Int. Journal of Math. Analysis, Vol. 5, 2011, no. 30, 1491 - 1496

Restriction Maps

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Abstract

In this paper, we prove the restriction maps define continuous linear operators on the Smirnov classes.

Mathematics Subject Classification: 30D55, 14Q05, 42B30

Keywords: Smirnov Classes, Restriction Maps

1 Introduction

An arc or closed curve γ is called $\sigma-$rectifiable if and only if it is a countable union of rectifiable arcs in $\mathbb{C}$, together with (∞) in the case when $\infty \in \gamma$. For instance, a parabola without ∞ is $\sigma-$rectifiable arc, and a parabola with ∞ is $\sigma-$rectifiable Jordan curve. Also a circle and an ellipse are rectifiable arcs and a line and a branch of a hyperbola are $\sigma-$rectifiable arcs. If γ is a $\sigma-$*rectifiable* arc in $\mathbb{C}$ and f is integrable with respect to *arc-length* measure the notation

$$\int_{\gamma} f(z)\,|dz|$$

will be used to denote the integral of f with respect to arc-length measure. The notation $L^p(\gamma)$ will denote the L^p space of normalized arc length measure on γ. Let us agree to use l for arc-length measure.

$\mathbb{T}$ will be used to denote the boundary of the open unit disc Δ. We will need the definition of Hardy and Smirnov classes of analytic functions. The reader is referred to [1], [2], [3], [4], and references therein for a basic account of the subject. $H^2(\Delta)$ is the set of all functions f analytic on Δ such that

$$\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left|f(re^{i\theta})\right|^2 d\theta < \infty. \tag{1.1}$$

We have additional definitions of $H^2(\Delta)$. $H^2(\Delta)$ is the set of all analytic functions f on Δ of the form

$$f(z)=\sum_{n=0}^{\infty} a_n z^n \qquad \text{with } a=(a_n)\in \ell^2. \tag{1.2}$$

For a more general simply-connected domain D in the sphere or extended plane $\overline{\mathbb{C}}=\mathbb{C}\cup(\infty)$ and a conformal mapping φ from D onto Δ (that is, a Riemann mapping function, abbreviation is RMF), a function g analytic in D is said to belong to the Smirnov class $E^2(D)$ if and only if

$$g=(f\circ\varphi)\varphi'^{1/2}$$

for some $f\in H^2(\Delta)$ where $\varphi'^{1/2}$ is an analytic branch of the square root of φ'.

Let $C=(C_1,C_2,C_3,...,C_N)$ be an N-tuple of closed distinct curves on the sphere $\overline{\mathbb{C}}$ and suppose that for each i, $1\leq i\leq N$, C_i is a circle, a line $\cup\{\infty\}$, an ellipse, a parabola $\cup\{\infty\}$ or a branch of a hyperbola $\cup\{\infty\}$. Let D_i be the complementary domain of C_i. Recall that a complementary domain of a closed $F\subseteq\overline{\mathbb{C}}$ is a maximal connected subset of $\overline{\mathbb{C}}-F$, which must be a domain. For $1\leq i\leq N$, suppose that

$$\varphi_i : D_i\to\Delta$$

is a conformal equivalence (i.e., RMF). Let $1\leq i\neq j\leq N$. Suppose that Γ is an open subarc of C_j and that $\Gamma\subseteq D_i$.

For $1\leq i\leq N$, let us keep the notations of $C_i, D_i, \varphi_i, \Gamma$ fixed until the end of the paper.

Definition 1 *A subarc $\gamma\subseteq\Gamma$ is said to be of* type I *if and only if*

$$\overline{\gamma}\subseteq D_i$$

(i.e., both of its end points a, b belong to D_i).

In this paper we prove:

Theorem 2 *If $\gamma\subseteq\Gamma$ is a* type I *subarc then the restriction*

$$f\to f|_{\gamma}$$

defines a continuous linear operator mapping $E^2(D_i)$ into $L^2(\gamma)$.

2 The Proof of Theorem 2

The following definition will simplify the language.

Definition 3 *Let $\gamma \subseteq \mathbb{C}$ be a simple $\sigma-$rectifiable arc contained in a simply connected domain $G \subseteq \overline{\mathbb{C}}$. We say that γ has the restriction property in G if and only if the map*

$$g \to g|_\gamma$$

defines a continuous linear operator mapping $E^2(G)$ into $L^2(\gamma)$.

Thus, the last sentence of Theorem 2 reads "γ has the restriction property in D_i".

Lemma 4 *(Invariance Lemma) Let $G_1, G_2 \subseteq \overline{\mathbb{C}}$ be simply connected domains and suppose that*

$$\gamma_1 \subseteq G_1 \cap \mathbb{C}, \gamma_2 \subseteq G_2 \cap \mathbb{C}$$

are simple $\sigma-$rectifiable arcs. If $\chi : G_1 \to G_2$ is a conformal equivalence onto G_2 and

$$\chi(\gamma_1) = \gamma_2$$

then γ_1 has the restriction property in G_1 if and only if γ_2 has the restriction property in G_2.

