Int. Journal of Math. Analysis, Vol. 5, 2011, no. 30, 1497 - 1506

Lyapunov Inequalities for Nonlinear p-Laplacian Problems with Weight Functions

M. M. Rodrigues

Departamento de Matemática
Universidade de Aveiro
3810-193 Aveiro, Portugal
mrodrigues@ua.pt

Abstract

In this paper we obtain Lyapunov inequality estimates for a single, cycled system and for a coupled system of p-Laplacian problems with weight functions on the one-dimensional case.

Mathematics Subject Classification: 34A34, 34B15

Keywords: p-Laplacian, Lyapunov inequality, boundary problem, ordinary differential equations

1 Introduction

Lyapunov inequalities have proved to be useful tools in oscillation theory, disconjugacy, eigenvalue problems and numerous other applications in the theory of differential and difference equations.

Eigenvalue problems for quasilinear operators of p-Laplace type have received considerable attention in the last years (see, e.g., [1, 3, 4, 6]).

The spectral counting function for the weighted p-Laplacian in one dimension was studied by Bonder and Pinasco in [5].

For the study of qualitative nature of solutions to ordinary linear differential equations, Lyapunov inequalities have proved to be useful tools.

The classical Lyapunov inequality states that, if $r : [a, b] \to \mathbb{R}$ is a positive continuous function and u is a solution of

$$-u''(x) = r(x)u(x), \quad x \in (a, b), \quad u(a) = u(b) = 0$$

then the following inequality holds

$$\int_a^b r(x)dx \geq \frac{4}{b-a}$$

(see [9]).

In [11] Pinasco obtained Lyapunov inequality for one-dimensional p-Laplacian problem

$$-(|u'|^{p-2}u)' = r(t)|u|^{p-2}u,\ t \in (a,b), \quad u(a) = u(b) = 0,$$

where $p > 1$ and $r \in C([a,b],(0,\infty))$ which is given by

$$\frac{2^p}{(b-a)^{p/q}} \leq \int_a^b r(t)dt$$

with $1/p + 1/q = 1$.

There have been many studies for various types of differential equations. Nápoli and Pinasco proved a Lyapunov type inequality for a monotone quasilinear operators generalizing the p-Laplacian, see [10]. A Lyapunov type inequality for the case of partial differential equations which have weight function in L^1 was proved by Cañada, Montero and Villegas, see [2].

In [12], Inbo Sim and Young-Hoon Lee obtained Lyapunov inequalities for a single equations as well as systems of one-dimensional p-Laplacian problems with singular weight functions, and for this propose they considered three specific classes of weight functions, i.e.,

$$\begin{aligned} \mathcal{A} &= \left\{ r \in C((a,b),[0,\infty)) : \int_a^{(a+b)/2} \varphi_p^{-1}\left(\int_s^{(a+b)/2} r(\tau)d\tau\right) ds \right. \\ &+ \left. \int_{(a+b)/2}^b \varphi_p^{-1}\left(\int_{(a+b)/2}^s r(\tau)d\tau\right) ds < \infty \right\}, \end{aligned}$$

where $\varphi_p(y) = |y|^{p-2}y$,

$$\mathcal{B} = \left\{ r \in C((a,b),[0,\infty)) : \int_a^b (s-a)^{p-1}(b-s)^{p-1} r(s)ds < \infty \right\}$$

and

$$\begin{aligned} \mathcal{C} &= \{ r \in C((a,b),[0,\infty)) : \text{ there are } \alpha, \beta > 0 \text{ such that } \alpha, \beta < p-1 \text{ and} \\ & \left. \int_a^b (s-a)^{\alpha}(b-s)^{\beta} r(s)ds < \infty \right\}. \end{aligned}$$

Here, we consider the following nonlinear problem for p-Laplacian with two weight functions

$$\begin{aligned} -\left(s(t)\varphi_p(u'(t))\right)' &= r(t)\varphi_p(u(t)), \quad t \in (a,b), \\ & u(a) = u(b) = 0 \end{aligned} \tag{1}$$

where

$$\varphi_p(y) = |y|^{p-2}y,\ p > 1,$$

$r \in C([a,b],(0,\infty)), s \in C^1([a,b],(0,\infty)),\ s(t) \geq k,$ for all $t \in [a,b]$ and $k > 0$.

