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On Locally Holomorphic Automorphisms of the Rigid Cubic Hypersurface in $\mathbb{C}^2$

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Abstract

We determine the real-analytic infinitesimal CR automorphisms of the rigid cubic hypersurface in $\mathbb{C}^2$ near the origin, and the connected component containing the identity transformation of all locally holomorphic automorphisms of this hypersurface near the origin. Then we obtain the connected component of the unit of the group of meromorphic automorphisms of this hypersurface.

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1 Introduction

It is well-known that the set of all locally holomorphic automorphisms of the hyperquadric $S_{n+1} = \{(z, w) \in \mathbb{C}^n \times \mathbb{C} \mid \operatorname{Im} w = |z|^2\}$ is the group of $\mathrm{SU}(n+1, 1)$ given by fractional linear transformations. This group plays an important role in the study of spherical CR manifold (c.f. [17]). It is interesting and important to determine all locally holomorphic automorphisms of a real submanifold in $\mathbb{C}^n$ [4]. A criterion for the finite dimensionality of the automorphism group of a hypersurface was given by Stanton [15, 16]. Baouendi, Ebenfelt and Rothschild [6, 7] also studied the condition under which the Lie algebra of locally defined infinitesimal CR automorphisms of a real submanifold is finite dimensional. On the other hand, Beloshapka [2] obtained a description of the Lie algebra of infinitesimal automorphisms of any quadric. Shevchenko [14] constructed canonical forms for nondegenerate CR-quadrics of codimension two in a complex space and gave a complete description of the algebra of infinitesimal holomorphic automorphisms. Ežov and Schmalz realized arbitrary automorphism of a non-degenerate $(n, 2)$-quadric by a rational map of degree not more than two [9]. For higher codimension, it's known that each (3,3)-quadric possessing non-linear automorphisms is equivalent to one of eight quadrics [11, 12], whose automorphism groups were determined in [1].

For higher degree model surface, Beloshapka consider the surface Q_3 in the space $\mathbb{C}^n \oplus \mathbb{C}^{n_2} \oplus \mathbb{C}^k$ with coordinates $(z \in \mathbb{C}^n, w_2 \in \mathbb{C}^{n_2}, w_3 \in \mathbb{C}^k)$, $n > 0,\ k > 0$, given by the equations $\operatorname{Im} w_2 = \langle z, \overline{z}\rangle, \quad \operatorname{Im} w_3 = 2 \operatorname{Re} \Phi(z, \overline{z})$, where $\Phi(z, \overline{z})$ is a homogeneous $\mathbb{C}^k$-valued form of degree three, and gave the structure of the automorphism algebra of the cubic (see [4] and references therein). See [3], [13] for results for the polynomial models of even higher codimension and degree.

All models above are homogeneous, i.e. the holomorphic automorphisms act on them transitively. Moreover, these models can be given the structure of nilpotent groups. Kolář [10] gives a complete description of local automorphism groups for Levi degenerate hypersurfaces of finite type in $\mathbb{C}^2$. In [18], we obtain the connected component of locally holomorphic automorphisms of a class of non-homogeneous rigid hypersurfaces in $\mathbb{C}^{N+1}$. Here we consider the rigid cubic hypersurface in $\mathbb{C}^2$, which is not homogeneous and is not included in [18].

Let M be a real *rigid* hypersurface through the origin in $\mathbb{C}^{n+1}$, i.e. there are coordinates $(z_1, \cdots, z_n, w)$ such that M is given by an equation of the following form

$$\operatorname{Im} w = F(z, \overline{z}).$$

In this paper, we will consider the cubic rigid hypersurfaces of the form

$$M = \left\{(z, w) \in \mathbb{C}^2 \mid \operatorname{Im} w = P_3(z, \overline{z})\right\},$$

where $P_3(z, \overline{z})$ is a real homogeneous polynomial in $z, \overline{z}$ of degree 3. After a holomorphic change of coordinates if necessary, each M of such type can be written in the following form

$$\mathcal{C} = \left\{(z, w) \in \mathbb{C}^2 \mid \operatorname{Im} w = 2 \operatorname{Re} z^2 \overline{z}\right\}. \tag{1.1}$$

or $\{(z, w) \in \mathbb{C}^2 \mid \operatorname{Im} w = 0\}$. So it is sufficient to consider $\mathcal{C}$.

By *a germ at the origin of holomorphic automorphism of* $\mathcal{C}$ we mean a local biholomorphism of $\mathbb{C}^2$ defined in a neighborhood U of the origin that maps $U \cap \mathcal{C}$ into $\mathcal{C}$. We denote by $\operatorname{Aut}(\mathcal{C}, 0)$ the set of germs at the origin of holomorphic automorphisms of $\mathcal{C}$. Also denote by $\operatorname{hol}(\mathcal{C}, 0)$ the set of real-analytic infinitesimal CR automorphisms of $\mathcal{C}$ at the origin, i.e. $\operatorname{hol}(\mathcal{C}, 0)$ consists of all germs at the origin of vector fields X tangent to $\mathcal{C}$ such that the local 1-parameter group of transformations generated by X are biholomorphic transformations of $\mathbb{C}^2$ preserving $\mathcal{C}$. From Proposition 12.4.22 in [5], $\operatorname{hol}(\mathcal{C}, 0)$ can be written in the following form

$$\operatorname{hol}(\mathcal{C}, 0) = \left\{X(z, w) = 2 \operatorname{Re}\left(f(z, w)\frac{\partial}{\partial z} + g(z, w)\frac{\partial}{\partial w}\right)\right\}, \tag{1.2}$$

where X is tangent to $\mathcal{C}$, $f(z,w)$ and $g(z,w)$ are holomorphic functions near the origin. Denote by $\mathrm{Aut}_0\,\mathcal{C}$ the set of germs in $\mathrm{Aut}(\mathcal{C},0)$ preserving the origin, and by $\mathrm{hol}_0\,\mathcal{C}$ the set of vector fields in $\mathrm{hol}(\mathcal{C},0)$ vanishing at the origin.

