Int. Journal of Math. Analysis, Vol. 5, 2011, no. 30, 1469 - 1476

Analogue of Brown-Pedersen' Quasi Invertibility for JB^*-Triples

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Abstract

L. G. Brown and G. K. Pedersen studied a notion of quasi-invertible elements in a C^*-algebra, which plays a vital rule in the study of geometry of the unit ball. We prove that a non-zero element u in a unital C^*-algebra is Brown-Pedersen' quasi invertible if and only if it is von Neumann regular and admits a generalized inverse v such that the Bergmann operator $B(u, v^*)$ vanishes. The notion is then extended to the general setting of JB^*-triples and some of its properties are observed. We also give a comparison of BP-quasi invertibility with some other notions of invertibility such as von Neumann regularity, Moore-Penrose invertibility and the quasi-invertibility studied by S. Perlis, N. Jacobson and K. McCrimmon.

Mathematics Subject Classification: 17Cxx, 46L05, 47A05, 15A09

Keywords: C^*-algebra, JB^*-algebra, JB^*-triple, von Neumann regular element, Bergmann operator, quasi invertible element

1 Introduction

R. M. Aron and R. H. Lohman [2] initiated the study of a geometric function, called λ-function, defined on the unit ball of a normed space and a normed space is said to have λ-property if each element of its unit ball is a convex combination of an extreme point of the ball with positive weight and a vector of norm at the most 1; they raised a question "what spaces of operators have the λ-property and what does the λ-function look like for these spaces?" To answer this question, G. K. Pedersen [13] introduced another related function, called λ_u-function, defined on the unit ball of a C^*-algebra and obtained its close relationship with regular approximations, uniatry approximations and unitary rank of elements of a C^*-algebra; these areas have been extensively developed in the context of C^*-algebras by some of the leaders in the field, most notably R. V. Kadison, G. K. Pedersen, U. Haagerup and L. G. Brown.

Later on, to complete the study of λ-function for C^*-algebras, L. G. Brown and G. K. Pedersen [3] jointly introduced a notion of quasi invertible elements in a C^*-algebra and then a C^*-algebra is called extremally rich if the Brown-Pedersen' quasi invertible (henceforth, BP-quasi invertible) elements are norm dense in the algebra. As is explained in [3, 4], the properties of BP-quasi invertible elements are analogous to those of usual invertible elements and the class of extremally rich C^*-algebras is pleasantly large that includes all C^*-algebras of topological stable rank 1, all von Neumann algebras, all purely infinite C^*-algebras and all Toeplitz algebras. Investigating these concepts in [4], Brown et al presented a theory of decompositions of elements of the unit ball in a C^*-algebra as convex combinations of extreme points of the ball and demonstrated that the relationships between the (extreme-convex) decomposition theory, λ-function and the distance from an element to the set of BP-quasi invertible elements are analogous with the relationships in the earlier C^*-algebra theory of unitary convex decompositions, λ_u-function and regular approximations.

A Jordan algebra analogue of C^*-algebras, introduced by I. Kaplansky, is called a Jordan C^*-algebra or JB^*-algebra (cf. [23]). A further generalization, called JB^*-triples, was introduced by W. Kaup, who proved that every bounded symmetric domain in a complex Banach space is biholomorphically equivalent to the open unit ball of a JB^*-triple; JB^*-triples include all C^*-algebras, JB^*-algebras, Hilbert spaces and spin factors (cf. [9, 10, 22]). One of the present authors has extended various C^*-algebra results on regular approximations, convex decompositions involving unitaries and λ_u-function to the general setting of JB^*-triples/JB^*-algebras (see [15, 16, 17, 18, 19, 20]). Of course, λ_u-function is not defined in JB^*-triples that are not JB^*-algebras; however, the λ-function is defined for any JB^*-triple. Since λ-function is a Banach space invariant, this should be studied when looking at JB^*-triples which have less algebraic structure but can have some unexpected applications to operator algebras.