The paper is organized as follows. Section (2) is devoted to estimate Lyapunov inequality for a one-dimensional p-Laplacian problem like (1) when $r \in \mathcal{A} \cap \mathcal{B}$. In Section (3), we estimate Lyapunov inequality for cycled system of one-dimensional p-Laplacian problem with $r \in \mathcal{C}$. At the end, in Section (4), we obtain Lyapunov inequality for coupled system of one-dimensional p-Laplacian problem when $r \in \mathcal{C}$.

2 Lyapunov inequality for a one-dimensional p-Laplacian problem

In this section, we estimate Lyapunov inequality for (1).

Here, we assume $r \in \mathcal{A} \cap \mathcal{B}$, $s \in C^1([a,b],(0,\infty))$ with $s(t) \geq k,$ for all $t \in [a,b]$ and $k > 0$. Hence, the solution u of the problem (1) means that $u \in C[a,b] \cap C^1[a,b]$, $\varphi_p(u'(t))$ is absolutely continuous in any compact subinterval of (a,b).

Theorem 2.1 *Let $r \in \mathcal{A} \cap \mathcal{B}$, $s \in C^1([a,b],(0,\infty))$ with $s(t) \geq k > 0$. If u is a positive solution of (1) then the following inequality hold*

$$\frac{k(b-a)^{p-1}}{2^{p-2}} \leq \int_a^b (t-a)^{p-1}(b-t)^{p-1}r(t)dt. \tag{2}$$

Proof: Integrating by parts (1) after multiplying both sides of (1) by u, we obtain

$$\int_a^b s(t)|u'(t)|^p dt = \int_a^b r(t)|u(t)|^p. \tag{3}$$

Clearly, by using Hölder's inequality, we get

$$|u(t)| \leq \int_a^t |u'(s)|ds \leq (t-a)^{(p-1)/p} \left(\int_a^t |u'(s)|^p ds\right)^{1/p}. \tag{4}$$

For $a \leq t \leq (a+b)/2$ which implies $t-a \leq (2/(b-a))(t-a)(b-t)$, and from (4) we obtain

$$|u(t)| \leq \left(\frac{2}{b-a}(t-a)(b-t)\right)^{(p-1)/p} \left(\int_a^{(a+b)/2} |u'(s)|^p ds\right)^{1/p}. \tag{5}$$

Thus, we have

$$|u(t)|^p \leq \left(\frac{2}{b-a}(t-a)(b-t)\right)^{(p-1)} \left(\int_a^{(a+b)/2} |u'(s)|^p ds\right). \qquad (6)$$

Similarly, by Hölder's inequality, we obtain

$$|u(t)| \leq \int_t^b |u'(s)|ds \leq (b-t)^{(p-1)/p} \left(\int_t^b |u'(s)|^p ds\right)^{1/p}. \qquad (7)$$

For $(a+b)/2 \leq t \leq b$ which implies $b-t \leq (2/(b-a))(t-a)(b-t)$ and therefore from the previous inequality, we have

$$|u(t)|^p \leq \left(\frac{2}{b-a}(t-a)(b-t)\right)^{(p-1)} \left(\int_{(a+b)/2}^b |u'(s)|^p ds\right). \qquad (8)$$

Adding the inequalities (6) and (8), we get

$$2|u(t)|^p \leq \left(\frac{2}{b-a}(t-a)(b-t)\right)^{(p-1)} \left(\int_a^b |u'(s)|^p ds\right). \qquad (9)$$

Let us remark that $s(t) \geq k > 0, t \in [a,b]$.

