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Common Unique Fixed Point Theorems for Compatible Map of Type(P)

Jyoti Nema

NRI Institute of Information Science and Technology Bhopal, (M.P) India
Jyoti_nema273@yahoo.co.in

K. Qureshi

NRI Institute of Information Science and Technology Bhopal, (M.P) India

Abstract

The aim of this paper to prove a unique common fixed point theorems for compatible mapping of type(P) for three and four self maps using contraction condition.

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1 Introduction

In 1922 the polish mathematician Banach proved a theorem which ensures under appropriate condition the existence and uniqueness of fixed point. His result is called Banach contraction principle. In 1986 G. Jungck defined compatible mapping. Others [1,2,3,4] also discussed few common fixed point theorems in complete metric space.

2 Preliminary Notes

Definition 2.1. According to G.Jungck[2] two self maps S and T of a Metric Space (X, d) are said to be compatible mappings if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Definition 2.2. Again G.Jungck ,P.P. Murthy and Y.J. Cho[3] introduced new concept i.e.compatible type(A), Two self maps S and T of a metric space

(X,d) are said to be compatible mappings of type (A), if $\lim_{n\to\infty} d(STx_n, TTx_n) = 0$ and $\lim_{n\to\infty} d(TSx_n, SSx_n) = 0$ when ever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Definition 2.3. By H.K Pathak and others[4] ,Two self maps S and T of a metric space (X,d) are said to be compatible mappings of type(P) if $\lim_{n\to\infty} d(SSx_n, TTx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

In 1993 G.Jungck and B.E.Rhoades[1] introduced compatible mapping and weakly commuting mapping. In this paper we extend above result.

3 Main Result

Theorem 3.1. *Let E, F and T be self mapping of a complete metric space (X,d) satisfying:*
(i) $E(X) \cup F(X) \subseteq T(X)$,
(ii)$d^2(Ex, Fy) \leq \alpha max\{d(Tx,Ty)d(Tx,Ex), d(Tx,Ty)d(Ty,Fy)\}$
$+\beta\{d(Tx,Fy)d(Ty,Ex)\}$,
for all $\alpha, \beta > 0, \beta < 1$ and $x, y \in X$,
(iii) The pair (E,T) or (F,T) is compatible of type (P),
(iv) If T is continuous,
Then E,F and T have a unique common fixed point.

Proof:Let $x_0 \in X$ arbitrary. Construct a sequence $\{Tx_n\}$, as follows.

$Tx_{2n+1} = Ex_{2n}; Tx_{2n+2} = Fx_{2n+1};$
and $n = 0, 1, 2, 3, \ldots$ From condition (ii) we have

$$d^2(Tx_{2n+1}, Tx_{2n+2}) = d^2(Ex_{2n}, Fx_{2n+1})$$
$$\leq \alpha max\{d(Tx_{2n}, Tx_{2n+1})d(Tx_{2n}, Ex_{2n}), d(Tx_{2n}, Tx_{2n+1})d(Tx_{2n+1}, Fx_{2n+1})\}$$
$$+ \beta\{d(Tx_{2n}, Fx_{2n+1})d(Tx_{2n+1}, Ex_{2n})\},$$
$$= \alpha max\{d(Tx_{2n}, Tx_{2n+1})d(Tx_{2n}, Tx_{2n+1}), d(Tx_{2n}, Tx_{2n+1})d(Tx_{2n+1}, Tx_{2n+2})\}$$
$$+ \beta\{d(Tx_{2n}, Tx_{2n+2})d(Tx_{2n+1}, Tx_{2n+1})\},$$
$$= \alpha max\{d(Tx_{2n}, Tx_{2n+1})d(Tx_{2n}, Tx_{2n+1}), d(Tx_{2n}, Tx_{2n+1})d(Tx_{2n+1}, Tx_{2n+2})\},$$
$$\Rightarrow d^2(Tx_{2n+1}, Tx_{2n+2}) \leq \alpha d(Tx_{2n}, Tx_{2n+1}) max\{d(Tx_{2n}, Tx_{2n+1}), d(Tx_{2n+1}, Tx_{2n+2})\},$$
$$\leq \alpha d(Tx_{2n}, Tx_{2n+1}) max\{d(Tx_{2n}, Tx_{2n+1}), d(Tx_{2n+1}, Tx_{2n+2})\}.$$