In this paper, we begin a study of BP-quasi invertibility in the general setting of JB^*-triples. We prove that a non-zero element u in a unital C^*-algebra is BP-quasi invertible if and only if it is von Neumann regular and admits a generalized inverse v such that the Bergmann operator $B(u, v^*)$ vanishes. The notion is then extended to the general setting of JB^*-triples and some of its properties are observed. We also give a comparison of BP-quasi invertibility with some other notions of invertibility such as von Neumann regularity and Moore-Penrose invertibility; it is observed that the BP-quasi invertibility is opposite to the quasi invertibility extensively studied by S. Perlis, N. Jacobson and K. McCrimmon for Jordan algebras.

2 Preliminaries

Our notation and basic terminology is standard as found in [12] or [9, 22]. Recall that the underlying binary product $\circ$ of a Jordan algebra $\mathcal{J}$ induces a triple product $\{xyz\} := (x \circ y) \circ z + (z \circ y) \circ x - (x \circ z) \circ y$ and this gives the construction of basic operators on $\mathcal{J}$, namely, $V_{x,y}z := \{xzy\}$ and $U_{x,y}z := \{xzy\}$; we shall use the short symbol U_x for the operator $U_{x,x}$. A JB^*-algebra is a Banach space $\mathcal{J}$ which is a complex Jordan algebra equipped with an involution $*$ satisfying $\|x \circ y\| \leq \|x\|\|y\|$ and $\|\{xx^*x\}\| = \|x\|^3$ for all $x, y \in \mathcal{J}$; if $\mathcal{J}$ has a unit e with $\|e\| = 1$ then it is called a unital JB^*-algebra. Thus, any C^*-algebra with associative product ab is a JB^*-algebra under the product $x \circ y = \frac{1}{2}(xy + yx)$.

A Jordan triple system (or Jordan triple, for short) is a vector space $\mathcal{J}$ over the real or the complex field K, endowed with a triple product $\{xyz\}$ which is linear and symmetric in outer variables x, z, anti-linear in the inner variable y and satisfies the Jordan triple identity: $\{xu\{yvz\}\} + \{\{xvy\}uz\} - \{yv\{xuz\}\} = \{x\{uyv\}z\}$ for all $u, v, x, y, z \in \mathcal{J}$. A JB^*-triple is a complex Banach space $\mathcal{J}$ together with a continuous, sesquilinear, operator-valued product $(x, y) \in \mathcal{J} \times \mathcal{J} \mapsto L_{x,y}$ on $\mathcal{J}$ given by $L_{x,y}z := \{xy^*z\}$ defines a Jordan triple system such that every operator $L_{z,z}$ is a positive hermitian operator on $\mathcal{J}$ satisfying the condition $\|\{xx^*x\}\| = \|x\|^3$ for all $x \in \mathcal{J}$ (cf. [9, page 504]).Thus, every C^*-algebra is a JB^*-triple under the triple product $\{xy^*z\} := \frac{1}{2}(xy^*z + zy^*x)$ and every JB^*-algebra is a JB^*-triple under the triple product $\{xy^*z\} := (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*$. The converse is not true; however, any JB^*-triple containing an invertible element is necessarily a JB^*-algebra (cf. [22]). A closed subspace $\mathcal{K}$ of a JB^*-algebra $(\mathcal{J}, \{, , \})$ is called a JB^*-subtriple of $\mathcal{J}$ if $\{xy^*z\} \in \mathcal{K}$ for all $x, y, z \in \mathcal{K}$. It is well known that any JB^*-triple can be realized as JB^*-subtriple of a JB^*-algebra (cf. [6]).

3 Main Results

Originally, L. G. Brown et al [3] defined BP-quasi invertible elements in a C^*-algebra as follows: *a non-zero element u of a unital C^*-algebra $\mathcal{A}$ is called BP-quasi invertible if there exists an orthogonal pair of closed ideals T and S in $\mathcal{A}$ such that $u+T$ is left invertible in the quotient algebra $\mathcal{A}/T$ and $u+S$ is right invertible in $\mathcal{A}/S$.* Here, by the orthogonality of the ideals S, T in $\mathcal{A}$ we mean $S\mathcal{A}T = \{0\} = T\mathcal{A}S$.