Multiplying both sides of (9) by $kr(t)$ and after some calculations, we have

$$\begin{aligned}\frac{(b-a)^{p-1}}{2^{p-2}} kr(t)|u(t)|^p &\leq r(t)\left((t-a)(b-t)\right)^{p-1} \left(\int_a^b k|u'(s)|^p ds\right). \\ &\leq r(t)\left((t-a)(b-t)\right)^{p-1} \left(\int_a^b s(s)|u'(s)|^p ds\right) \qquad (10)\end{aligned}$$

Integrating (10) on $[a,b]$ and using

$$\int_a^b s(t)|u'(t)|^p dt = \int_a^b r(t)|u(t)|^p dt$$

(see (3)), we obtain

$$\begin{aligned}\frac{(b-a)^{p-1}}{2^{p-2}} \int_a^b s(t)|u'(t)|^p dt &= \int_a^b \frac{k(b-a)^{p-1}}{2^{p-2}} r(t)|u(t)|^p dt \\ &\leq \int_a^b r(t)\left((t-a)(b-t)\right)^{p-1} \left(\int_a^b s(s)|u'(s)|^p ds\right) dt.\end{aligned}$$

Therefore,

$$\frac{k(b-a)^{p-1}}{2^{p-2}} \leq \int_a^b r(t)\left((t-a)(b-t)\right)^{p-1} dt \qquad (11)$$

which proves our result.

■

3 Lyapunov inequality for cycled system of one-dimensional p-Laplacian problem

This section is devoted for estimate Lyapunov inequality for cycled system of one-dimensional p-Laplacian problem when $r_i \in \mathcal{C}$, $s_i \in C^1([a,b],(0,\infty))$ with $s_i(t) \geq k_i$, for all $t \in [a,b]$ and $k > 0$, for $i = 1, \dots, n$.

Here, we consider the following cycled system

$$\begin{aligned}
&(s_1(t)\varphi_p(u_1'(t)))' = r_1(t)\varphi_p(u_2(t)), \quad t \in (a,b),\\
&(s_2(t)\varphi_p(u_2'(t)))' = r_2(t)\varphi_p(u_3(t)), \quad t \in (a,b),\\
&\dots\\
&\left(s_{n-1}(t)\varphi_p(u_{n-1}'(t))\right)' = r_{n-1}(t)\varphi_p(u_n(t)), \quad t \in (a,b),\\
&(s_n(t)\varphi_p(u_n'(t)))' = r_n(t)\varphi_p(u_1(t)), \quad t \in (a,b),
\end{aligned}$$

$$u_1(a) = \dots u_n(a) = 0 = u_1(b) = \dots = u_n(b) \tag{12}$$

with $u_i \in C[a,b] \cap C^1[a,b]$, $\varphi_p(u_i'(t))$ is absolutely continuous in any compact subinterval of (a,b).

Theorem 3.1 *Let $s_i \in C^1([a,b],(0,\infty))$, $s_i(t) \geq k_i > 0$, and $r_i \in \mathcal{C}$, $i = 1, \dots, n$. If $(u_1, u_2, \dots, u_n)$ is a positive solution of (12) then*

$$\prod_{i=1}^{n} k_i^2 \frac{[(b-a)^{p-1}]^n}{[2^{p-2}]^n} \leq \int_a^b ((t-a)(b-t))^{p-1} r_1(t)dt \dots \int_a^b ((t-a)(b-t))^{p-1} r_n(t)dt. \tag{13}$$

Proof: We will make the proof only for the case $n = 2$. The general case can be proved by repeating the procedure that we will present below.

Taking into account (9) and (10), for $i = 1, 2$, we obtain

$$|u_i(t)|^{p-1} \leq \frac{2^{(p-2)(p-1)/p}}{k_i(b-a)^{(p-1)^2/p}} ((t-a)(b-t))^{(p-1)^2/p} \left(\int_a^b s_i(s)|u_i'(s)|^p ds\right)^{(p-1)/p}, \tag{14}$$

i.e.,

$$|u_i(t)| \leq \frac{2^{(p-2)/p}}{k_i(b-a)^{(p-1)/p}} ((t-a)(b-t))^{(p-1)/p} \left(\int_a^b s_i(s)|u_i'(s)|^p ds\right)^{1/p}. \tag{15}$$

Integrating the first equation of (12) on $[a,b]$ after multiplying by u_1 and using (14), (15), we have

$$\begin{aligned}\int_a^b s_1(t)|u_1'(t)|^p dt &\leq \int_a^b r_1(t)|u_2(t)|^{p-1}|u_1(t)|dt \\ &\leq \frac{2^{p-2}}{k_2k_1(b-a)^{p-1}}\int_a^b ((t-a)(b-t))^{p-1} r_1(t)dt \\ &\times \left(\int_a^b s_2(s)|u_2'(s)|^p ds\right)^{(p-1)/p}\left(\int_a^b s_1(s)|u_1'(s)|^p ds\right)^{1/p}.\end{aligned} \tag{16}$$