In this paper, we obtain an explicit formula of $\mathrm{hol}(\mathcal{C},0)$ and also the connected component of the identity transformation of $\mathrm{Aut}(\mathcal{C},0)$, which is denoted by $\mathrm{Aut}_{\mathrm{id}}(\mathcal{C},0)$.

Theorem 1.1. *(I). Suppose* $X = 2\,\mathrm{Re}[f(z,w)\frac{\partial}{\partial z} + g(z,w)\frac{\partial}{\partial w}] \in \mathrm{hol}(\mathcal{C},0)$, *then locally in a neighborhood of the origin, the functions* f, g *can be written in the following form*

$$\begin{cases} f(z,w) &= \alpha_1 z + i\alpha_2, \\ g(z,w) &= 3\alpha_1 w + 2\alpha_2 z^2 + \alpha_3, \end{cases} \tag{1.3}$$

where $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$.
(II). $(F,G) \in \mathrm{Aut}_{\mathrm{id}}(\mathcal{C},0)$ *if nd only if functions* F *and* G *can be written in the following form*

$$\begin{cases} F(z,w) &= \lambda z + i\gamma_1, \\ G(z,w) &= \lambda^3 w + 2\lambda^2\gamma_1 z^2 + 2i\lambda\gamma_1^2 z + \gamma_2, \end{cases} \tag{1.4}$$

where $\lambda \in \mathbb{R}_+,\ \gamma_1, \gamma_2 \in \mathbb{R}$.

Obviously, $\mathrm{Aut}(\mathcal{C},0)$ is not a group. We denote by $\mathrm{Aut}^{\mathrm{m}}\,\mathcal{C}$ the set of all meromorphic automorphisms of $\mathcal{C}$, which is a large class of locally holomorphic automorphisms. It is a group under composition. Denote by $\mathrm{Aut}^{\mathrm{m}}_{\mathrm{id}}\,\mathcal{C}$ the connected component of the unit of $\mathrm{Aut}^{\mathrm{m}}\,\mathcal{C}$. We obtain the following description of $\mathrm{Aut}^{\mathrm{m}}_{\mathrm{id}}\,\mathcal{C}$.

Corollary 1.2. $(F,G) \in \mathrm{Aut}^{\mathrm{m}}_{\mathrm{id}}\,\mathcal{C}$ *if nd only if* F *and* G *are given by* (1.4).

In Section 2, we will prove Theorem 1.1 (I). In Section 3, we find the connected component of the identity transformation of $\mathrm{Aut}(\mathcal{C},0)$, and complete the proof of Theorem 1.1 and Corollary 1.1.

To give the complete description of all locally holomorphic automorphisms of the hypersurface $\mathcal{C}$, we need to consider locally holomorphic automorphisms near other point $(z_0,w_0) \in \mathcal{C}$, near which the surface can be transformed by some biholomorphic mapping into

$$\mathrm{Im}\, w = 4\,\mathrm{Re}\, z_0|z|^2 + 2\,\mathrm{Re}\, z^2\overline{z}, \tag{1.5}$$

with (z_0,w_0) into the origin. When $\mathrm{Re}\, z_0 \neq 0$, the right hand side of (1.5) is not homogeneous. We can also determine the real-analytic infinitesimal CR automorphisms of (1.5) near the origin, which is more complicated than that of $\mathcal{C}$ and will appear elsewhere.

2 Real-analytic infinitesimal CR automorphisms of $\mathcal{C}$

Any real cubic homogeneous polynomial $P_3(z,\overline{z})$ ($z \in \mathbb{C}$) can be written as

$$P_3(z,\overline{z}) = a_1(z^3+\overline{z}^3) + a_2(z^2\overline{z} + z\overline{z}^2),$$

where $a_1, a_2 \in \mathbb{R}$. The term $2a_1 \operatorname{Re} z^3$ in $\operatorname{Im} w = P_3(z,\overline{z})$ can be easily eliminated by the transformation $w \mapsto w + 2ia_1 z^3$. We may assume $a_2 \neq 0$, otherwise the resulting hyperplane $Q = \{(z,w) \mid \operatorname{Im} w = 0\}$ has an infinite-dimensional automorphism group. Then under a dilation, M in (1) can be transformed into the canonical form $\mathcal{C}$ in (1.1), which is defined by the equation

$$\rho(z,w,\overline{z},\overline{w}) = z^2\overline{z} + z\overline{z}^2 - \frac{w-\overline{w}}{2i} = 0. \tag{2.1}$$

Since any $X = 2\operatorname{Re}[f(z,w)\frac{\partial}{\partial z} + g(z,w)\frac{\partial}{\partial w}] \in \operatorname{hol}(\mathcal{C},0)$ is tangent to $\mathcal{C}$, i.e. $X\rho = 0$, we have

$$\operatorname{Re}\left[ig(z,u+2i\operatorname{Re} z^2\overline{z}) + 2(2|z|^2+\overline{z}^2)f(z,u+2i\operatorname{Re} z^2\overline{z})\right] = 0, \tag{2.2}$$

for $w = u + 2i\operatorname{Re} z^2\overline{z}$ and $(z,u) \in U$, where U is a small neighborhood of the origin in $\mathbb{C}\times\mathbb{R}$.