Proof. By symmetry, we need only prove one of the implication. So let us suppose that γ_1 has the restriction property in G_1. Let

$$S_1 : E^2(G_1) \to L^2(\gamma_1)$$

be the (continuous) restriction operator

$$S_1 f = f|_{\gamma_1} (f \in E^2(G_1))$$

and let S_2 be the restriction operator

$$S_2 g = g|_{\gamma_2} (g \in E^2(G_2)).$$

As it can be easily seen, the formula

$$U_\chi g(z) = g(\chi(z))\chi'(z)^{1/2} \qquad (g \in E^2(G_2), z \in G_1)$$

defines a unitary operator mapping $E^2(G_2)$ onto $E^2(G_1)$. The same formula, restricted to elements of $L^2(\gamma_2)$ defines a unitary operator W_χ mapping $L^2(\gamma_2)$ onto $L^2(\gamma_1)$:

$$W_\chi g(z) = g(\chi(z))\chi'(z)^{1/2} \qquad (g \in L^2(\gamma_2), z \in \gamma_1).$$

The isometric property of W_χ follows by integrating by substitution:

$$\begin{aligned} \|W_\chi g\|^2_{L^2(\gamma_1)} &= \int_{\gamma_1} |g(\chi(z))|^2 \, |\chi'(z)| \, |dz| \\ &= \int_{\gamma_2} |g(\zeta)|^2 \, |d\zeta| \end{aligned}$$

here $z = \chi^{-1}(\zeta)$. It is easily verified that

$$W_\chi^{-1} = W_{\chi^{-1}}.$$

Clearly

$$S_1 U_\chi = W_\chi S_2,$$

so that

$$S_2 = W_\chi^{-1} S_1 U_\chi.$$

So S_1 is continuous implies S_2 is continuous.

Corollary 5 *Γ has the restriction property in D_i, if and only if $\varphi_i(\Gamma)$ has the restriction property in Δ, for some RMF $\varphi_i : D_i \to \Delta$.*

Proof. Immediate.

A *subarc* γ of Γ has the restriction property in D_i if and only if $\varphi_i(\gamma)$ has the restriction property in Δ. The item iv of the next Lemma is exactly the Theorem 2.

Lemma 6 *Let γ be a subarc of Γ and suppose that φ_i, θ_i are Riemann mapping functions for D_i.*

i *$\varphi_i(\gamma)$ has the restriction property in Δ if and only if $\theta_i(\gamma)$ has the restriction property in Δ;*

ii *$\varphi_i(\gamma)$ is rectifiable if and only if $\theta_i(\gamma)$ is rectifiable;*

iii *if γ is of* type I *then $\overline{\varphi_i(\gamma)} \subseteq \Delta$ and $\varphi_i(\gamma)$ is rectifiable;*

iv *if γ is of* type I *it has the restriction property in D_i.*

Proof. In i and ii we need only prove one of the implications. By the Riemann mapping theorem

$$\theta_i = \chi \circ \varphi_i$$

where χ is a conformal automorphism of Δ, so i follows from the invariance Lemma 4. Now χ is analytic in a neighbourhood of $\overline{\Delta}$ because,

$$\chi(z) = \omega \frac{z-\lambda}{\overline{\lambda} z - 1} \qquad (|\omega| = 1, |\lambda| < 1),$$

so if $\varphi_i(\gamma)$ is rectifiable

$$\begin{aligned} l(\theta_i(\gamma)) &= \int_{\theta_i(\gamma)} |dz| \\ &= \int_{\varphi_i(\gamma)} |\chi'(z)|\,|dz| < \infty. \end{aligned}$$

Now suppose γ is of type I then

$$\overline{\gamma} \subseteq D_i$$

so

$$\varphi_i(\overline{\gamma}) = \overline{\varphi_i(\gamma)} \subseteq \Delta.$$

We shall construct an RMF θ_i for D_i such that $\theta_i(\gamma)$ is rectifiable. It will follow from ii that $\varphi_i(\gamma)$ is rectifiable as well.

Choose an $a \in \mathbb{C} - \overline{D_i}$ and let

$$\mu(z) = \frac{1}{z-a}.$$

Then $\mu(D_i)$ is a (bounded) simply connected domain and since

$$\begin{aligned} l(\mu(\gamma)) &= \int_\gamma |\mu'(z)|\,|dz| \\ &= \int_\gamma \frac{|dz|}{|z-a|^2} < \infty, \end{aligned}$$

$\mu(\gamma)$ is rectifiable and, clearly

$$\overline{\mu(\gamma)} \subseteq \mu(D_i).$$

Let $\nu : \mu(D_i) \to \Delta$ be a RMF for $\mu(D_i)$, then

$$\theta_i = \nu \circ \mu$$

is a RMF for D_i and

$$\theta_i(\gamma) = \nu(\mu(\gamma))$$

is rectifiable so by ii, iii is proved.

Finally, if R is the operator which restricts an element of $H^2(\Delta)$ to $\varphi_i(\gamma)$ then, by Cauchy's integral formula,

$$\begin{aligned} Rf(z) &= \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\widetilde{f}(\zeta)}{\zeta - z} d\zeta \\ &= \frac{1}{2\pi}\int_{\mathbb{T}} \frac{\widetilde{f}(\zeta)}{1 - \overline{\zeta} z}\,|d\zeta| \qquad (f \in H^2(\Delta), z \in \varphi_i(\gamma)) \end{aligned}$$

So R is essentially an integral operator with a continuous kernel ($\mathbb{T}$ and $\varphi_i(\gamma)$ have finite measure), hence R is in fact compact. ■

References

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Received: February, 2011