So, we get

$$\begin{aligned}\left(\int_a^b s_1(t)|u_1'(t)|^p dt\right)^{(p-1)/p} &\leq \frac{2^{p-2}}{k_2k_1(b-a)^{p-1}}\int_a^b ((t-a)(b-t))^{p-1} r_1(t)dt \\ &\times \left(\int_a^b s_2(s)|u_2'(s)|^p ds\right)^{(p-1)/p}.\end{aligned} \tag{17}$$

Similarly, for the second equation of (12), we obtain

$$\begin{aligned}\left(\int_a^b s_2(t)|u_2'(t)|^p dt\right)^{(p-1)/p} &\leq \frac{2^{p-2}}{k_1k_2(b-a)^{p-1}}\int_a^b ((t-a)(b-t))^{p-1} r_2(t)dt \\ &\times \left(\int_a^b s_1(s)|u_1'(s)|^p ds\right)^{(p-1)/p}.\end{aligned} \tag{18}$$

Thus, we have

$$\int_a^b ((t-a)(b-t))^{p-1}r_1(t)dt\int_a^b ((t-a)(b-t))^{p-1}r_2(t)dt \geq \frac{(k_2k_1)^2((b-a)^{p-1})^2}{(2^{p-2})^2}. \tag{19}$$

■

Corollary 3.2 *Assume* $s_i \in C^1([a,b],(0,\infty))$, $s_i(t) \geq k_i > 0$, *and* $r_i = r \in \mathcal{C}$, $k_i = k$ *for* $i = 1,2,\dots,n$. *If* $(u_1,u_2,\dots,u_n)$ *is a positive solution of (12) then we have*

$$\frac{k^2(b-a)^{p-1}}{2^{p-2}} \leq \int_a^b ((t-a)(b-t))^{p-1}r(t)dt. \tag{20}$$

4 Lyapunov inequality for coupled system of one-dimensional p-Laplacian problem

Here, we consider the following coupled system

$$\begin{aligned}
&(s_1(t)\varphi_p(u_1'(t)))' + r_1(t)\left(\varphi_p(u_1(t)) + \varphi_p(u_2(t)) + \ldots + \varphi_p(u_n(t))\right) = 0, \quad t \in (a,b),\\
&(s_2(t)\varphi_p(u_2'(t)))' + r_2(t)\left(\varphi_p(u_1(t)) + \varphi_p(u_2(t)) + \ldots + \varphi_p(u_n(t))\right) = 0, \quad t \in (a,b),\\
&\ldots\\
&(s_n(t)\varphi_p(u_n'(t)))' + r_n(t)\left(\varphi_p(u_1(t)) + \varphi_p(u_2(t)) + \ldots + \varphi_p(u_n(t))\right) = 0, \quad t \in (a,b),\\
&\\
&u_1(a) = \ldots u_n(a) = 0 = u_1(b) = \ldots = u_n(b),
\end{aligned} \tag{21}$$

where $r_i \in \mathcal{C}$, $s_i \in C^1([a,b],(0,\infty))$, $s_i(t) \geq k_i > 0$, for $i = 1,\ldots,n$. We can give a definition for solution of (21) as the definition for a solution of (12) and it is known that all positive solutions for (21) are of class $C^1(a,b)$.

Theorem 4.1 *Let $s_i \in C^1([a,b],(0,\infty))$, $s_i(t) \geq k_i > 0$, $M = \max_{i=1,\ldots,n}\{1/k_i\}$, $r_i \in \mathcal{C}$, $i = 1,\ldots,n$. If $(u_1,u_2,\ldots,u_n)$ is a positive solution of (21) then*

$$\frac{(b-a)^{p-1}}{(M+(n-1)M^2)2^{p-2}} \leq \int_a^b ((t-a)(b-t))^{p-1} r_1(t)dt + \ldots + \int_a^b ((t-a)(b-t))^{p-1} r_n(t)dt. \tag{22}$$

Proof: As in proof of Theorem (3.1), we only show the case $n = 2$.

