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Newton Polygons of Polynomial Ordinary Differential Equations

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Abstract

In this paper we show some properties of the Newton polygon of a polynomial ordinary differential equation. We give the relation between the Newton polygons of a differential polynomial and its partial derivatives. Newton polygons of evaluations of differential polynomials are also described.

1 Introduction

Newton polygons construct a useful tool for solving polynomial ordinary differential equations. Earlier, Briot and Bouquet [1] used Newton polygon methods for solving first order and first degree ordinary differential equations. Recently, Grigoriev and Singer [8], Cano [2, 3] and Della Dora et. al. [6] described Newton polygon algorithms for computing Puiseux series and series with real exponents as solutions of polynomial differential equations. Newton polygons are also useful for computing Newton-Puiseux expansions for the roots of polynomials and for factoring polynomials over fields of formal power series [4, 5].

In this paper, we describe some useful properties of Newton polygons of differential polynomials. Section 2 gives the definition of the Newton polygon of a polynomial ordinary differential equation. Section 3 describes the relation between the Newton polygons of a differential polynomial and its partial derivatives. Section 4 establishes Newton polygon of different evaluations of differential polynomials.

Let K be a field and $\overline{K}$ be an algebraic closure of K. Let L and $\mathcal{L}$ be the two following fields:

$$L = \cup_{\nu \in \mathbb{N}^*} K((x^{\frac{1}{\nu}})), \quad \mathcal{L} = \cup_{\nu \in \mathbb{N}^*} \overline{K}((x^{\frac{1}{\nu}}))$$

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which are the fields of fraction-power series of x over K (resp. $\overline{K}$), i.e., the fields of Puiseux series of x with coefficients in K (resp. $\overline{K}$). Each element $\psi \in L$ (resp. $\psi \in \mathcal{L}$) can be represented in the form $\psi = \sum_{i\in\mathbb{Q}} c_i x^i, \quad c_i \in K$ (resp. $c_i \in \overline{K}$). The order of ψ is defined by $ord(\psi) := \min\{i \in \mathbb{Q}, c_i \neq 0\}$. The fields L and $\mathcal{L}$ are differential fields with the differentiation operator

$$\frac{d}{dx}(\psi) = \sum_{i\in\mathbb{Q}} i c_i x^{i-1}.$$

Let $y_0, \dots, y_n$ be new variables algebraically independent over K and $F(y_0, \dots, y_n)$ be a polynomial in the variables $y_0, \dots, y_n$ with coefficients in L. This polynomial defines an ordinary differential equation $F(y, \frac{dy}{dx}, \dots, \frac{d^n y}{dx}) = 0$ which will be denoted by $F(y) = 0$. We can write F in the form:

$$F = \sum_{i\in\mathbb{Q},\alpha\in A} f_{i,\alpha} x^i y_0^{\alpha_0} \cdots y_n^{\alpha_n}, \quad f_{i,\alpha} \in K$$

where $\alpha = (\alpha_0, \dots, \alpha_n)$ belongs to a finite subset A of $\mathbb{N}^{n+1}$. The order of F is defined by $ord(F) := \min\{i \in \mathbb{Q}; f_{i,\alpha} \neq 0 \text{ for a certain } \alpha\}$. We define the degree of F w.r.t. x by $\deg_x(F) = \max\{i \in \mathbb{Q};\ f_{i,\alpha} \neq 0 \text{ for a certain } \alpha\}$ (it can be equal to $+\infty$).

2 Newton polygons of polynomial differential equations

We define the Newton polygon of F as follow. For every couple $(i, \alpha) \in \mathbb{Q} \times A$ such that $f_{i,\alpha} \neq 0$ (i.e., every existing term in F), we mark the point

$$P_{i,\alpha} := (i - \alpha_1 - 2\alpha_2 - \cdots - n\alpha_n, \alpha_0 + \alpha_1 + \cdots + \alpha_n) \in \mathbb{Q} \times \mathbb{N}.$$

We denote by $P(F)$ the set of all the points $P_{i,\alpha}$. The convex hull of these points and the point $(+\infty, 0)$ in the plane $\mathbb{R}^2$ is denoted by $\mathcal{N}(F)$ and is called the Newton polygon of the differential equation $F(y) = 0$ in the neighborhood of $x = 0$. If $\deg_{y_0,\dots,y_n}(F) = m$, then $\mathcal{N}(F)$ is located between the two horizontal lines $y = 0$ and $y = m$. For each $(a, b) \in \mathbb{Q}^2 \setminus \{(0, 0)\}$, we define the set

$$N(F, a, b) := \{(u, v) \in P(F),\ \forall (u', v') \in P(F), \quad au' + bv' \geq au + bv\}.$$