We proceed to characterize the BP-quasi invertible elements of a C^*-algebra in some terms that are available in the general context of JB^*-triples. For any elements x, y, z in a JB^*-triple $\mathcal{J}$, note that the operators $P_x(z) := \{xz^*x\}$ and $L_{x,y}(z) := \{xy^*z\}$ are the Jordan triple system analogue of the Jordan algebra operators U_x and $V_{x,y}$ (see above). For any fixed element $x, y \in \mathcal{J}$, there is an other basic operator, called Bergmann operator, defined on $\mathcal{J}$ by $B(x,y) := I - 2L_{x,y} + P_xP_y$ where I denotes the identity operator on $\mathcal{J}$. Clearly, $L_{x,y}z = V_{x,y^*}z$, $P_xz = U_xz^*$ and so $B(x,y)z = z - 2V_{x,y^*}z + U_xU_{y^*}z$ for any elements x, y, z in a JB^*-algebra. In case of a C^*-algebra, these operators take the forms $L_{x,y}z = \frac{1}{2}(xy^*z + zy^*x)$, $P_xz = xz^*x$ and $B(x,y)z = (1-xy^*)z(1-y^*x)$.

Recall that an element x in a Jordan algebra $\mathcal{J}$ is called von Neumann regular if $U_xy = x$ for some $y \in \mathcal{J}$; this condition is equivalent to $xyx = x$ if $\mathcal{J}$ is a C^*-algebra. In such a case y is called a generalized inverse of x (cf. [7, 12]). The following result gives a characterization of BP-quasi invertible elements of a C^*-algebras in terms of von Neumann regularity and Bergmann operator.

Theorem 3.1 *For any non-zero element u in a C^*-algebra $\mathcal{A}$ with unit 1, the following statements are equivalent:*
(i). u is BP-quasi invertible in $\mathcal{A}$;
(ii). there exist two elements $a, b \in \mathcal{A}$ such that $(1-ua)\mathcal{A}(1-bu) = \{0\}$;
(iii). u is von Neumann regular with generalized inverse v such that the Bergmann operator $B(u, v^*)$ vanishes.

Proof. For the equivalence of (i) and (ii), see [1, pages 611-612].

$(iii) \Rightarrow (ii)$: Immediate if we take $a = b = v$ (recall that $B(x,y)z = (1-xy^*)z(1-y^*x)$).

$(ii) \Rightarrow (iii)$: If statement (ii) holds then $(1-ua)u(1-bu) = \{0\}$. By taking $v = a + b - aub$, we obtain $uvu = u(a+b-aub)u = uau + ubu - uaubu = -(1-ua)u(1-bu) + u = 0 + u = u$. So that u is von Neumann regular element in $\mathcal{A}$ with generalized inverse v. Further, we observe that $(1-ua)(1-ub) = 1 - ua - ub + uaub = 1 - u(a+b+aub) = (1-uv)$; similarly, $(1-vu) = (1-au)(1-bu)$. Hence, for any $x \in \mathcal{A}$, we have $B(u,v^*)x = (1-uv^{**})x(1-v^{**}u)$ *(see above)* $= (1-uv)x(1-vu) = (1-$

$ua)((1-ub)x(1-au))(1-bu) = \{0\}$ since $(1-ub)x(1-au) \in \mathcal{A}$. Thus, $B(u, v^*)$ vanishes.

We recall that an element x in a JB^*-triple $\mathcal{J}$ is called von Neumann regular if $P_x y = x$ for some $y \in \mathcal{J}$; such an element y is called a generalized inverse of x (cf. [5, 7]). We shall denote the set of all von Neumann regular elements in $\mathcal{J}$ by $\widetilde{\mathcal{J}}$. If $\mathcal{J}$ is a JB^*-algebra, then $x = P_x y = \{xy^*x\}$ and so y is a generalized inverse of x in $\mathcal{J}$ considered as a JB^*-triple if and only if y^* is a generalized inverse of x in the JB^*-algebra $\mathcal{J}$. In particular, x has generalized inverse y^* in a C^*-algebra $\mathcal{A}$ if and only if y is generalized inverse of x in $\mathcal{A}$ considered as JB^*-triple (see [5, page 198]).