Multiplying the first equation of (21) by u_1 and integrating on $[a,b]$, and using (10), (14), (15), we have

$$\begin{aligned}
\int_a^b s_1(t)|u_1'(t)|^p dt &\leq \int_a^b r_1(t)|u_1|^p dt + \int_a^b r_1(t)|u_2(t)|^{p-1}|u_1(t)|dt\\
&\leq \frac{2^{(p-2)}}{k_1(b-a)^{(p-1)}} \int_a^b ((t-a)(b-t))^{(p-1)} r_1(t)dt \int_a^b s_1(s)|u_1'(s)|^p ds\\
&+ \frac{2^{(p-2)}}{k_2k_1(b-a)^{(p-1)}} \int_a^b ((t-a)(b-t))^{(p-1)} r_1(t)dt\\
&\times \left(\int_a^b s_2(s)|u_2'(s)|^p\right)^{(p-1)/p} ds \left(\int_a^b s_1(s)|u_1'(s)|^p\right)^{1/p} ds.
\end{aligned} \tag{23}$$

Similarly, from the second equation of (21), we have

$$
\begin{aligned}
\int_a^b s_2(t)|u_2'(t)|^p dt &\leq \int_a^b r_2(t)|u_2(t)|^p dt + \int_a^b r_2(t)|u_1(t)|^{p-1}|u_2(t)|dt \\
&\leq \frac{2^{(p-2)}}{k_2(b-a)^{(p-1)}} \int_a^b ((t-a)(b-t))^{(p-1)} r_2(t)dt \int_a^b s_2(s)|u_2'(s)|^p ds \\
&+ \frac{2^{(p-2)}}{k_2 k_1 (b-a)^{(p-1)}} \int_a^b ((t-a)(b-t))^{(p-1)} r_2(t)dt \\
&\times \left(\int_a^b s_1(s)|u_1'(s)|^p\right)^{(p-1)/p} ds \left(\int_a^b s_2(s)|u_2'(s)|^p\right)^{1/p} ds. \qquad (24)
\end{aligned}
$$

Let us denote

$$
\begin{aligned}
X &= \int_a^b s_1(t)|u_1'(t)|^p dt, \\
Y &= \int_a^b s_2(t)|u_2'(t)|^p dt, \\
C_1 &= \frac{2^{(p-2)}}{k_1(b-a)^{(p-1)}} \int_a^b ((t-a)(b-t))^{(p-1)} r_1(t)dt, \\
C_2 &= \frac{2^{(p-2)}}{k_2(b-a)^{(p-1)}} \int_a^b ((t-a)(b-t))^{(p-1)} r_2(t)dt. \qquad (25)
\end{aligned}
$$

From (23), (24) and (25) we have

$$
\begin{aligned}
X &\leq C_1 X + \frac{1}{k_2} C_1 X^{1/p} Y^{(p-1)/p}, \\
Y &\leq C_2 Y + \frac{1}{k_1} C_2 Y^{1/p} X^{(p-1)/p}, \qquad (26)
\end{aligned}
$$

respectively. Inequalities (26) implies

$$
\begin{aligned}
X &\leq C_1(X+Y) + \frac{1}{k_2} C_1 \left(X^{1/p} Y^{(p-1)/p} + Y^{1/p} X^{(p-1)/p}\right), \\
Y &\leq C_2(X+Y) + \frac{1}{k_1} C_2 \left(X^{1/p} Y^{(p-1)/p} + Y^{1/p} X^{(p-1)/p}\right), \qquad (27)
\end{aligned}
$$

respectively. Therefore, we have

$$
X + Y \leq (C_1 + C_2)(X+Y) + \left(\frac{1}{k_2} C_1 + \frac{1}{k_1} C_2\right) \left(X^{1/p} Y^{(p-1)/p} + Y^{1/p} X^{(p-1)/p}\right). \qquad (28)
$$

Since $\left(X^{1/p} Y^{(p-1)/p} + Y^{1/p} X^{(p-1)/p}\right) \leq X + Y$ (see [7], page 38), we get

$$
X + Y \leq (C_1 + C_2)(X+Y) + \left(\frac{1}{k_2} C_1 + \frac{1}{k_1} C_2\right)(X+Y). \qquad (29)
$$

Hence, we obtain

$$1 \leq (M+M^2)\frac{2^{(p-2)}}{(b-a)^{(p-1)}}\left(\int_a^b ((t-a)(b-t))^{(p-1)} r_1(t)dt + \int_a^b ((t-a)(b-t))^{(p-1)} r_2(t)dt\right), \tag{30}$$

where $M = \max_{i=1,\dots,n} \{1/k_i\}$.