A point $P_{i,\alpha} \in P(F)$ is a vertex of the Newton polygon $\mathcal{N}(F)$ if there exist $(a, b) \in \mathbb{Q}^2 \setminus \{(0, 0)\}$ such that $N(F, a, b) = \{P_{i,\alpha}\}$. We remark that $\mathcal{N}(F)$ has a finite number of vertices. A pair of different vertices $e = (P_{i,\alpha}, P_{i',\alpha'})$ forms an edge of $\mathcal{N}(F)$ if there exist $(a, b) \in \mathbb{Q}^2 \setminus \{(0, 0)\}$ such that $e \subset N(F, a, b)$. We denote by $E(F)$ (resp. $V(F)$) the set of all the edges e (resp. all the vertices

p) of $\mathcal{N}(F)$ for which $a > 0$ and $b \geq 0$ in the previous definitions. It is easy to prove that if $e \in E(F)$, then there exists a unique pair $(a(e), b(e)) \in \mathbb{Z}^2$ such that $GCD(a(e), b(e)) = 1,\ a(e) > 0,\ b(e) \geq 0$ and $e \subset N(F, a(e), b(e))$ where GCD is an abbreviation of "Greatest Common Divisor". By the inclination of a line we mean the negative inverse of its geometric slope. If $e \in E(F)$, we can prove that the fraction $\mu_e = \frac{b(e)}{a(e)} \in \mathbb{Q}$ is the inclination of the straight line passing through the edge e. If $p \in V(F)$ and $N(F, a, b) = \{p\}$ for a certain $(a, b) \in \mathbb{Q}^2 \setminus \{(0, 0)\}$, then the fraction $\mu = \frac{b}{a} \in \mathbb{Q}$ is the inclination of a straight line which intersects $\mathcal{N}(F)$ exactly in the vertex p.

For each edge $e \in E(F)$, we define the univariate polynomial (in a new variable Z)

$$H_{(F,e)}(Z) = \sum_{P_{i,\alpha} \in N(F, a(e), b(e))} f_{i,\alpha} Z^{\alpha_0 + \alpha_1 + \cdots + \alpha_n} (\mu_e)_1^{\alpha_1} \cdots (\mu_e)_n^{\alpha_n} \in K[Z],$$

where $(\mu_e)_k := \mu_e(\mu_e - 1) \cdots (\mu_e - k + 1)$ for any positive integer k. We call $H_{(F,e)}(Z)$ the *characteristic* polynomial of F associated to the edge $e \in E(F)$. Its degree is at most $m = \deg_{y_0,\ldots,y_n}(F)$.

If $\psi \in \mathcal{L}$ is a solution of the differential equation $F(y) = 0$ such that $ord(\psi) = \mu$, i.e., ψ has the form $\psi = \sum_{i \in \mathbb{Q},\, i \geq \mu} c_i x^i$, $c_i \in \overline{K}$, then there exists an edge $e \in E(F)$ such that $\mu_e = \mu$ and $H_{(F,e)}(c_\mu) = 0$, i.e. c_μ is a root of the polynomial $H_{(F,e)}$ in $\overline{K}$. This condition is called a necessary initial condition to have a solution of $F(y) = 0$ in the form of ψ (see Lemma 1 of [3]). Namely, $H_{(F,e)}(c_\mu)$ is equal to the coefficient of the lowest term in the expansion of $F(\psi(x))$ with indeterminates μ and c_μ.

For each vertex $p = (u, v) \in V(F)$, let $\mu_1 < \mu_2$ be the inclinations of the adjacent edges at p in $\mathcal{N}(F)$. It is easy to prove that for all rational number $\mu = \frac{b}{a},\ a \in \mathbb{N}^*,\ b \in \mathbb{N}$ such that $N(F, a, b) = \{p\}$, we have $\mu_1 < \mu < \mu_2$. We associate to p the polynomial

$$h_{(F,p)}(\mu) = \sum_{P_{i,\alpha} = p} f_{i,\alpha} (\mu)_1^{\alpha_1} \cdots (\mu)_n^{\alpha_n} \in K[\mu],$$

which is called the *indicial* polynomial of F associated to the vertex p (here μ is considered as an indeterminate). Let $H_{(F,p)}(Z) = Z^v h_{(F,p)}(\mu)$ defined as above for edges $e \in E(F)$.

