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A Sequence Space Defined by a Sequence of Modulus Functions

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Abstract. In the present paper we study a sequence space $m(F, \varphi, p)$ defined by a sequence of modulus functions and examine some topological properties of this space.

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1. Introduction and Preliminaries

A function $f : [0, \infty) \to [0, \infty)$ is said to be a modulus function if it satisfies the following:

1. $f(x) = 0$ if and only if $x = 0$;
2. $f(x + y) \leq f(x) + f(y)$ for all $x \geq 0$, $y \geq 0$;
3. f is increasing;
4. f is continuous from right at 0.

It follows that f must be continuous everywhere on $[0, \infty)$. The modulus function may be bounded or unbounded. For example, if we take $f(x) = \frac{x}{x+1}$, then $f(x)$ is bounded. If $f(x) = x^p$, $0 < p < 1$, then $f(x)$ is unbounded. For more details see ([1], [2], [5] etc.).

Let w be the set of all sequences, real or complex numbers $x = (x_k)$ normed by $||x|| = \sup_k |x_k|$, where $k \in \mathbb{N}$, the set of positive integers.

Let $\mathcal{C}$ denote the space whose elements are finite sets of distinct positive integers. Given any elements σ of $\mathcal{C}$, we denote by $c(\sigma)$ the sequence $\{c_n(\sigma)\}$ which is such that $c_n(\sigma) = 1$ if $n \in \sigma, c_n(\sigma) = 0$ otherwise. Further

$$\mathcal{C}_s = \Big\{\sigma \in \mathcal{C} : \sum_{n=1}^{\infty} c_n(\sigma) \leq s\Big\},$$

be the set of those σ whose support has cardinality at most s see[3]. Throughout the paper φ_n denotes a non-decreasing sequence of positive numbers such that $n\varphi_{n+1} \leq (n+1)\varphi_n$ for all $n \in \mathbb{N}$.

If $x = (x_k)$ is a sequence, then $S(x)$ denotes the set of all permutation of the elements of (x_k). A sequence space E is said to be symmetric if $S(x) \subset E$ for all $x \in E$. A sequence space E is said to be solid if $(y_n) \in E$ whenever $(x_n) \in E$ and $|y_n| \leq |x_n|$ for all $n \in \mathbb{N}$.
A BK-space is a Banach sequence space E in which the coordinate maps are continuous, i.e if $(x_k^{(n)}) \in E$, then

$$||(x_k^{(n)}) - (x_k)|| \to 0 \ \text{ as } \ n \to \infty$$
$$\Rightarrow |(x_k^{(n)}) - (x_k)| \to 0 \ \text{ as } \ n \to \infty, \ \text{ for each fixed } k.$$

The space $m(\varphi)$ was defined and introduced by Sargent [4] as follows:

$$m(\varphi) = \Big\{x = \{x_k\} \in w : \sup_{s\geq 1} \sup_{\sigma \in \mathcal{C}_s} \Big(\frac{1}{\varphi_s} \sum_{k\in\sigma} |x_k|\Big) < \infty\Big\}.$$

The space $m(\varphi)$ was extended to (m, φ, p) by Tripathy and Sen [6] as follows:

$$m(\varphi, p) = \Big\{x = \{x_k\} \in w : \sup_{s\geq 1} \sup_{\sigma \in \mathcal{C}_s} \Big(\frac{1}{\varphi_s} \sum_{k\in\sigma} |x_k|^p\Big) < \infty\Big\}.$$

Let $F = (f_k)$ be a sequence of modulus functions. In this paper we define the sequence space $m(F, \varphi, p)$ as :

$$m(F, \varphi, p) = \Big\{x = (x_k) \in w : \sup_{s\geq 1} \sup_{\sigma \in \mathcal{C}_s} \frac{1}{\varphi_s} \big\{ \sum_{k\in\sigma} \big[f_k\big(\frac{|x_k|}{\rho}\big)\big]^p\big\}^{\frac{1}{p}} < \infty, \ \text{ for some } \ \rho > 0\Big\}.$$

2. Main Results

Theorem 1. *The space $m(F,\varphi,p)$ is complete.*

Proof. Let $\{x^{(n)}\}$ be a Cauchy sequence in $m(F,\varphi,p)$. Then

$$\sup_{s\geq 1}\sup_{\sigma\in C_s}\frac{1}{\varphi_s}\Big\{\sum_{k\in\sigma}\big[f_k\big(\frac{|x_k^{(n)}|}{\rho}\big)\big]^p\Big\}^{1/p}<\infty,$$

for some $\rho>0$ and for all $n\in\mathbb{N}$. For each $\epsilon>0$, there exists a positive integer n_0 such that

$$\|x^{(m)}-x^{(n)}\|_{m(F,\varphi,p)}<\epsilon,\ \text{ for all }\ m,n\geq n_0.$$