Proof of Theorem 1.1(I). The theorem is proved by solving equation (2.2) in the class of formal power series. This method was originally used by Beloshapka [4] for homogeneous models.

Let $X = 2\operatorname{Re}\left[f(z,w)\frac{\partial}{\partial z} + g(z,w)\frac{\partial}{\partial w}\right] \in \operatorname{hol}(\mathcal{C},0)$. By Taylor's expansion with respect to variable w at the point $w=u$, we have

$$\begin{aligned} f\left(z,u+2i\operatorname{Re} z^2\overline{z}\right) &= \sum_{m=0}^{\infty} f^{(m)}(z,u)\frac{i^m(z^2\overline{z}+z\overline{z}^2)^m}{m!}, \\ g\left(z,u+2i\operatorname{Re} z^2\overline{z}\right) &= \sum_{m=0}^{\infty} g^{(m)}(z,u)\frac{i^m(z^2\overline{z}+z\overline{z}^2)^m}{m!}, \end{aligned} \tag{2.3}$$

where $f^{(m)}(z,u)$, $g^{(m)}(z,u)$ indicate differentiation with respect to w. Since $f(z,u)$ and $g(z,u)$ are holomorphic in z, we can write

$$f(z,u) = \sum_{k=0}^{\infty} f_k(z,u), \qquad g(z,u) = \sum_{k=0}^{\infty} g_k(z,u), \tag{2.4}$$

where

$$f_k(tz,u) = t^k f_k(z,u), \qquad g_k(tz,u) = t^k g_k(z,u).$$

In fact, $f_k(z,u) = \frac{1}{k!}\frac{\partial^k f}{\partial z^k}(0,u)z^k$. Now substitute (2.3) and (2.4) into (2.2), we find that

$$
\begin{aligned}
0 = &\frac{i}{2}\sum_{k=0}^{\infty} g_k(z,u) + ig_k'(z,u)(z^2\overline{z}+z\overline{z}^2) - \frac{1}{2}g_k''(z,u)(z^4\overline{z}^2+2z^3\overline{z}^3+z^2\overline{z}^4)+\cdots\\
&-\frac{i}{2}\sum_{k=0}^{\infty} \overline{g_k(z,u)} - i\overline{g_k'(z,u)}(z^2\overline{z}+z\overline{z}^2) - \frac{1}{2}\overline{g_k''(z,u)}(z^4\overline{z}^2+2z^3\overline{z}^3+z^2\overline{z}^4)+\cdots\\
&+2|z|^2\sum_{k=0}^{\infty} f_k(z,u) + if_k'(z,u)(z^2\overline{z}+z\overline{z}^2) - \frac{1}{2}f_k''(z,u)(z^4\overline{z}^2+2z^3\overline{z}^3+z^2\overline{z}^4)+\cdots\\
&+2|z|^2\sum_{k=0}^{\infty} \overline{f_k(z,u)} - i\overline{f_k'(z,u)}(z^2\overline{z}+z\overline{z}^2) - \frac{1}{2}\overline{f_k''(z,u)}(z^4\overline{z}^2+2z^3\overline{z}^3+z^2\overline{z}^4)+\cdots\\
&+\overline{z}^2\sum_{k=0}^{\infty} f_k(z,u) + if_k'(z,u)(z^2\overline{z}+z\overline{z}^2) - \frac{1}{2}f_k''(z,u)(z^4\overline{z}^2+2z^3\overline{z}^3+z^2\overline{z}^4)+\cdots\\
&+z^2\sum_{k=0}^{\infty} \overline{f_k(z,u)} - i\overline{f_k'(z,u)}(z^2\overline{z}+z\overline{z}^2) - \frac{1}{2}\overline{f_k''(z,u)}(z^4\overline{z}^2+2z^3\overline{z}^3+z^2\overline{z}^4)+\cdots,
\end{aligned}
\tag{2.5}
$$

where the prime indicates differentiation with respect to w, and the dots denote the terms of $z^j\overline{z}^l$ with $l \geq 3$.