Keeping in view the above discussion and Theorem 3.1, we define an exact analogue of BP-quasi invertibility in the general setting of JB^*-triples as follows: *a non-zero element x in a JB^*-triple $\mathcal{J}$ is BP-quasi-invertible if it is von Neumann regular with generalized inverse y such that the Bergmann operator $B(x, y)$ vanishes.* Thus, an element x in a JB^*-algebra $\mathcal{J}$ is BP-quasi invertible if and only if there is some $y \in \mathcal{J}$ such that y^* is a generalized inverse of x and the Bergmann operator $B(x, y) = 0$; in such a case, $y*$ is called BP-quasi inverse of x. The set of BP-quasi invertible elements in a JB^*-triple $\mathcal{J}$ is denoted by $\mathcal{J}_q^{-1}$.

Next, we observe some properties of BP-quasi invertible elements. The following result gives the invariance of such elements under the involution of a JB^*-algebra.

Theorem 3.2 *If y^* is BP-quasi inverse of x in a JB^*-algebra $\mathcal{J}$, then y is BP-quasi inverse of x^*.*

Proof. Suppose $x \in \mathcal{J}_q^{-1}$ with BP-inverse y^*. Then x is von Neumann regular with generalized inverse y^* such that $B(x, y) = 0$. In particular, $\{xy^*x\} = U_x y^* = x$ and so $U_{x^*} y = \{x^*yx^*\} = \{xy^*x\}^* = x^*$ since $(x \circ y)^* = x^* \circ y^*$. Therefore, x^* is von Neumann regular with generalized inverse y. Moreover, $B(x, y) = 0$ implies $B(x, y)z^* = 0$, $\forall z \in \mathcal{J}$. Hence, for any $z \in \mathcal{J}$, $B(x^*, y)z = (I - 2V_{x^*,y} + U_{x^*}U_y)z = ((I - 2V_{x,y^*} + U_x U_{y^*})z^*)^* = (B(x, y^*)z^*)^* = 0^* = 0$. Thus, y is BP-quasi inverse of x^*.

Recall that an element x in a JB^*-algebra $\mathcal{J}$ with unit 1 is said to be invertible if there exists $x^{-1} \in \mathcal{J}$ (called the inverse of x) satisfying $x \circ x^{-1} = 1$ and $x^2 \circ x^{-1} = x$; and that $u \in \mathcal{J}$ is called a unitary if u^* is the inverse of u. Our next result shows that BP-quasi invertible elements in a JB^*-algebra include all the unitary elements.

Theorem 3.3 *In any JB^*algebra $\mathcal{J}$, $\mathcal{U}(\mathcal{J}) \subset \mathcal{J}_q^{-1}$, where $\mathcal{U}(\mathcal{J})$ denotes the set of all unitary elements in $\mathcal{J}$.*

Proof. Let $u \in \mathcal{U}(\mathcal{J})$. Then $\{uu^*z\} = z$ for all $z \in \mathcal{J}$ (cf. [22]). In particular, $\{uu^*u\} = u$ and hence u^* is a generalized inverse of u in the JB^*-algebra $\mathcal{J}$.

Further, for any $z \in \mathcal{J}$, $B(u,u)z = z - 2\{uu^*z\} + \{u\{u^*zu^*\}u\} = z - 2z + z = 0$, for all $z \in \mathcal{J}$. We conclude that u is BP-quasi invertible.

The following result improves the above Theorem 3.3 and it justifies the notion of BP-quasi invertible elements, too.

Theorem 3.4 *Any invertible element in a (unital) JB^*-algebra $\mathcal{J}$ is BP-quasi invertible.*

Proof. Let an element $x \in \mathcal{J}$ be invertible, then $\{xx^{-1}z\} = z$ for all $z \in \mathcal{J}$ (by the Shirshov-Cohn theorem with inverses [11]). In particular, $\{xx^{-1}x\} = x$. Thus x^{-1} is a generalized inverse of x in the JB^*-algebra $\mathcal{J}$. Moreover, $U_x^{-1} = U_{x^{-1}}$ by [8, Theorem 13] and so $U_x U_{x^{-1}} = U_x U_x^{-1} = I$. Hence, $B(x,(x^{-1})^*)z = (I - 2V_{x,x^{-1}} + U_x U_{x^{-1}})z = z - 2\{xx^{-1}z\} + U_x U_x^{-1} z = z - 2\{xx^{-1}z\} + Iz = z - 2z + z = 0$. Thus, x is BP-quasi invertible in the algebra $\mathcal{J}$.

