega K^*)^*\lambda^*$
we assume that the system is observable in Ω, then KK^*KK^* is invertible, and

$$\theta^* = (KK^*KK^*)^{-1}KK^*z - \frac{1}{2}(KK^*KK^*)^{-1}K\chi_\omega^*\lambda^*$$

so y^* is given by
$y^* = \chi_\omega K^*(KK^*KK^*)^{-1}KK^*z - \frac{1}{2}\chi_\omega K^*(KK^*KK^*)^{-1}K\chi_\omega^*\lambda^*$
then

$$y^* = R_\omega K^*z - \frac{1}{2}R_\omega\chi_\omega^*\lambda^*$$

with $R_\omega = \chi_\omega K^*(KK^*KK^*)^{-1}K$ and using (14), we have

$$-\langle \lambda^*, y - y^*\rangle \geq 0, \forall y \in [\alpha(.), \beta(.)]$$

so $\langle (\rho\lambda^* + y^*) - y^*, y - y^*\rangle \leq 0, \forall y \in [\alpha(.), \beta(.)]$ and $\forall\rho > 0$
then

$$y^* = P_{[\alpha(.),\beta(.)]}(\rho\lambda^* + y^*)$$

Corollary 4.2 *If the system (6) together with the output (7) is exactly observable in Ω and the function*

$$L_\omega = [(K\chi_\omega^*)^*K\chi_\omega^*]^{-1}(K\chi_\omega^*)^*KK^*KK^*[(\chi_\omega K^*)^*\chi_\omega K^*]^{-1}(\chi_\omega K^*)^*$$

is coercive, then for ρ convenably chosen, the system (11) has a unique solution (λ^*, y^*).

Proof. We have

$$y^* = \chi_\omega K^*(KK^*KK^*)^{-1}KK^*z - \frac{1}{2}\chi_\omega K^*(KK^*KK^*)^{-1}K\chi_\omega^*\lambda^*$$

then

$$\lambda^* = -2L_\omega y^* + 2[(K\chi_\omega^*)^*K\chi_\omega^*]^{-1}(K\chi_\omega^*)^*KK^*z$$

So if $(\theta^*, y^*, \lambda^*)$ is a saddle point of L then the system (11) is equivalent to

$$\begin{cases} \lambda^* = -2L_\omega y^* + 2[(K\chi^*_\omega)^* K\chi^*_\omega]^{-1}(K\chi^*_\omega)^* KK^* z \\ y^* = P_{[\alpha(.),\beta(.)]}(-2\rho L_\omega y^* + 2\rho[(K\chi^*_\omega)^* K\chi^*_\omega]^{-1}(K\chi^*_\omega)^* KK^* z + y^*) \end{cases}$$

It follows that y^* is a fixed point of the function

$$\begin{array}{rcl} F_\rho : [\alpha(.),\beta(.)] & \rightarrow & [\alpha(.),\beta(.)] \\ y & \rightarrow & P_{[\alpha(.),\beta(.)]}(-2\rho L_\omega y + 2\rho[(K\chi^*_\omega)^* K\chi^*_\omega]^{-1}(K\chi^*_\omega)^* KK^* z + y) \end{array}$$

The operator L_ω is coercive, i.e.

$$\exists m > 0 \text{ such that } \langle L_\omega y, y\rangle \geq m\|y\|^2 \quad \forall y \in L^2(\omega)$$

It follows that $\forall t_1, t_2 \in [\alpha(.),\beta(.)]$

$$\begin{array}{rcl} \|F_\rho(t_2) - F_\rho(t_1)\|^2 & \leq & \| - 2\rho L_\omega(t_2 - t_1) + (t_2 - t_1)\|^2 \\ & \leq & 4\rho^2\|L_\omega\|^2\|t_2 - t_1\|^2 + \|t_2 - t_1\|^2 - 4\rho\langle L_\omega(t_2 - t_1), (t_2 - t_1)\rangle \\ & \leq & 4\rho^2\|L_\omega\|^2\|t_2 - t_1\|^2 + \|t_2 - t_1\|^2 - 4\rho m\|t_2 - t_1\|^2 \\ & \leq & (4\rho^2\|L_\omega\|^2 + 1 - 4\rho m)\|t_2 - t_1\|^2 \end{array}$$

and we deduce that if

$$\rho < \frac{m}{\|L_\omega\|^2}$$

then F_ρ is a contractor map, which implies the uniqueness of y^* and λ^*.