In the following, we call a term is *of type* (k,l) if it has the form $h(u)z^k\overline{z}^l$ for some function h. Now we collect terms of type (k,l). Firstly, we consider the terms of type $(k,0)$ in (2.5). Note that $g_k(z,u)$ and $f_k(z,u)$ are terms of type $(k,0)$ and $\overline{f_k(z,u)}$, $\overline{g_k(z,u)}$ are terms of type $(0,k)$. Consider terms of type $(2,0)$ for example. Since terms in the second to the fifth rows in (2.5) contain the factors $\overline{z}^l$ with $l \geq 1$, terms of type $(2,0)$ only appear in the first and the last row. Furthermore, in these rows, the terms concerning $f_j^{(k)}(z,u)$ and $g_j^{(k)}(z,u)$ $(k \geq 1)$ contain the the factors $\overline{z}^l$ with $l \geq 1$. So they only exist in the first summations in these two rows, i.e. $\frac{i}{2}g_2(z,u)$, and $\overline{f_0(z,u)}z^2$. So on the right hand side in equation (2.5):

$$
\begin{aligned}
&(0,0): \quad -\operatorname{Im} g_0(z,u), \quad (1,0): \quad \frac{i}{2}g_1(z,u),\\
&(2,0): \quad \frac{i}{2}g_2(z,u) + \overline{f_0(z,u)}z^2, \quad (k,0): \quad \frac{i}{2}g_k(z,u), \quad k>2.
\end{aligned}
\tag{2.6}
$$

So we have

$$
\operatorname{Im} g_0(z,u) = 0, \quad g_2(z,u) = 2i\overline{f_0(z,u)}z^2, \quad g_k(z,u) = 0 \ \ (k \neq 0,2). \tag{2.7}
$$

To determine $f_k(z,u)$, consider the terms of type $(k,1)$ $(k \geq 1)$ in (2.5):

$$\begin{aligned}
&(1,1): \ \ 4\,\mathrm{Re}\, f_0(z,u)|z|^2,\\
&(2,1): \ \ -\frac{1}{2}\left(g_0'(z,u)+\overline{g_0'(z,u)}\right)z^2\overline{z}+2f_1(z,u)|z|^2+\overline{f_1(z,u)}z^2,\\
&(3,1): \ \ -\frac{1}{2}g_1'(z,u)z^2\overline{z}+2f_2(z,u)|z|^2,\\
&(4,1): \ \ -\frac{1}{2}g_2'(z,u)z^2\overline{z}+2f_3(z,u)|z|^2-i\overline{f_0'(z,u)}z^4\overline{z},\\
&(k,1): \ \ -\frac{1}{2}g_{k-2}'(z,u)z^2\overline{z}+2f_{k-1}(z,u)|z|^2, \qquad (k>4).
\end{aligned} \tag{2.8}$$

Then (2.8) and (2.7) imply that

$$\begin{aligned}
&\mathrm{Re}\, f_0(z,u)=0, \qquad f_3(z,u)=i\overline{f_0'(z,u)}z^3, \qquad f_k(z,u)=0, \quad (k\neq 0,1,3)\\
&-g_0'(z,u)|z|^2+2f_1(z,u)\overline{z}+\overline{f_1(z,u)}z=0,.
\end{aligned} \tag{2.9}$$

To determine $f_0'(z,u)$, we consider the terms of type $(3,2)$ and obtain

$$-\frac{1}{2}g_2'(z,u)z\overline{z}^2+2i\left(f_0'(z,u)-\overline{f_0'(z,u)}\right)z^3\overline{z}^2+f_3(z,u)\overline{z}^2-i\overline{f_0'(z,u)}z^3\overline{z}^2=0.$$

By substituting (2.7) into the above equation, we get

$$2if_0'(z,u)-3i\overline{f_0'(z,u)}=0.$$

Then by (2.9), we get

$$f_0'(z,u)=0, \quad f_3(z,u)=0.$$

It remains to determine f_1. Let's consider terms of type $(4,2)$ in (2.5), i.e.

$$\begin{aligned}
0=&-\frac{1}{2}g_3'(z,u)z\overline{z}^2-\frac{i}{4}\left(g_0''(z,u)-\overline{g_0''(z,u)}\right)z^4\overline{z}^2\\
&+2if_1'(z,u)z^3\overline{z}^2+f_4(z,u)\overline{z}^2-i\overline{f_1'(z,u)}z^4\overline{z}.
\end{aligned} \tag{2.10}$$

Applying (2.7) and (2.9) to (2.10) to get

$$2if_1'(z,u)\overline{z}-i\overline{f_1'(z,u)}z=0. \tag{2.11}$$

Taking the conjugates in both sides in (2.11) and by simple calculation, we get

$$f_1'(z,u)=0.$$

Then by (2.9), we have

$$g_0''(z,u)=0.$$

Now we conclude that

$$\begin{aligned} &\operatorname{Re} f_0(z,u)=0, \quad f_0'(z,u)=0, \quad f_1'(z,u)=0,\\ &\operatorname{Im} g_0(z,u)=0, \quad g_0''(z,u)=0, \quad g_2(z,u)=2i\overline{f_0(z,u)}z^2,\\ &f_k(z,u)=0, \quad (k\geq 2), \quad g_l(z,u)=0, \quad (l\neq 0,2). \end{aligned} \tag{2.12}$$

(2.12) implies that f and g must be written in the following form

$$\begin{aligned} f(z,w)&=f_0(z,w)+f_1(z,w)=i\alpha+cz,\\ g(z,w)&=g_0(z,w)+g_2(z,w)=\gamma+\beta w+2\alpha z^2, \end{aligned} \tag{2.13}$$

for some $\alpha,\beta,\gamma\in\mathbb{R},\ c\in\mathbb{C}$.