It is well known that every unitary element in a JB^*-algebra is an extreme point of the closed unit ball (cf. [10]). Thus, the following result gives another improvement of Theorem 3.3.

Theorem 3.5 *Every extreme point of the closed unit ball in a JB^*-algebra is BP-quasi invertible.*

Proof. Any extreme point v of the closed unit ball in a JB^*-algebra $\mathcal{J}$ is a regular tripotent; this means $P_v v = \{vv^*v\} = v$ with $B(v,v) = 0$ (see [10, Lemma 3.2 and Proposition 3.5]). Hence, v is von Neumann regular in the JB^*-algebra $\mathcal{J}$ with generalized inverse v^* such that the Bergmann operator $B(v,v)$ vanishes. Thus v is BP-quasi invertible in $\mathcal{J}$.

Next, we recall that an element x in a JB^*-triple $\mathcal{J}$ is called Moore-Penrose invertible (in short, MP-invertible) with MP-inverse $y \in \mathcal{J}$ if $P_x(y) = x$, $P_y(x) = y$ and both operators $L_{x,y}$ and $L_{y,x}$ are Hermitian (cf. [21]). We denote the set of all MP-invertible elements in $\mathcal{J}$ by $\mathcal{J}^\dagger$.

It is clear from the definitions that all MP-invertible elements and all BP-quasi invertible element in any JB^*-algebra are von Neumann regular. In case of C^*-algebras $\mathcal{A}$, Harte et. al proved that $\widetilde{\mathcal{A}} \subseteq \mathcal{A}^\dagger$ (see [7, Theorem 6]). We conclude the following result:

Theorem 3.6 *For any C^*-algebra $\mathcal{A}$, $\mathcal{U}(\mathcal{A}) \subset \mathcal{E}(\mathcal{A}) \subset \mathcal{A}_q^{-1}$ and $\mathcal{U}(\mathcal{A}) \subset \mathcal{A}^{-1} \subset \mathcal{A}_q^{-1} \subset \widetilde{\mathcal{A}} = \mathcal{A}^\dagger$, where $\mathcal{E}(\mathcal{A})$ denotes the set of extreme points of the closed unit ball in $\mathcal{A}$.*

Notice that all the set inclusions (except the last equality) in the above result are proper: it is well known that there exist C^*-algebras having non-invertible (hence, non-unitary) extreme points of the closed unit ball and invertible elements may not belong to the unit ball. The last inclusion, namely, $\mathcal{A}_q^{-1} \subset \widetilde{\mathcal{A}}$ is also proper; for example, consider any proper idempotent element $a \in \mathcal{A}$

which is, of course, von Neumann regular but not BP-quasi invertible because $(1-a^2)\mathcal{A}(1-a^2) = (1-a)\mathcal{A}(1-a) = \{0\}$ since $1-a \in \mathcal{A}$ (see Theorem 3.1). In case of a JB^*-algebra $\mathcal{J}$, we at this stage are not sure about the inclusion $\widetilde{\mathcal{J}} \subseteq \mathcal{J}^\dagger$; however, it is clear from the above analysis that all the other inclusions (appeared in Theorem 3.6) remain true in case of JB^*-algebras.

We close this article by comparing the BP-quasi invertibility with another standard notion of quasi invertibility; this notion of quasi invertibility was originally introduced by S. Perlis and later on extensively studied by N. Jacobson, K. McCrimonn and their colleagues (cf. [12, 14]: an element x in a JB^*-algebra $\mathcal{J}$ is quasi invertible with quasi inverse $y \in \mathcal{J}$ if the Bergmann operator $B(x, y)$ is invertible on $\mathcal{J}$. Thus, by Theorem 3.1, BP-quasi invertibility is opposite to Perlis-Jacobson quasi invertibility in the sense of invertibility of the Bergmann operators.