This implies that

$$\sup_{s\geq 1}\sup_{\sigma\in C_s}\frac{1}{\varphi_s}\Big\{\sum_{k\in\sigma}\big[f_k\big(\frac{|x_k^{(m)}-x_k^{(n)}|}{\rho}\big)\big]^p\Big\}^{1/p}<\epsilon \tag{1}$$

for some $\rho>0$ and for all $m,n\geq n_0$. Hence $|x_k^{(m)}-x_k^{(n)}|<\epsilon\ \varphi_1$ for all $m,n\geq n_0$ and for all $k\in\mathbb{N}$, showing that for each fixed $k(1\leq k<\infty)$,the sequence $\{x_k^{(n)}\}$ is a Cauchy sequence in $\mathbb{C}$.
Let $x_k^{(n)}\to x_k$ as $n\to\infty$. We define $x=(x_1,x_2,\cdots)$. We need to show that $x\in m(F,\varphi,p)$ and $x^{(n)}\to x$. From (1) we get , for each fixed s.

$$\sum_{k\in\sigma}\big[f_k\big(\frac{|x_k^{(m)}-x_k^{(n)}|}{\rho}\big)\big]^p<\epsilon^p\ \varphi_s^p,\quad \text{for some }\ \rho>0,\ \text{ for all, }\ m,n\geq n_0\ \text{ and }\ \sigma\in\mathcal{C}_s.$$

Taking $n\to\infty$, we get

$$\sum_{k\in\sigma}\big[f_k\big(\frac{|x_k^{(m)}-x_k|}{\rho}\big)\big]^p<\epsilon^p\ \varphi_s^p,\quad \text{for some }\ \rho>0,\ \text{ for all }\ m,n\geq n_0\ \text{ and }\ \sigma\in\mathcal{C}_s.$$

This implies that

$$\sup_{s\geq 1}\sup_{\sigma\in C_s}\frac{1}{\varphi_s}\Big\{\sum_{k\in\sigma}\big[f_k\big(\frac{|x_k^{(m)}-x_k|}{\rho}\big)\big]^p\Big\}^{1/p}<\epsilon \tag{2}$$

for some $\rho>0$, for all $m,n\geq n_0$.

$$\Rightarrow x^{(n)}-x\in m(F,\varphi,p),\ \text{ for all }\ n\geq n_0.$$

Hence $x=x^{(n_0)}+x-x^{(n_0)}\in m(F,\varphi,p)$ as $m(F,\varphi,p)$ is a linear space. From (2), $\|x^{(n)}-x\|_{m(F,\varphi,p)}<\epsilon$, for all $m,n\geq n_0$, which implies that $\|x^{(n)}-x\|_{m(F,\varphi,p)}\to 0$ as $n\to\infty$. Hence $m(F,\varphi,p)$ $(1\leq p<\infty)$ is a Banach space.

Theorem 2. *The space $m(F,\varphi,p)$ is a BK-space.*

Proof. Suppose that $\|x^{(n)}-x\|_{m(F,\varphi,p)} \to 0$ as $n\to\infty$. For each $\epsilon>0$ there exists $n_0\in\mathbb{N}$ such that

$$\|x^{(n)}-x\| < \epsilon \quad \text{for all} \;\; n\geq n_0.$$

This implies that

$$\sup_{s\geq 1}\sup_{\sigma\in C_s}\frac{1}{\varphi_s}\Big\{\sum_{k\in\sigma}\big[f_k\big(\frac{|x_k^{(n)}-x_k|}{\rho}\big)\big]^p\Big\}^{1/p} < \epsilon \quad \text{for some} \;\; \rho>0, \quad \text{for all} \;\; n\geq n_0.$$

Consequently

$|x_k^{(n)}-x_k| < \epsilon\varphi_1$, for all $n\geq n_0$ and for all k. So $|x_k^{(n)}-x_k|\to 0$ as $n\to\infty$. This completes the proof.

Corollary 3.(i) *The space $m(F,\varphi,p)$ is a symmetric space. If $x\in m(F,\varphi,p)$ and $v\in S(x)$, then*

$$\|v\|_{m(F,\varphi,p)} = \|x\|_{m(F,\varphi,p)}.$$

(ii) The space $m(F,\varphi,p)$ is a normal space.

Proof. It is obvious.

Theorem 4. $m(\varphi)\subseteq m(F,\varphi,p)$.