Since $X=2\operatorname{Re}[f(z,w)\frac{\partial}{\partial z}+g(z,w)\frac{\partial}{\partial w}]\in\operatorname{hol}(\mathcal{C},0)$, f and g given by (2.13) satisfies (2.2), i.e.,

$$\operatorname{Re}\left[i\left(\gamma+\beta w+2\alpha z^2\right)+\left(4|z|^2+2\overline{z}^2\right)(i\alpha+cz)\right]=0,$$

for $w=u+2i\operatorname{Re} z^2\overline{z}$ and $(z,u)\in U$. Then

$$\begin{aligned} 0&=\operatorname{Re}\left[i\beta\left(u+2i\operatorname{Re} z^2\overline{z}\right)+i\gamma+2i\alpha z^2+4i\alpha|z|^2+4cz^2\overline{z}+2i\alpha\overline{z}^2+2cz\overline{z}^2\right]\\ &=\operatorname{Re}\left[(4c-\beta)z^2\overline{z}+(2c-\beta)z\overline{z}^2\right]. \end{aligned}$$

Thus, $2c+\overline{c}-\beta=0$, and so $c\in\mathbb{R},\ \beta=3c$. Let α_1,α_2 and $\alpha_3\in\mathbb{R}$ denote c,α and γ, respectively. Then (1.3) follows from (2.13). This proves Theorem 1.1(I). □

3 Locally holomorphic automorphisms of $\mathcal{C}$

Let $(z,w)\mapsto(F_t(z,w),G_t(z,w))$ be a one-parameter group generated by $X=2\operatorname{Re}[f(z,w)\frac{\partial}{\partial z}+g(z,w)\frac{\partial}{\partial w}]\in\operatorname{hol}(\mathcal{C},0)$ with $F_0(z,w)=z,\ G_0(z,w)=w$, i.e. F_t and G_t are solutions to the initial problem of the following ordinary differential equation,

$$\left\{\begin{array}{rcl} \frac{dF_t}{dt} &=& f(F_t,G_t)=\alpha_1F_t+i\alpha_2,\\ \frac{dG_t}{dt} &=& g(F_t,G_t)=3\alpha_1G_t+2\alpha_2F_t^2+\alpha_3,\\ F_0(z,w) &=& z,\\ G_0(z,w) &=& w. \end{array}\right. \tag{3.1}$$

Proposition 3.1. *The transformations* $(F_t(z,w),G_t(z,w))$ *satisfying* (3.1) *are locally holomorphic automorphisms of* $\mathcal{C}$.

Proof. We need to show

$$\rho\left(F_t(z,w), G_t(z,w), \overline{F_t(z,w)}, \overline{G_t(z,w)}\right) = 0.$$

By definition (2.1) of ρ and equation (3.1), we have

$$\begin{aligned}
&\frac{d\rho(F_t, G_t, \overline{F_t(z,w)}, \overline{G_t(z,w)})}{dt} \\
&= \left(2|F_t|^2 + \overline{F}_t^2\right)\frac{dF_t}{dt} + \left(F_t^2 + 2|F_t|^2\right)\frac{d\overline{F}_t}{dt} - \frac{1}{2i}\left(\frac{dG_t}{dt} - \frac{d\overline{G}_t}{dt}\right) \\
&= \left(2|F_t|^2 + \overline{F}_t^2\right)(\alpha_1 F_t + i\alpha_2) + \left(2|F_t|^2 + F_t^2\right)\left(\alpha_1 \overline{F}_t - i\alpha_2\right) \\
&\quad - \frac{1}{2i}\left[\left(3\alpha_1 G_t + 2\alpha_2 F_t^2 + \alpha_3\right) - \left(3\alpha_1 \overline{G}_t + 2\alpha_2 \overline{F}_t^2 + \alpha_3\right)\right] \\
&= 3\alpha_1 \rho(F_t, G_t, \overline{F}_t, \overline{G}_t).
\end{aligned} \tag{3.2}$$

Since

$$\rho(F_0, G_0, \overline{F_0}, \overline{G_0}) = \rho(z, w, \overline{z}, \overline{w}) = 0, \qquad (z,w) \in \mathcal{C}, \tag{3.3}$$

it follows from (3.2), (3.3) and the uniqueness of solutions of ordinary differential equations that

$$\rho(F_t, G_t, \overline{F_t}, \overline{G_t}) = 0,$$

for any t. The proposition is proved. $\square$

Proposition 3.2. *The transformation $(F, G) : \mathcal{C} \mapsto \mathcal{C}$ generated by any $X \in \mathrm{hol}(\mathcal{C}, 0)$ can be written in the following form*

$$\left\{\begin{array}{rcl} F(z,w) &=& \lambda z + i\gamma_1, \\ G(z,w) &=& \lambda^3 w + 2\lambda^2\gamma_1 z^2 + 2i\lambda\gamma_1^2 z + \gamma_2, \end{array}\right. \tag{3.4}$$

where $\lambda \in \mathbb{R}_+,\ \gamma_1, \gamma_2 \in \mathbb{R}$.