Two case arise

Case I. If

$d^2(Tx_{2n+1}, Tx_{2n+2}) \leq \alpha d^2(Tx_{2n}, Tx_{2n+1}),$

$\Rightarrow d(Tx_{2n+1}, Tx_{2n+2}) \leq k_1 d(Tx_{2n}, Tx_{2n+1}).$ where $k_1 = \alpha^{\frac{1}{2}}, k_1 < 1.$

Case II. If

$d^2(Tx_{2n+1}, Tx_{2n+2}) \leq \alpha d(Tx_{2n}, Tx_{2n-1}) d(Tx_{2n+1}, Tx_{2n+2}),$

$\Rightarrow d(Tx_{2n+1}, Tx_{2n+2}) \leq k_2 d(Tx_{2n}, Tx_{2n-1}).$ where $k_2 = \alpha, k_2 < 1.$

From Case I and Case II

$\Rightarrow d(Tx_{2n+1}, Tx_{2n+2}) \leq kd(Tx_{2n}, Tx_{2n+1})$ where $k = max\{k_1, k_2\},$

In general $d(Tx_{n+1}, Tx_n) \leq k^n d(Tx_0, Tx_1).$

Thus $\{Tx_n\}$ is a Cauchy sequence. Since X is complete,$\exists z \in X$ such that $Tx_n \to z$. It follows that the sequences $\{Ex_{2n}\}$ and $\{Fx_{2n+1}\}$ also converge to z. First suppose that the pair (E, T) is compatible of type P, Then from condition (ii) we have

$d^2(EEx_{2n}, Fx_{2n+1}) \leq \alpha max\{d(TEx_{2n}, Tx_{2n+1}) d(TEx_{2n}, EEx_{2n}), d(TEx_{2n}, Tx_{2n+1})$
$d(Tx_{2n+1}, Fx_{2n+1})\} + \beta d\{(TEx_{2n}, Fx_{2n+1}) d(Tx_{2n+1}, EEx_{2n})\}.$

Since T is continuous,$TTx_{2n} \to Tz, TEx_{2n} \to Tz$ as $n \to \infty$, the pair (E, T) is compatible of type (P), then $TTx_{2n} \to Tz$ and $EEx_{2n} \to Tz$ as $n \to \infty$.

Letting$n \to \infty$ and we get

$d^2(Tz, z) \leq \alpha max\{d(Tz, z) d(Tz, Tz), d(Tz, z)(d(z, z)\} + \beta\{d(Tz, z) d(z, Tz)\},$

$= \beta d(Tz, z) d(Tz, z),$

$\Rightarrow d(Tz, z) \leq \beta^{1/2} d(Tz, z).$ Let $\beta^{\frac{1}{2}} = k$ $\{$Since $\beta < 1$ therefore $k < 1\}$

$\Rightarrow (1 - k) d(Tz, z) \leq 0$ but $k \neq 0$.Therefore $Tz = z$.

Again from (ii), we have

$d^2(Ez, Fx_{2n+1}) \leq \alpha max\{d(Tz, Tx_{2n+1}) d(Tz, Ez) d(Tz, Tx_{2n+1}) d(Tx_{2n+1}, Fx_{2n+1})\}$
$+ \beta\{d(Tz, Fx_{2n+1}) d(Tx_{2n+1}, Ez)\}$. Letting as $n \to \infty$,

$\Rightarrow d^2(Ez, z) \leq \alpha max\{d(z, z) d(Ez, z), d(z, z) d(z, z)\} + \beta\{d(z, z) d(z, Ez)\},$

$\Rightarrow d(Ez, z) \leq 0$. But $d(Ez, z) \geq 0$.