Remark 2.1 *Let $p = (u, v) \in V(F)$ and e be the edge of $\mathcal{N}(F)$ descending from p, then $h_{(F,p)}(\mu_e)$ is the coefficient of the monomial Z^v in the expansion of the characteristic polynomial of F associated to e.*

3 Newton polygons of partial derivatives of differential polynomials

Write F in the form $F = F_0 + \cdots + F_m$ where $m = \deg_{y_0,\dots,y_n}(F)$ and $F_s = \sum_{i\in\mathbb{Q},\,|\alpha|=s} f_{i,\alpha}x^i y_0^{\alpha_0}\cdots y_n^{\alpha_n}$ is the homogeneous part of F of degree s with respect to the indeterminates $y_0,\dots,y_n$, $\alpha = (\alpha_0,\dots,\alpha_n) \in A$ and $|\alpha| = \alpha_0 + \cdots + \alpha_n$ is the norm of α. Then the ordinate of any point of $P(F_s)$ is equal to s and

$$P(F) = \cup_{0\le s\le m} P(F_s).$$

Let $0 \le j \le n$. If there exists an integer $k \ge 1$ such that for all $1 \le s \le m$ (such that $F_s \ne 0$), $D_{s,j} := \deg_{y_j}(F_s) \ge k$ then we can easily prove that $P(\frac{\partial^k F}{\partial y_j^k})$ is the translation of $P(F)$ defined by the point $(kj,-k)$, i.e.,

$$P\Big(\frac{\partial^k F}{\partial y_j^k}\Big) = P(F) + \{(kj,-k)\} \text{ and then } \mathcal{N}\Big(\frac{\partial^k F}{\partial y_j^k}\Big) = \mathcal{N}(F) + \{(kj,-k)\}.$$

For any $(a,b) \in \mathbb{Q}^2 \setminus \{(0,0)\}$, we have

$$N\Big(\frac{\partial^k F}{\partial y_j^k}, a, b\Big) = N(F,a,b) + \{(kj,-k)\}.$$

Thus the edges of $\mathcal{N}\Big(\frac{\partial^k F}{\partial y_j^k}\Big)$ are exactly the translation of those of $\mathcal{N}(F)$. For each $e \in E\Big(\frac{\partial^k F}{\partial y_j^k}\Big)$, its *characteristic* polynomial is

$$H_{(\frac{\partial^k F}{\partial y_j^k},e)}(Z) = \sum_{P_{i,\alpha}\in N(F,a(e),b(e))} f_{i,\alpha} Z^{\alpha_0+\alpha_1+\cdots+\alpha_n-k}(\alpha_j)_k(\mu_e)_1^{\alpha_1}\cdots(\mu_e)_j^{\alpha_j-k}\cdots(\mu_e)_n^{\alpha_n} \in K[Z].$$

For each $p \in V\Big(\frac{\partial^k F}{\partial y_j^k}\Big)$, its *indicial* polynomial is

$$h_{(\frac{\partial^k F}{\partial y_j^k},p)}(\mu) = \sum_{P_{i,\alpha}=p} f_{i,\alpha}(\alpha_j)_k(\mu_e)_1^{\alpha_1}\cdots(\mu_e)_j^{\alpha_j-k}\cdots(\mu_e)_n^{\alpha_n} \in K[\mu].$$

Let $0 \le j_1 \ne j_2 \le n$. If there exist integers $k_1, k_2 \ge 1$ such that for all $1 \le s \le m$, $D_{s,j_2} \ge k_2$ and $\deg_{y_{j_1}}\Big(\frac{\partial^{k_2} F_s}{\partial y_{j_2}^{k_2}}\Big) \ge k_1$ then

$$P\Big(\frac{\partial^{k_1+k_2} F}{\partial y_{j_1}^{k_1} y_{j_2}^{k_2}}\Big) = P(F) + \{(k_1j_1 + k_2j_2, -k_1-k_2)\}$$

and then

$$\mathcal{N}\Big(\frac{\partial^{k_1+k_2} F}{\partial y_{j_1}^{k_1} y_{j_2}^{k_2}}\Big) = \mathcal{N}(F) + \{(k_1j_1 + k_2j_2, -k_1-k_2)\}.$$