Proof. Let's solve the ordinary differential equation (3.1). When $\alpha_1 \neq 0$, the first equation in (3.1) with $F_0(z,w) = z$ implies

$$F_t = \left(z + i\frac{\alpha_2}{\alpha_1}\right)e^{\alpha_1 t} - i\frac{\alpha_2}{\alpha_1}. \tag{3.5}$$

Then substitute (3.5) into the second equation in (3.1) to get

$$\frac{dG_t}{dt} = 3\alpha_1 G_t + 2\alpha_2\left[\left(z + i\frac{\alpha_2}{\alpha_1}\right)^2 e^{2\alpha_1 t} - 2i\frac{\alpha_2}{\alpha_1}\left(z + i\frac{\alpha_2}{\alpha_1}\right)e^{\alpha_1 t} - \left(\frac{\alpha_2}{\alpha_1}\right)^2\right] + \alpha_3,$$

with $G_0(z,w)=w$. This is a linear ordinary equation of first order. It is easy to see that

$$G_t=\left[w+2\frac{\alpha_2}{\alpha_1}z^2+2i\left(\frac{\alpha_2}{\alpha_1}\right)^2z+\frac{\alpha_1^2\alpha_3-2\alpha_2^3}{3\alpha_1^3}\right]e^{3\alpha_1 t}-2\frac{\alpha_2}{\alpha_1}\left(z+i\frac{\alpha_2}{\alpha_1}\right)^2e^{2\alpha_1 t}$$
$$+2i\left(\frac{\alpha_2}{\alpha_1}\right)^2\left(z+i\frac{\alpha_2}{\alpha_1}\right)e^{\alpha_1 t}-\frac{\alpha_1^2\alpha_3-2\alpha_2^3}{3\alpha_1^3}.$$

Denote $\lambda=e^{\alpha_1 t},\ \beta_1=\frac{\alpha_2}{\alpha_1},\ \beta_2=\frac{\alpha_3}{\alpha_1}$. Then,

$$F_t(z,w)=\lambda z+i\beta_1(\lambda-1),$$
$$G_t(z,w)=\lambda^3w+2\lambda^2\beta_1(\lambda-1)z^2+2i\lambda\beta_1^2(\lambda-1)^2z-\frac{2}{3}\beta_1^3(\lambda-1)^3+\frac{1}{3}\beta_2(\lambda^3-1).$$

It is easy to see that when $\lambda=1$, i.e. $t=0$, then $F_0(z,w)=z,\ G_0(z,w)=w$. When $\lambda\neq 1$, let $\beta_3=\beta_1(\lambda-1)$, $\beta_4=-\frac{2}{3}\beta_1^3(\lambda-1)^3+\frac{1}{3}\beta_2(\lambda^3-1)$, then

$$F(z,w)=\lambda z+i\beta_3,\quad G(z,w)=\lambda^3w+2\lambda^2\beta_3z^2+2i\lambda\beta_3^2z+\beta_4. \tag{3.6}$$

When $\alpha_1=0$, from the first equation in (3.1) with initial data $F_0(z,w)=z$, we obtain

$$F_t=z+i\alpha_2t. \tag{3.7}$$

By substituting (3.7) into the second equation in (3.1),

$$\frac{dG_t}{dt}=2\alpha_2(z^2+2i\alpha_2tz-\alpha_2^2t^2)+\alpha_3,$$

with $G_0(z,w)=w$. Hence, $G_t=w+2\alpha_2tz^2+2i(\alpha_2t)^2z-\frac{2}{3}\alpha_2^3t^3+\alpha_3t$. Denote $\beta_5=\alpha_2t,\ \beta_6=-\frac{2}{3}\alpha_2^3t^3+\alpha_3t$, then

$$F_t(z,w)=z+i\beta_5,\quad G_t(z,w)=w+2\beta_5z^2+2i\beta_5z+\beta_6. \tag{3.8}$$

From (3.6) and (3.8), we have (3.4), which can be written as the form (3.6) when $\lambda\neq 1$ and (3.8) when $\lambda=1$. The proposition is proved. $\square$

Let $M\subset\mathbb{C}^N$ be a CR submanifold and $p_0\in M$. Then M is said to be *of finite type* m *at* p_0 if the tangent space of M at point p_0 is spanned by commutators of length up to m of sections of $T^{1,0}M\oplus T^{0,1}M$ and is not spanned by commutators of length up to $m-1$. By *of finite type* we mean the type at each point $p\in M$ is less than a fixed positive integer.

The complex tangential subbundles $T^{1,0}\mathcal{C}$ of the CR manifold $\mathcal{C}$ is spanned by $L = \frac{\partial}{\partial z} + 2i(2|z|^2 + \overline{z}^2)\frac{\partial}{\partial w}$ and $T^{0,1}\mathcal{C} = \overline{T^{1,0}\mathcal{C}}$, which is spanned by $\overline{L} = \frac{\partial}{\partial \overline{z}} - 2i(2|z|^2 + z^2)\frac{\partial}{\partial \overline{w}}$. We have

$$[L, \overline{L}] = -4i(\overline{z} + z)T, \quad \big[L, [L, \overline{L}]\big] = -4iT, \tag{3.9}$$

where $T = \frac{\partial}{\partial w} + \frac{\partial}{\partial \overline{w}}$.