Therefore $d(Ez, z) = 0$ and hence $Ez = z$.

Again from condition (ii), we have

$d^2(Ex_{2n}, Fz) \leq \alpha max\{d(Tx_{2n}, Tz) d(Tx_{2n}, Ex_{2n}), d(Tx_{2n}, Tz) d(Tz, Fz)\}$
$+ \beta\{d(Tx_{2n}, Fz) d(Tz, Ex_{2n})\}$, Letting as $n \to \infty$, we have

$\Rightarrow d^2(z, Fz) \leq \alpha max\{d(z, z) d(z, z), d(z, z) d(z, Fz)\} + \beta\{d(z, Fz) d(z, z)\},$

$\Rightarrow d(z, Fz) = 0$. Hence $z = Fz$.Thus $z = Fz = Ez = Tz$.

Showing that z is a common fixed point of E, F and T. Similarly we can prove that$z$ is a common fixed point of E, F and T when the pair (F, T) is compatible of type (P).

Uniqueness: Let z and w be two common fixed points of E, F and T, so $z = Ez = Fz = Tz$ and $w = Ew = Fw = Tw$ From condition (ii), we have

$$\begin{aligned} &d^2(Ez, Fw) \leq \alpha max\{d(Tz, Tw)d(Tz, Ez), d(Tz, Tw)d(Tw, Fw)\} \\ &+ \beta\{d(Tz, Fw)d(Tw, Ez)\}, \\ &= \alpha max\{d(z, w)d(z, z), d(z, w)d(w, w)\} + \beta\{d(z, w)d(w, z)\}, \\ &= \beta d(z, w)d(w, z), \\ &\Rightarrow (1 - \beta)d^2(z, w) \leq 0, \end{aligned}$$

$\Rightarrow d(z, w) < 0$ a contradiction, hence the proof.

Theorem 3.2. *:Let S, I, J and T be four self mapping of a complete metric space (X, d) into itself satisfying the conditions:*
(i) $S(X) \subseteq J(X), T(X) \subseteq I(X)$,
(ii) $d^2(Sx, Ty) \leq \alpha max\{d(Ix, Jy)(d(Ix, Sx), d(Ix, Sx)d(Jy, Ty)\}$
$+\beta\{d(Ix, Ty)d(Jy, Sx)\}$,
for all $\alpha, \beta > 0, \alpha + \beta < 1$ and $x, y \in X$.
(iii) one of $S, I, T$and J is continuous. And if
(iv) the pairs (S, I) and (T, J) are compatible of type (P).
Then S, I, T and J have a unique common fixed point.

Proof:Let x_0 in X be arbitrary.Define sequence $\{x_n\}$ and $\{y_n\}$ in X.
$y_{2n} = Sx_{2n} = Jx_{2n+1}$ and $y_{2n+1} = Tx_{2n+1} = Ix_{2n+2}$,
Using condition (ii), we have

$$\begin{aligned} &d^2(y_{2n}, y_{2n+1}) = d^2(Sx_{2n}, Tx_{2n+1}) \leq \alpha max\{d(Ix_{2n}, Jx_{2n+1})d(Ix_{2n}, Sx_{2n}), \\ &\qquad d(Ix_{2n}, Sx_{2n})d(Jx_{2n+1}, Tx_{2n+1})\} + \beta\{d(Ix_{2n}, Tx_{2n+1})d(Jx_{2n+1}, Sx_{2n})\}, \\ &\leq \alpha max\{d(y_{2n-1}, y_{2n})d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n})d(y_{2n}, y_{2n+1})\} \\ &+ \beta\{d(y_{2n-1}, y_{2n+1})d(y_{2n}, y_{2n})\}, \\ &= \alpha max\{d(y_{2n-1}, y_{2n})d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n})d(y_{2n}, y_{2n+1})\}, \\ &\Rightarrow d^2(y_{2n}, y_{2n+1}) \leq \alpha d(y_{2n}, y_{2n-1})max\{d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1})\}, \end{aligned}$$