For any $(a,b) \in \mathbb{Q}^2 \setminus \{(0,0)\}$, we have

$$N\Big(\frac{\partial^{k_1+k_2}F}{\partial y_{j_1}^{k_1} y_{j_2}^{k_2}}\Big) = N(F,a,b) + \{(k_1 j_1 + k_2 j_2, -k_1 - k_2)\}.$$

For each $e \in E\Big(\frac{\partial^{k_1+k_2}F}{\partial y_{j_1}^{k_1} y_{j_2}^{k_2}}\Big)$, its *characteristic* polynomial is $H_{(\frac{\partial^{k_1+k_2}F}{\partial y_{j_1}^{k_1} y_{j_2}^{k_2}},e)}(Z) =$

$$\sum_{P_{i,\alpha}\in N(F,a(e),b(e))} f_{i,\alpha} Z^{\alpha_0+\alpha_1+\cdots+\alpha_n-k_1-k_2} (\alpha_{j_1})_{k_1} (\alpha_{j_2})_{k_2} (\mu_e)_1^{\alpha_1} \cdots (\mu_e)_{j_1}^{\alpha_{j_1}-k_1} \cdots (\mu_e)_{j_2}^{\alpha_{j_2}-k_2} \cdots (\mu_e)_n^{\alpha_n}.$$

Theorem 3.1 *Let $k \geq 1$ be an integer such that for all $1 \leq s \leq m$ (such that $F_s \neq 0$) and for all $0 \leq j \leq n$ we have $D_{s,j} \geq k$. For any $e \in E(F)$, the k-th derivative of $H_{(F,e)} \in K[Z]$ is given by the formula*

$$H^{(k)}_{(F,e)}(Z) = \sum_{0\leq k_0,\ldots,k_n\leq n} (\mu_e)_1^{k_1} \cdots (\mu_e)_n^{k_n} H_{(\frac{\partial^k F}{\partial y_0^{k_0}\cdots \partial y_n^{k_n}},e)}(Z).$$

where the sum ranges over all the partitions $(k_0,\ldots,k_n)$ of k, i.e., $k_0+\cdots+k_n = k$.

Proof. By induction on k taking into account the above discussion. □

4 Newton polygons of evaluations of differential polynomials

Let F be a differential polynomial as in Section 1. Let $0 \neq c \in \overline{K}$, $\mu \in \mathbb{Q}$ and $G(y) = F(cx^\mu + y)$ be the differential polynomial obtained from F by replacing y_k by $c(\mu)_k x^{\mu-k} + y_k$ for all $0 \leq k \leq n$ (where $(\mu)_0 := 1$). In this section, we will construct the Newton polygon of the differential equation $G(y) = 0$ for different values of c and μ.

For each differential monomial $m(y) = f_{i,\alpha} x^i y_0^{\alpha_0} \cdots y_n^{\alpha_n}$ of F with corresponding point $p \in P(F)$, compute

$$m(cx^\mu + y) = f_{i,\alpha} x^i (cx^\mu + y_0)^{\alpha_0} (c\mu x^{\mu-1} + y_1)^{\alpha_1} \cdots (c(\mu)_n x^{\mu-n} + y_n)^{\alpha_n}.$$

Remark that the corresponding points of the differential monomials of $m(cx^\mu + y)$ have ordinate less or equal than $s = \alpha_0 + \alpha_1 + \cdots + \alpha_n$ and lie in the line passing through p with inclination μ. There are two possibilities for μ:

Theorem 4.1 *If $\mu = \mu_e$ is the inclination of an edge $e \in E(F)$. For any $0 \leq s \leq m = \deg_{y_0,\ldots,y_n}(F)$, the vertex of $\mathcal{N}(G)$ of ordinate s corresponds to the differential monomial of G with coefficient equals to*

$$q_s(c, \mu_e) := \sum_{0\leq k_0\leq\cdots\leq k_n\leq n} \frac{1}{k_0!\cdots k_n!} H_{\left(\frac{\partial^s F}{\partial y_0^{k_0}\cdots\partial y_n^{k_n}}, e\right)}(c).$$