Thus (30) can be rewritten by

$$\int_a^b ((t-a)(b-t))^{(p-1)} r_1(t)dt + \int_a^b ((t-a)(b-t))^{(p-1)} r_2(t)dt \geq \frac{(b-a)^{p-1}}{(M+M^2)2^{p-2}}. \tag{31}$$

■

Corollary 4.2 *Assume $s_i \in C^1([a,b],(0,\infty))$, $s_i(t) \geq k_i > 0$, and $r_i = r \in \mathcal{C}$, $k_i = k$ for $i = 1, 2, \dots, n$. If $(u_1, u_2, \dots, u_n)$ is a positive solution of (21), then we have*

$$\frac{1}{n/k+(n-1)n/k^2}\frac{(b-a)^{p-1}}{2^{p-2}} \leq \int_a^b ((t-a)(b-t))^{p-1} r(t)dt. \tag{32}$$

Acknowledgments

The author was supported by “CIDMA - Centro de Investigação e Desenvolvimento em Matemática e Aplicações” of the University of Aveiro. The author wishes to express his gratitude to *Fundação para a Ciência e a Tecnologia* for the support of his work via the grant `SFRH/BPD/73537/2010`.

References

[1] A. Anane, M. Moussa, and O. Chakrone, *Spectrum of one dimensional p-Laplacian opearator with indefinitive weight*, Electron. J.Qual. Theory Differ. Equ., **17**, (2002), 1–11.

[2] A. Cañada, J.A. Montero and S. Villegas, *Lyaunov inequlities for partial equations*, J. of Fun. Anal.237, **1**, (2006), 176–193.

[3] M. Del Pino, P. Drábek, and R. Manásevich, *The Fredholm alternative at the first eigenvalue for the one-dimensional p-Laplacian*, C.R. Acad. Sci. Paris Ser.I Math.327, **5**, (1998), 461–465.

[4] P.Drábek and R. Manásevich, *On the closed solution to some nonhomogeneous eigenvalue problems with p- Laplacian*, Diff. Int. Eq.12, **6**, (1999), 773–788.

[5] J. Fernandez Bonder and J. P. Pinasco, *Asymptotic behavior of the eigenvalue of the one dimensional weighted p-Laplace operator*, Ark. Mat., **41**, (2003), 266-280.

[6] M. Guedda and L. Véron, *Bifurcation phenomena associated to the p-Laplace operator*, Trans. Amer. Math. Soc. 310, **1**, (1988), 419–431.

[7] G.H. Hardy, J.E. Littlewood, and G. Polya, Inequalities, Cambridge University Press, Cambridge, UK, 1978.

[8] J. Henderson and H. Wang, *Nonlinear eigenvalue problems for quasilinear systems*, Computers and Mathematics with Applications, **49**, (2005), 1941–1949.

[9] A. Liapounoff, *Problème Général de la Stabilité du Mouvement*, Annals of Mathematics Studies, **17**, Princeton University Press, New Jersey, 1947.

[10] P.L. Nápoli and J.P. Pinasco, *A Lyapunov inequality for monotone quasilinear operators*, Diff. and Int. Eq., **8(10)**, (2005), 1193–1200.

[11] J.P. Pinasco, *Lower bounds for eigenvalues of the one-dimensional p-Laplacian*, Abst. and Appl. Anal., **2**, (2004), 147–153.

[12] I. Sim and Y.H. Lee, *Lyapunov Inequalities for one-dimendional p-Laplacian problems with a singular weight function*, J. of Ineq. and Appl., (2010), 1–9.

[13] W. Walter, *Sturm-Liouville theory for the radial Δ_p-operator*, Math.Z. 227, **1**, (1998), 175–185.

Received: March, 2011