Two case arise
Case I. If
$d^2(y_{2n}, y_{2n+1}) \leq \alpha d^2(y_{2n}, y_{2n-1})$,
$\Rightarrow d(y_{2n}, y_{2n+1}) \leq k_1 d(y_{2n}, y_{2n-1})$. where $k_1 = \alpha^{\frac{1}{2}}, k_1 < 1$.
Case II. If

$d^2(y_{2n}, y_{2n+1}) \leq \alpha d(y_{2n}, y_{2n+1})d(y_{2n}, y_{2n-1}),$
$\Rightarrow d(y_{2n}, y_{2n+1}) \leq k_2 d(y_{2n}, y_{2n-1}).$ where $k_2 = \alpha, k_2 < 1.$
From Case I and Case II
$\Rightarrow d(y_{2n}, y_{2n+1}) \leq kd(y_{2n}, y_{2n-1}).$ where $k = max\{k_1, k_2\}$and $k < 1$

In general $d(y_{n+1}, y_n) \leq k^n d(y_0, y_1).$
More over for any integer $p > 0$, we get
$d(y_n, y_{n+p}) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + . + d(y_{n+p+1}, y_{n+p}),$
$\leq (1 + k + k^2 + . + k_{p-1})d(y_n, y_{n+1}),$
$\leq (\frac{1}{1-K})k^n d(y_0, y_1).$

This means that $d(y_n, y_{n+p}) \to 0$ as $n \to \infty$. Thus $\{y_n\}$ is a Cauchy sequence in X and since X is a complete metric space, the sequence $\{y_n\}$ converges to a limit z in X. Hence the subsequences $\{y_{2n}\} = \{Sx_{2n}\} = \{Jx_{2n+1}\}$ and $\{y_{2n+1}\} = \{Tx_{2n+1}\} = \{Ix_{2n+2}\}$ also converge to the limit point z. Suppose that the mapping I is continuous, then $IIx_{2n} \to Iz$ and $ISx_{2n} \to Iz$ as $n \to \infty$. Since the pair (S, I) is compatible of type (P) we get $SSx_{2n} \to Iz$ as $n \to \infty$. Now by (ii)

$$d^2(S^2x_{2n}, Tx_{2n+1}) \leq \alpha max\{d(ISx_{2n}, Jx_{2n+1})d(ISx_{2n}, SSx_{2n}), d(ISx_{2n}, SSx_{2n}) \\ d(Jx_{2n+1}, Tx_{2n+1})\} + \beta\{d(ISx_{2n}, Tx_{2n+1})d(Jx_{2n+1}, SSx_{2n})\},$$

letting $n \to \infty$, using the compatible of type (P) of the pair (S, I) we get

$$d^2(Iz, z) \leq \alpha max\{d(Iz, z)d(Iz, Iz), d(Iz, Iz)d(z, z)\} + \beta\{d(Iz, z)d(z, Iz)\}, \\ = \beta d(Iz, z)d(Iz, z),$$

$\Rightarrow d(Iz, z) \leq \beta^{\frac{1}{2}} d(Iz, z).$ Let $\beta^{\frac{1}{2}} = k$ {since $\beta < 1$ Therefore $k < 1$}
$\Rightarrow (1 - k)d(Iz, z) \leq 0.$ $d(Iz, z) = 0$ i.e. $Iz = z$.

Further

$$d^2(Sz, Tx_{2n+1}) \leq \alpha max\{d(Iz, Jx_{2n+1})d(Iz, Sz), d(Iz, Sz)d(Jx_{2n+1}, Tx_{2n+1})\} \\ + \beta\{d(Iz, Tx_{2n+1})d(Jx_{2n+1}, Sz)\}.$$ As $n \to \infty$ we get

$\Rightarrow d^2(Sz, z) \leq \alpha max\{d(z, z)d(z, Sz), d(z, Sz)d(z, z)\} + \beta\{d(z, z)d(z, Sz)\},$
$\Rightarrow d(Sz, z) \leq 0,$
so we have $d(Sz, z) = 0$ and hence $Sz = z$.Thus $Sz = Iz = z$.