where the sum ranges over all the partitions $(k_0, \ldots, k_n)$ of s i.e., $k_0 + \cdots + k_n = s$. Its x-coordinate is the minimum of the quantities $i + \mu(\alpha_0 + \cdots + \alpha_n - s) - \alpha_1 - 2\alpha_2 - \cdots - n\alpha_n$ for $i \in \mathbb{Q}$ and $\alpha \in A$.
If $\mu_1 < \mu < \mu_2$ where μ_1 and μ_2 are the inclinations of the two adjacent edges of a vertex $p = (u, v) \in V(F)$, then for any $0 \leq s \leq m$, the vertex of $\mathcal{N}(G)$ of ordinate s corresponds to the differential monomial of G with coefficient equals to

$$c^{v-s} \sum_{0\leq k_0\leq\cdots\leq k_n\leq n} \frac{1}{k_0!\cdots k_n!} h_{\left(\frac{\partial^s F}{\partial y_0^{k_0}\cdots\partial y_n^{k_n}}, p\right)}(\mu).$$

where the sum ranges over all the partitions $(k_0, \ldots, k_n)$ of s.

Proof. Let $\mu = \mu_e$ for $e \in E(F)$ and compute

$$G(y) = \sum_{i\in\mathbb{Q},\alpha\in A} f_{i,\alpha} x^i (cx^\mu + y_0)^{\alpha_0} (c\mu x^{\mu-1} + y_1)^{\alpha_1} \cdots (c(\mu)_n x^{\mu-n} + y_n)^{\alpha_n}.$$

For each $0 \leq s \leq m$, compute G_s the homogeneous part of G of degree s in $y_0, \ldots, y_n$. We remark that for fixed i and α, all the differential monomials of G_s have the same corresponding point which is

$$(i + \mu(\alpha_0 + \cdots + \alpha_n - s) - \alpha_1 - 2\alpha_2 - \cdots - n\alpha_n,\ s).$$

The x-coordinate of the vertex of $\mathcal{N}(G)$ of ordinate s is the minimum of the x-coordinates $i+\mu(\alpha_0+\cdots+\alpha_n-s)-\alpha_1-2\alpha_2-\cdots-n\alpha_n$ for $i \in \mathbb{Q}$ and $\alpha \in A$. This minimum is realized by the points $P_{i,\alpha} \in N(F, a(e), b(e))$. This proves the lemma taking into account the formula for the *characteristic* polynomial of the derivatives of F in Section 3 (see Theorem 3.1). □

The following Corollary is a generalization of Lemma 2.2 of [7] which deals with the Newton polygon of the *Riccatti* equation associated to a linear ordinary differential equation.

Corollary 4.2 *Let $\mu = \mu_e$ be the inclination of an edge $e \in E(F)$. The edges of $\mathcal{N}(G)$, situated above the edge e are the same as in $\mathcal{N}(F)$. Let an integer $s_1 \geq 0$ be such that $q_s(c, \mu_e) = 0$ for all $0 \leq s < s_1$ and $q_{s_1}(c, \mu_e) \neq 0$ then $\mathcal{N}(G)$ has a vertex of ordinate s_1 and it has at least s_1 edges with inclination greater than μ_e.*

Proof. Let p_{s_1} be the vertex of $\mathcal{N}(G)$ of ordinate s_1 and x-coordinate $x_{p_{s_1}}$ the minimum of the values $i+\mu(\alpha_0+\cdots+\alpha_n-s_1)-\alpha_1-2\alpha_2-\cdots-n\alpha_n$ for $i\in\mathbb{Q}$ and $\alpha\in A$ (by Theorem 4.1). We have $q_{s_1-1}(c,\mu_e)=0$, then the x-coordinate of the vertex p_{s_1-1} of ordinate s_1-1 is strictly less than the minimum of the values $i+\mu(\alpha_0+\cdots+\alpha_n-s_1+1)-\alpha_1-2\alpha_2-\cdots-n\alpha_n$ for $i\in\mathbb{Q}$ and $\alpha\in A$. Thus the inclination of the edge joining p_{s_1} and p_{s_1-1} is greater than μ. □

Corollary 4.3 *Let $\mu=\mu_e$ be the inclination of an edge $e\in E(F)$. If $H_{(F,e)}(c)=0$ then the intersection point of the straight line passing through e with the x-axis is not a vertex of $\mathcal{N}(G)$ and $\mathcal{N}(G)$ has an edge with inclination greater than μ_e.*

Proof. We have $q_0(c,\mu_e)=H_{(F,e)}(c)=0$, then $s_1\geq 1$. This proves the corollary by applying Corollary 4.2. □

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