Proof. Suppose that $x\in m(\varphi)$. Then

$$\|x\|_{m(\varphi)} = \sup_{s\geq 1}\sup_{\sigma\in C_s}\frac{1}{\varphi_s}\big\{\sum_{k\in\sigma}|x_k|\big\} = K < \infty.$$

Hence for each fixed s,

$$\sum_{k\in\sigma}|x_k| \leq K\varphi_s, \sigma\in C_s, \;\; \text{for some} \;\; \rho>0$$

so that

$$\sup_{s\geq 1}\sup_{\sigma\in C_s}\frac{1}{\varphi_s}\Big\{\sum_{k\in\sigma}\big[f_k\big(\frac{|x_k|}{\rho}\big)\big]^p\Big\}^{1/p} \leq K, \;\; \text{for some} \;\; \rho>0.$$

Thus $x\in m(F,\varphi,p)$. Hence $m(\varphi)\subseteq m(F,\varphi,p)$.

Theorem 5. $m(F,\varphi,p)\subseteq m(F,\psi,p)$ *if and only if* $\sup_{s\geq 1}\big(\frac{\varphi_s}{\psi_s}\big) < \infty$.

Proof. Let $\sup_{s\geq 1}\big(\frac{\varphi_s}{\psi_s}\big) = K < \infty$. Then $\varphi_s \leq K\psi_s$. Now if $(x_k) \in m(F,\varphi,p)$, then

$$\sup_{s\geq 1}\sup_{\sigma\in C_s}\Big[\frac{1}{\varphi_s}\Big\{\sum_{k\in\sigma}\big[f_k\big(\frac{|x_k|}{\rho}\big)\big]^p\Big\}^{1/p}\Big] < \infty, \quad \text{for some} \quad \rho > 0.$$

This implies that

$$\sup_{s\geq 1}\sup_{\sigma\in C_s}\Big[\frac{1}{K\psi_s}\Big\{\sum_{k\in\sigma}\big[f_k\big(\frac{|x_k|}{\rho}\big)\big]^p\Big\}^{1/p}\Big] < \infty, \quad \text{for some} \ \ \rho > 0,$$

so that $\|x\|_{m(F,\psi,p)} < \infty$. Hence $m(F,\varphi,p) \subseteq m(F,\psi,p)$.
Conversely, suppose that $m(F,\varphi,p) \subseteq m(F,\psi,p)$. We need to show that

$$\sup_{s\geq 1}\big(\frac{\varphi_s}{\psi_s}\big) = \sup_{s\geq 1}\big(\eta_s\big) < \infty.$$

Let $\sup_{s\geq 1}(\eta_s) = \infty$. Then there exists a subsequence $\big(\eta_{s_i}\big)$ of $\big(\eta_s\big)$ such that $\lim_{i\to\infty}\big(\eta_{s_i}\big) = \infty$. Then for $(x_k) \in m(F,\varphi,p)$ we have

$$\begin{aligned}\sup_{s\geq 1}\sup_{\sigma\in C_s}\Big[\frac{1}{\psi_s}\Big\{\sum_{k\in\sigma}\big[f_k\big(\frac{|x_k|}{\rho}\big)\big]^p\Big\}^{1/p}\Big] &\geq \sup_{s\geq 1}\sup_{\sigma\in C_s}\Big[\frac{\psi_{s_i}}{\varphi_{s_i}}\Big\{\sum_{k\in\sigma}\big[f_k\big(\frac{|x_k|}{\rho}\big)\big]^p\Big\}^{1/p}\Big]\\ &= \infty.\end{aligned}$$

for some $\rho > 0$. This implies that $(x_k) \notin m(F,\psi,p)$, a contradiction which completes the proof.

Theorem 6. $\ell^p \subseteq m(F,\varphi,p) \subset \ell^\infty$.
Proof. Since $m(F,\varphi,p) = \ell^p$ for $f_k(x) = x$ and $\varphi_n = 1$, for all $n \in \mathbb{N}$, it follows that $\ell^p \subseteq m(F,\varphi,p)$. Next, let $x \in m(F,\varphi,p)$. Then

$$\sup_{s\geq 1}\sup_{\sigma\in C_s}\Big[\frac{1}{\varphi_s}\Big\{\sum_{n\in\sigma}\big[f_k\big(\frac{|x_k|}{\rho}\big)\big]^p\Big\}^{1/p}\Big] = K < \infty, \ \ \text{for some} \ \ \rho > 0.$$

This implies that $|x_k| \leq K\varphi_1$, for all $k \in \mathbb{N}$, so that $x \in \ell^\infty$. Thus $m(F,\varphi,p) \subset \ell^\infty$.

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