Since $S(X) \subseteq J(X)$, there is a point $z' \in X$such that $z = Sz = Jz'$. Now we prove that $Jz' = Tz'$. Now by (ii)

$$d^2(Sz, Tz') \leq \alpha max\{d(Iz, Jz')d(Iz, Sz), d(Iz, Sz)d(Jz', Tz')\} \\ + \beta\{d(Iz, Tz')d(Jz', Sz)\}, \\ = \alpha max\{d(z, z)d(z, z), d(z, z)d(z, Tz')\} + \beta\{d(z, Tz')d(z, z)\},$$

$\Rightarrow d^2(z, Tz') \leq 0$.This implies $Tz' = z$, hence $z = Jz' = Tz'$.

Take $y_n = z'$ for $n \geq 1$, then $Ty_n \to Tz' = z$ and $Jy_n \to Jz' = z$ as $n \to \infty$. Since the pair (T, J) is compatible of type (P) we get $TTz' = JJz'$, hence $Tz = Jz$. Now

$$\begin{aligned}
&d^2(Sz, Tz) \leq \alpha max\{d(Iz, Jz)d(Iz, Sz), d(Iz, Sz)d(Jz, Tz)\} \\
&+ \beta\{d(Iz, Tz)d(Jz, Sz)\}, \\
&= \alpha max\{d(z, Tz)d(z, z), d(z, z)d(Tz, Tz)\} + \beta\{d(z, Tz)d(Tz, z)\}, \\
&= \beta\{d(Tz, z)d(Tz, z)\}, \\
&\Rightarrow d(Tz, z) \leq \beta^{\frac{1}{2}} d(Tz, z). \qquad \text{Let } \beta^{\frac{1}{2}} = k \text{ but } \beta < 1. \text{ Therefore } k < 1 \\
&\Rightarrow (1 - k)d(Tz, z) \leq 0.
\end{aligned}$$

Hence $d(Tz, z) = 0$ i.e. $Tz = z$ and $z = Tz = Jz$. So z is a common fixed point of S, I, J and T. When continuity of I is assumed. The proof that z is a common fixed point of S, I, J and T is similar. When J is continuous. Now suppose that S is continuous, then $SSx_{2n}, SIx_{2n} \to Sz$ as $n \to \infty$.Now by (ii)

$$\begin{aligned}
&d^2(SIx_{2n}, Tx_{2n+1}) \leq \alpha max\{d(IIx_{2n}, Jx_{2n+1})d(IIx_{2n}, SIx_{2n}), d(IIx_{2n}, SIx_{2n}) \\
&\quad d(Jx_{2n+1}, Tx_{2n+1})\} + \beta\{d(IIx_{2n}, Tx_{2n+1})d(Jx_{2n+1}, SIx_{2n})\}, \{ \text{ as } n \to \infty, \} \\
&\Rightarrow d^2(Sz, z) \leq \alpha max\{d(Sz, z)d(Sz, Sz), d(Sz, Sz)d(z, z)\} + \beta\{d(Sz, z)d(z, Sz)\}, \\
&\qquad \leq \beta d(Sz, z)d(Sz, z). \\
&\Rightarrow d(Sz, z) \leq \beta^{\frac{1}{2}} d(Sz, z). \qquad \text{Let } \beta^{\frac{1}{2}} = k \text{ but } \beta < 1. \text{ Therefore } k < 1 \\
&\Rightarrow (1 - k)d(Sz, z) \leq 0.
\end{aligned}$$