Let $\mathcal{T}$ denote the group of transformations (F_1, G_1) of $\mathcal{C}$ with F_1 and G_1 given by

$$F_1(z, w) = z + it_1, \quad G_1(z, w) = w + 2t_1 z^2 + 2it_1^2 z + t_2, \tag{3.10}$$

for some $t_1, t_2 \in \mathbb{R}$. It is easy to see that the inverse $(F_1, G_1)^{-1}$ is the transformations (F_2, G_2) in the following form

$$F_2(z, w) = z - it_1, \quad G_2(z, w) = w - 2t_1 z^2 + 2it_1^2 z - t_2,$$

where $t_1, t_2 \in \mathbb{R}$. We have the following decomposition of $\mathrm{Aut}(\mathcal{C}, 0)$.

Proposition 3.3. *$\mathrm{Aut}(\mathcal{C}, 0) = \mathcal{T} \circ \mathrm{Aut}_0\, \mathcal{C}$.*

Proof. Suppose H is an arbitrary element in $\mathrm{Aut}(\mathcal{C}, 0)$. We claim that H maps $(0, 0)$ to (it_1, t_2) for some $t_1, t_2 \in \mathbb{R}$. Let

$$P := \{(it, s) \mid t, s \in \mathbb{R}\}.$$

It is easy to see that $P \subset \mathcal{C}$ by definition. By (3.9), we see that $\mathcal{C}$ is strictly pseudoconvex at the points (z, w) with $\mathrm{Re}\, z \neq 0$, and is of type 3 at the points in P. Since the type is preserved under locally biholomorphic transformations, there does not exist a biholomorphic transformation mapping $(0, 0)$ to (z_0, w_0) with $\mathrm{Re}\, z_0 \neq 0$. Since $H \in \mathrm{Aut}(\mathcal{C}, 0)$, hence, $H(0, 0) = (it_1, t_2)$ for some $t_1, t_2 \in \mathbb{R}$. From Proposition 3.2, there indeed exists a holomorphic automorphism $H_1 = (F_1, G_1)$ of $\mathcal{C}$, with F_1, G_1 given by (3.10), mapping $(0, 0)$ to (it_1, t_2). So we have $H = H_1 \circ H_2$, with $H_2 := H_1^{-1} \circ H \in \mathrm{Aut}(\mathcal{C}, 0)$. Since

$$H_2(0) = H_1^{-1} \circ H(0) = H_1^{-1} \circ H_1(0) = 0.$$

Therefore, $H_2 \in \mathrm{Aut}_0\, \mathcal{C}$. Hence,

$$\mathrm{Aut}(\mathcal{C}, 0) \subset \mathcal{T} \circ \mathrm{Aut}_0\, \mathcal{C}.$$

Clearly, $\mathrm{Aut}(\mathcal{C}, 0) \supset \mathcal{T} \circ \mathrm{Aut}_0\, \mathcal{C}$. Consequently, $\mathrm{Aut}(\mathcal{C}, 0) = \mathcal{T} \circ \mathrm{Aut}_0\, \mathcal{C}$. The proposition is proved. $\square$

Proof of Theorem 1.1(II). From (3.9) we can see that the real-analytic hypersurface $\mathcal{C}$ is of finite type. Hence, by Corollary 1.6 in [8], $\mathrm{Aut}_0\,\mathcal{C}$ is a Lie group. It is obvious that $\mathrm{hol}_0\,\mathcal{C}$ is its Lie algebra. Therefore $\mathrm{hol}_0\,\mathcal{C}$ can generate a connected component of $\mathrm{Aut}_0\,\mathcal{C}$. From Theorem 1.1(I), for any $X = 2\,\mathrm{Re}[f(z,w)\frac{\partial}{\partial z} + g(z,w)\frac{\partial}{\partial w}] \in \mathrm{hol}_0\,\mathcal{C}$, locally in a neighborhood of the origin, the functions f and g can be written in the following form

$$f(z,w) = \alpha_1 z, \qquad g(z,w) = 3\alpha_1 w,$$

for some $\alpha_1 \in \mathbb{R}$. Then from Proposition 3.2, the transformation generated by X can be written as

$$\delta_\lambda(z,w) = (\lambda z, \lambda^3 w),$$

for some $\lambda \in \mathbb{R}_+$. Consequently, $\delta = \{\delta_\lambda \mid \lambda \in \mathbb{R}_+\}$ is a connected component of $\mathrm{Aut}_0\,\mathcal{C}$. Then by Proposition 3.3, $\mathcal{T} \circ \delta$ is a connected component of $\mathrm{Aut}(\mathcal{C}, 0)$, the elements in which can be written as (1.4). Clearly, the identity transformation is in this component. This proves Theorem 1.1(II) and hence Theorem 1.1. $\square$

Proof of Corollary 1.2. We claim that

$$\mathrm{Aut}^{\mathrm{m}}\,\mathcal{C} \subset \mathrm{Aut}(\mathcal{C}, 0) \circ \mathcal{T}. \tag{3.11}$$

In fact, for an arbitrarily chosen element $H = (\widetilde{F}, \widetilde{G}) \in \mathrm{Aut}^{\mathrm{m}}\,\mathcal{C}$, without loss of generality, we may assume that it is not defined near the origin. Let S be the analytic variety of singular points of H. Since hyperplane P ($\subset \mathcal{C}$) defined by (3) is totally real and $\dim_{\mathbb{C}} S = 1$, we see that $P \setminus (P \cap S)$ is not empty and is an open subset of $\mathcal{C}$. Then H is well-defined in a neighborhood of some point $(it_1, t_2) \in P \setminus (P \cap S)$, which we denote by U. Let $\Phi^{-1} = (F_2, G_2) \in \mathcal{T}$ with F_2, G_2 given by (3), then Φ^{-1} maps U into a neighborhood of the origin. Therefore, $H \circ \Phi$ is well-defined in $\Phi^{-1}(U)$, which is a neighborhood of the origin. Namely, $H \circ \Phi \in \mathrm{Aut}(\mathcal{C}, 0)$. Consequently, $H = H \circ \Phi \circ \Phi^{-1} \in \mathrm{Aut}(\mathcal{C}, 0) \circ \mathcal{T}$. So we have (3.11).