4.1 Numerical approach

From proposition (4.1) it follows that the solution of the problem (8) arises to compute the saddle points of L, which is equivalent to solving the problem

$$\inf_{(\theta,y)\in L^2(0,T,I\!R^q)\times[\alpha(.),\beta(.)]} \left(\sup_{\lambda\in L^2(\omega)} L(\theta, y, \lambda) \right)$$

To achieve this we use an algorithm of Uzawa type ([9]).
Let $T = KK^*KK^*$, if we choose two functions $(y^*_0, \lambda^*_1) \in [\alpha(.),\beta(.)] \times L^2(\omega)$ and

$$\theta^*_n = T^{-1}KK^*z - \frac{1}{2}T^{-1}K\chi^*_\omega\lambda^*_n, \quad y^*_n = P_{[\alpha(.),\beta(.)]}(\rho\lambda^*_n + y^*_{n-1})$$

and

$$\lambda^*_{n+1} = \lambda^*_n + (\chi_\omega K^*\theta^*_n - y^*_n)$$

then we obtain the following algorithm

Step 1 :The initial state y_0^*, the subregion ω, the location of the sensor D, the function of measure distribution f.

- Choose the function $y_0^* \in L^2(\omega)$ and $\lambda_1^* \in L^2(\omega)$.
- Threshold accuracy ε.

Step 2 : Repeat

- ▶ Solve $T(\theta_n^*) = KK^*z - \dfrac{1}{2}K\chi_\omega^*\lambda_n^*$, $n \geq 1$.
- ▶ Calculate $y_n^* = P_{[\alpha(.),\beta(.)]}(\rho\lambda_n^* + y_{n-1}^*)$, $n \geq 1$.
- ▶ Calculate $\lambda_{n+1}^* = \lambda_n^* + (\chi_\omega K^*\theta_n^* - y_n^*), n \geq 1$.

Until $||y_{n+1}^* - y_n^*||_{L^2(\omega)} \leq \varepsilon$.

Step 3 : The functions θ_n^* and y_n^* lead to the initial state y^* to be reconstructed in ω.

4.2 Simulation results

In this section we give a numerical example that leads to results related to the choice of the subregion, the initial condition and the sensor location. In $\Omega =]0,1[$, consider the one dimensional system

$$\begin{cases} \dfrac{\partial y(x,t)}{\partial t} = \dfrac{\partial^2 y(x,t)}{\partial x^2} & \Omega\times]0,T[\\ y(0,t) = y(1,t) = 0 &]0,T[\\ y(x,0) = y_0(x) & \Omega \end{cases} \tag{16}$$

augmented with the output function

$$z(t) = y(b,t); \quad b \in \Omega \tag{17}$$

The initial state to be reconstructed is $y_0(x) = (x^2(x-1)^2 - 2x(x-1))/2$.
Let $\alpha(x) = x^2(x-1)^2$, $\beta(x) = -2x(x-1)$. Applying the previous algorithm, we obtain

Global case $\omega = \Omega$

- If the sensor is located in $b = 0.52$, we obtain the following result

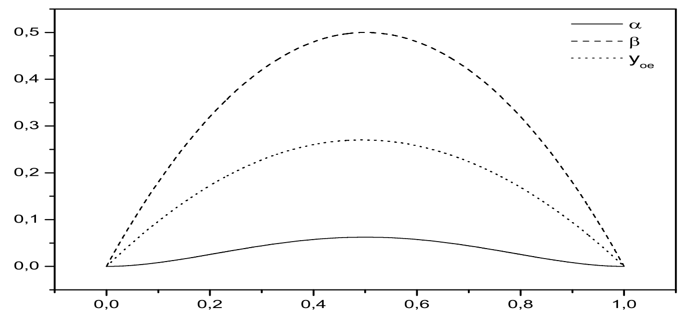

Figure 1 : The estimated initial state, $\alpha(.)$ and $\beta(.)$ in ω.

We note that the estimated state is between $\alpha(.)$ and $\beta(.)$ and then the sensor is $[\alpha(.), \beta(.)]$-strategic in ω.

- If the sensor is located in $b = 0.26$ then we have

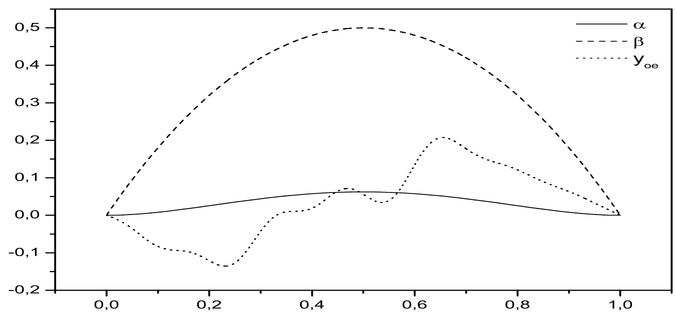

Figure 2 : The estimated initial state, $\alpha(.)$ and $\beta(.)$ in ω.

Figs. 1 and 2 show that if the sensor is $[\alpha(.), \beta(.)]$-strategic in Ω then the estimated initial state is between $\alpha(.)$ and $\beta(.)$ in the whole domain Ω.