Hence $d(Sz, z) = 0$ i.e. $Sz = z.S(X) \subseteq J(X)$, there is a point $u \in X$such that $z = Sz = Ju$. Now by (ii)

$$\begin{aligned}
&d^2(SIx_{2n}, Tu) \leq \alpha max\{d(IIx_{2n}, Ju)d(IIx_{2n}, SIx_{2n}), d(IIx_{2n}, SIx_{2n})d(Ju, Tu)\} \\
&\qquad + \beta\{d(IIx_{2n}, Tu)d(Ju, SIx_{2n})\}, \qquad \text{letting } n \to \infty, \text{ we get} \\
&d^2(z, Tu) = d^2(Sz, Tu) \leq \alpha max\{d(z, z)d(z, z), d(z, z)d(z, Tu)\} + \beta\{d(z, Tu)d(z, z)\}. \\
&\Rightarrow d(z, Tu) = 0 \text{ and hence } z = Tu. \text{ Hence } z = Tu = Ju.
\end{aligned}$$

Let $y_n = u$, then $Ty_n \to Tu = z$ and $Jy_n \to Ju = z$. Since (T, J) is compatible of type (P), $d(TTy_n, JJy_n) = 0$. This gives $Tz = Jz$. Further

$$\begin{aligned}
&d^2(Sx_{2n}, Tz) \leq \alpha max\{d(Ix_{2n}, Jz)d(Ix_{2n}, Sx_{2n}), d(Ix_{2n}, Sx_{2n})(d(Jz, Tz)\} \\
&\qquad + \beta\{d(Ix_{2n}, Tz)d(Jz, Sx_{2n})\}, \qquad \text{letting } n \to \infty, \text{we get} \\
&\Rightarrow d^2(z, Tz) \leq \alpha max\{d(z, Tz)d(z, z), d(z, z)d(Tz, Tz)\} + \beta\{d(z, Tz)d(Tz, z)\}, \\
&\qquad = \beta d(z, Tz)d(z, Tz). \qquad \text{Let} \beta^{\frac{1}{2}} = k \text{ but} \beta < 1. \text{ Therefore } k < 1 \\
&\Rightarrow (1 - k)d(Tz, z) \leq 0.
\end{aligned}$$

Hence $d(Tz, z) = 0$ i.e.$Tz = z$. Hence$z = Tz$ and $z = Jz = Tz$.

Since $T(X) \subseteq I(X)$, There is a point $v \in X$ such that$z = Tz = Iv$. Now we prove $Sv = z$
$d^2(Sv, Tz) \leq \alpha max\{d(Iv, Jz)d(Iv, Sv), d(Iv, Sv)d(Jz, Tz)\}$
$+ \beta\{d(Iv, Tz)d(Jz, Sv)\},$
$\leq \alpha max\{d(z, z)d(z, Sv), d(z, Sv)d(z, z)\} + \beta\{d(z, z)d(z, Sv)\},$
$\Rightarrow d(Sv, z) \leq 0$ and $Sv = z$.
Take $y_n = v$ then $Sy_n \to Sv = z, Iy_n \to Iv = z$. Since (S, I) is compatible of type (P) we get $\lim_{n\to\infty} d(IIy_n, SSy_n) = 0$. This implies that $Sz = z$.Hence z is a common fixed point of S, I, J and T when S is a continuous. The proof is similar that z is common fixed point of S, I, J and T, when T is continuous.

Uniqueness: Let z and w be two common fixed point of S, I, J and T i.e. $z = Sz = Iz = Tz = Jz$ and $w = Sw = Iw = Tw = Jw$. From condition (ii)

$$\begin{aligned} d^2(Sz, Tw) &\leq \alpha max\{d(Iz, Jw)d(Iz, Sz), d(Iz, Sz)d(Jw, Tw)\} \\ &\quad + \beta\{d(Iz, Tw)d(Jw, Sz)\}, \\ &= \alpha max\{d(z, w)d(z, z), d(z, z)d(w, w)\} + \beta\{d(z, w)d(w, z)\}, \\ &= \beta\{d(z, w)d(w, z)\}, \\ &\Rightarrow (1-k)d(z, w) \leq 0. \\ &\Rightarrow d(z, w) = 0. \text{ i.e. } w = z. \end{aligned}$$

Hence z is unique common fixed point.

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