References

[1] P. Ara, G. K. Pedersen and F. Perera, An infinite analogue of rings with stable rank one, *Journal of Algebra,* **vol. 230** (2000), 608-655.

[2] R. M. Aron and R. H. Lohman, A geometric function determined by extreme points of the unit ball of a normed space, *Pac. J. Math.,* **127** (1987), 209-231.

[3] L. G. Brown and G. K. Pedersen, On the geometry of the unit ball of a C*-algebra, *J. Reine Angew. Math.,* **vol. 469** (1995), 113-147.

[4] L. G. Brown and G. K. Pedersen, Approximation and convex decomposition by extremals in C*-algebras, *Mathematica Scandinavica,* **vol. 81** (1997), 69-85.

[5] M. Burgos, A. Kaidi, A. Morales, A. M. Peralta and M. Ramirez, von Neumann regularity and quadratic conorms in JB*-triples and C*-algebras, *Acta Mathematica,* **vol.24 (2)** (2008), 185-200.

[6] Y. Friedmann and B. Russo, The Gelfand-Naimark theorem for JB^*-triples, *Duke Math. J.,* **53** (1986), 139-148.

[7] R. E. Harte and M. Mbekhta, On generalized inverses in C*-algebras, *Studia Mathematica,* **vol. 103**(1992), 71-77.

[8] N. Jacobson, *Structure and representation of Jordan algebras,* vol. 39 of American Mathematical Society Colloquium Publications, American Mathematical Society, Providence, RI, USA, 1968.

[9] W. Kaup, A Riemann mapping theorem for bounded summetric domains in complex Banach spaces, *Math. Z.,* **183** (1983), 503-529.

[10] W. Kaup and H. Upmeier, Jordan algebras and symmetric Siegel domains in Banach spaces, *Mathematische Zeitschrift,* **vol. 157**, **no. 2** (1977), 179-200.

[11] K. McCrimmon, Macdonald's theorem with inverses, *Pac. J. Math.,* **21** (1967), 315-325.

[12] K. McCrimmon,*A taste of Jordan algebras,* Springer-Verlag, New York, Inc., 2004.

[13] G. K. Pedersen, λ-function in operator algebras, *J. Operator Theory,* **26** (1991), 345-381.

[14] S. Perlis, A characterization of the radical of an algebra, *Bul. Amer. Math. Soc.,* **vol. 48** (1942), 128-132.

[15] Siddiqui, A. A., JB*-algebras of topological stable rank 1,*International Journal of Mathematics and Mathematical Sciences,* (2007), Article ID 37186, 24 pages, 2007. doi:10.1155/2007/37186

[16] Siddiqui, A. A., Self-adjointness in unitary isotopes of JB*-algebras, *Arch. Math.,* **87**(2006), 350358.

[17] Siddiqui, A. A., On unitaries in JB^*-algebras, *Indian J. Math.,* **vol. 48, no. 1** (2006), 35-48.

[18] Siddiqui, A. A., Average of two extreme points in JBW*-triples, *Proc. Japan Acad. Ser. A, Math. Sci.,* **83** (2007), 176178.

[19] Siddiqui, A. A., A proof of Russo-Dye theorem for JB^*-algebras, *New York J. Math.,* **16** (2010), 53-60.

[20] Siddiqui, A. A., λ_u-function in JB*-algebras,*to appear in New York J. Math.,* (2011).

[21] Tevelen, E., Moore-Penrose inverse, parabolic subgroups and Jordan pairs, *Journal of Lie Theory,* **Vol. 12, no. 2** (2002), 461-481.

[22] H. Upmeier, *Symmetric Banach Manifolds and Jordan C*-algebras*, Elsevier Science Publishers B.V., 1985.

[23] J. D. M. Wright, Jordan C*-algebras, *Michigan Mathematical Journal,* **vol. 24** (1977), 291-302.

Received: February, 2011