Regional case $\omega =]0.38, 0.62[$

- If the sensor is located in $b = 0.49$ then we obtain

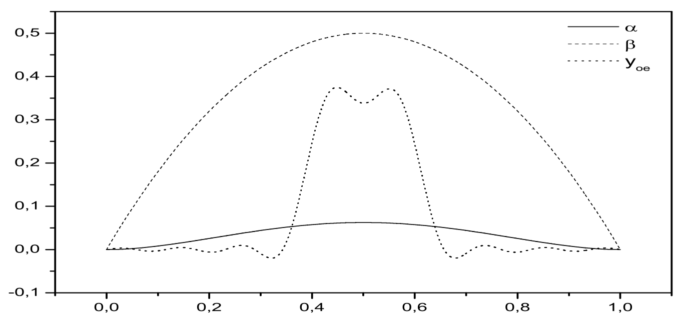

Figure 3 : The estimated initial state, $\alpha(.)$ and $\beta(.)$ in ω.

- If the sensor is located in $b = 0.01$ then we obtain

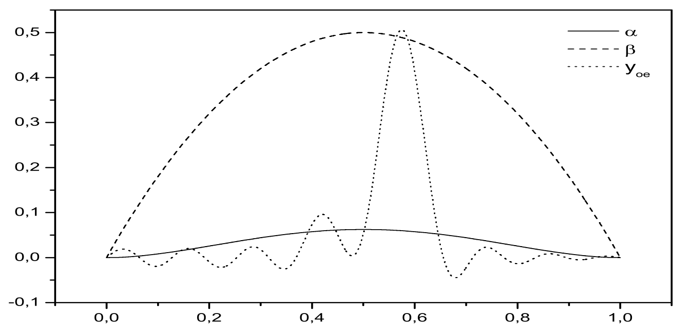

Figure 4 : The estimated initial state, $\alpha(.)$ and $\beta(.)$ in ω.

Figs. 3 and 4 show that if the sensor $(b, \delta(b-\cdot))$ is $[\alpha(.), \beta(.)]$-strategic in ω then the estimated initial state is between $\alpha(.)$ and $\beta(.)$ in the subregion ω. The estimated initial state is obtained with reconstruction error $\varepsilon = 5.20 \times 10^{-6}$.

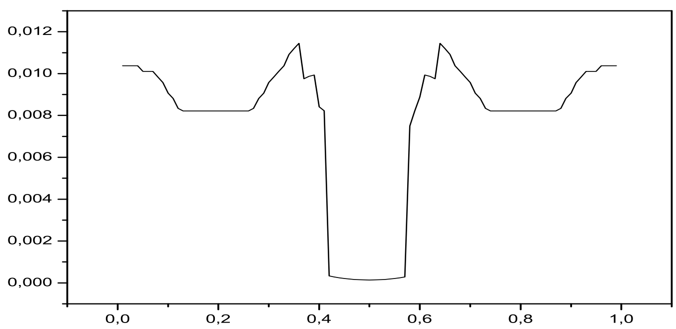

Figure 5 : The reconstruction error with respect to the sensor location b.

We note that there exists the best location of the sensor allowing a good reconstruction of the initial state.

5 Conclusion

The constrained observability of parabolic system is considered. The developed methods, based on regional observability tools in connection with Lagrangian and subdifferential techniques, lead to a computational algorithm which is illustrated by numerical example and simulations. The case where ω is a part of the boundary of the system evolution domain is under consideration and the work will be the subject of the future paper.

References

[1] R. F. Curtain and H. Zwart, An Introduction to Infinite Dimensional Linear Systems Theory, Texts in Applied Mathematics, Springer-Verlag, 21, New York, 1995.

[2] R. F. Curtain and A. J. Pritchard, Infinite Dimensional Linear Systems Theory, Springer-Verlag, Berlin, 1978.

[3] E. Zerrik, Analyse régionale des systèmes distribués, PhD thesis Mohamed V University, Rabat, 1993.

[4] E. Zerrik, A. Boutoulout and A. El Jai, Actuators and regional boundary controllability of parabolic systems, International Journal of Systems Science, 31(2000), 73-82.

[5] M. Amouroux, A. El Jai and E. Zerrik, Regional observabiliy of distributed systems, International Journal of Systems Science, 25(1994), 301-313.

[6] E. Zerrik and F. Ghafrani, Minimum energy control subject to output constraints: numerical approach, IEE Proceedings-Control Theory and Application, 149(2002), 105-110.

[7] A. El Jai and A. J. Pritchard, Sensors and Actuators in Distributed Systems Analysis, Ellis Horwood series in Applied Mathematics, J. Wiley, 1988.

[8] J.P. Aubin, L'analyse non linéaire et ses motivations économiques, Masson, Paris, 1984.

[9] M. Fortin and R. Glowinski, Méthodes de Lagrangien Augmenté. Applications à la résolution numérique des problèmes aux limites, Dunod, Paris, 1982.

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