From Theorem 1.1(II), we find that $\mathcal{T} \circ \delta$ is a connected component of $\mathrm{Aut}(\mathcal{C}, 0)$. Therefore, $\mathcal{T} \circ \delta \circ \mathcal{T} = \mathcal{T} \circ \delta$ is a connected component of $\mathrm{Aut}(\mathcal{C}, 0) \circ \mathcal{T}$. Obviously,

$$\mathcal{T} \circ \delta \subset \mathrm{Aut}^{\mathrm{m}}\,\mathcal{C}. \tag{3.12}$$

From (3.11) and (3.12) we can see that $\mathcal{T} \circ \delta$ is a connected component of $\mathrm{Aut}^{\mathrm{m}}\,\mathcal{C}$. Clearly, the identity transformation is in this component. The corollary is proved. $\square$

Remark. Using the same method as above, we can generalize all the results to a kind of hypersurfaces of higher degree:

$$\mathcal{M} = \left\{(z,w) \in \mathbb{C}^2 \mid \mathrm{Im}\, w = 2\,\mathrm{Re}\, z^n \overline{z}\right\}.$$

References

[1] A.F. Arbatskii, On the structure of groups of nonlinear automorphisms of $(3,3)$-quadrics, *Math. Notes* **59** (1996), no. 1-2, 116-122.

[2] V.K. Beloshapka, Holomorphic transformations of a quadric, *Math. USSR-Sb.* **72** (1992), 189-205.

[3] V.K. Beloshapka, Polynomial models of real manifolds, *Izv. Math.* **65** (2001), no. 4, 641-657.

[4] V.K. Beloshapka, Real submanifolds of a complex space: their polynomial models, automorphisms, and classification problems, *Russian Math. Surveys* **57** (2002), no.1, 1-41.

[5] M.S. Baouendi, P. Ebenfelt and L.P. Rothschild, *Real submanifolds in complex space and their mappings,* Princeton Math. Series 47, Princeton Univ. Press, Princeton, NJ,1999.

[6] M.S. Baouendi, P. Ebenfelt and L.P. Rothschild, CR automorphisms of real analytic manifolds in complex space, *Comm. Anal. Geom.* **6** (1998), no. 2, 291-315.

[7] M.S. Baouendi and L.P. Rothschild, Local holomorphic equivalence of real analytic submanifolds in $\mathbb{C}^N$, Several complex variables (Berkeley, CA, 1995-1996), 1-24, Math. Sci. Res. Inst. Publ. **37**, Cambridge Univ. Press, Cambridge, 1999.

[8] P. Ebenfelt, B. Lamel and D. Zaitsev, Finite jet determination of local analytic CR automorphisms and their parametrization by 2-jets in the finite type case, *Geom. Funct. Anal.* **13** (2003), no. 3, 546-573.

[9] V. Ežov and G. Schmalz, Holomorphic automorphisms of quadrics, *Math. Z.* **216** (1994), no. 3, 453-470.

[10] Kolář, M., Local symmetries of finite type hypersurfaces in $\mathbb{C}^2$, *Sci. China Ser. A*, **49** (2006), no. 11, 1633–1641.

[11] N.F. Palinchak, Quadrics of high codimension, *Math. Notes,* **55** (1994), no. 5-6, 512-516.

[12] N.F. Palinchak, Real quadrics of codimension 3 in $\mathbf{C}^6$ and their nonlinear automorphisms, *Izv. Math.* **59** (1995), no. 3, 597-617.

[13] E.N. Shananina, Polynomial models of degree 5 and algebras of their automorphisms, *Math. Notes.* **75** (2004), no. 5-6, 702-716.

[14] S.N. Shevchenko, Description of the algebra of infinitesimal automrophisms of quadrics of codimension two and their classification, Math. Notes. **55** (1994), no. 5-6, 534-542.

[15] N.K. Stanton, Infinitesimal CR automorphisms of rigid hypersurfaces, *Amer. J. Math.* **117** (1995), no. 1, 141-167.

[16] N.K. Stanton, Infinitesimal CR automorphisms of real hypersurfaces, *Amer. J. Math.* **118** (1996), no. 1, 209-233.

[17] W. Wang, The Teichmüller distance on the space of spherical CR structures, *Sci. China Ser. A* **49** (2006), no. 11, 1523-1538.

[18] Q.Y. Wu, On holomorphic automorphisms of a class of non-homogeneous rigid hypersurfaces in $\mathbb{C}^{N+1}$, *Chinese Ann. Math., Ser. B* **31** (2010), no. 2, 201-210.

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