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Existence and Multiplicity of Symmetric Positive Solutions for Singular Second-Order m-Point Boundary Value Problem

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Abstract

In this paper, we are concerned with the existence of symmetric positive solutions for singular second-order differential equation. Under the suitable conditions, the theorems about the existence of symmetric positive solutions are obtained by using Krasnoselskii's fixed-point theorem.

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1 Introduction

Recent years have witnessed a very rapid growth of developments of ordinary differential equations(see[1],[2],[3] and references therein). As far as we know, few results on the symmetric positive solutions are obtained(see[4]). in this paper, we study the existence of symmetric positive solution for boundary value problems as follows,

$$u''(t) + \lambda a(t) f(t, u(t)) = 0, 0 < t < 1, \tag{1}$$

$$u(t) = u(1-t), u'(0) - u'(1) = \Sigma_{i=1}^{m-2} b_i u(\xi_i), \tag{2}$$

where $\lambda > 0$ is a positive parameter, $a(t) : (0,1) \to [0,\infty)$ is continuous, symmetric, and may be singular at $t = 0$ or $t = 1$. $f : [0,1] \times [0,\infty] \to [0,\infty)$ is continuous and $f(\cdot, u)$ is symmetric on $[0,1]$ for all $u \in [0,\infty)$. $\xi_i \in (0,1)$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $b_i \in [0,\infty)$ with $0 < \sum_{i=1}^{m-2} b_i < 1$.

Here, the existence of symmetric positive solutions for the equation (1) with the boundary conditions (2) are discussed when the conditions are imposed on f and $a(t)$.

2 Preliminaries

In this section, we present some preliminary lemmas that will be used in the proof of our results.

Lemma 2.1. Let $y \in C[0,1]$ be symmetric on $[0,1]$, then it can be easily proof that the m-point BVP

$$u''(t) + y(t) = 0, 0 < t < 1, \tag{3}$$

$$u(t) = u(1-t), u'(0) - u'(1) = \sum_{i=1}^{m-2} b_i u(\xi_i). \tag{4}$$

has a unique symmetric solution $u(t) = \int_0^1 G(t,s)y(s)ds$, where $G(t,s) = G_1(t,s) + G_2(s)$, here

$$G_1(t,s) = \begin{cases} t(1-s), 0 \le t \le s \le 1, \\ s(1-t), 0 \le s \le t \le 1, \end{cases}$$

and

$$G_2(s) = (\sum_{i=1}^{m-2} b_i)^{-1} \begin{cases} 1 + \sum_{i=1}^{m-2} b_i s(\xi_i - 1), 0 \le s \le \xi_1, \\ 1 + \sum_{i=j+1}^{m-2} b_i(\xi_i - s) - \sum_{i=1}^{m-2} b_i \xi_i (1-s), \xi_j \le s \le \xi_{j+1}, \\ (j = 1, 2, \cdots, m-3), \\ 1 - \sum_{i=1}^{m-2} b_i \xi_i (1-s), \xi_{m-2} \le s \le 1. \end{cases}$$

Lemma 2.2. Let $t, s \in [0,1]$, then $LG(s,s) \le G(t,s) \le G(s,s)$, where $L = \frac{4m_{G_2}}{4m_{G_2}+1}$, and $m_{G_2} = min\{G_2(\xi_1), \cdots, G_2(\xi_{m-2})\}$.

Proof. Note that $G_2(s) > 0$ and $G_2(s)$ is continuous. It is obvious that $G_2(s) \ge m_{G_2}$ for $s \in [0,1]$. Thus

$$\begin{aligned} G(t,s) = G_1(t,s) + G_2(s) \ge G_2(s) = \frac{1}{4m_{G_2}+1} G_2(s) + \frac{4m_{G_2}}{4m_{G_2}+1} G_2(s) \\ \ge \frac{1}{4} \cdot \frac{4m_{G_2}}{4m_{G_2}+1} + \frac{4m_{G_2}}{4m_{G_2}+1} G_2(s) \ge s(1-s) \frac{4m_{G_2}}{4m_{G_2}+1} + \frac{4m_{G_2}}{4m_{G_2}+1} G_2(s) \\ \ge LG_1(s,s) + LG_2(s) = LG(s,s). \end{aligned}$$

It is easy to know that $G(s,s) \geq G(t,s)$ for $t, s \in [0,1]$. The proof of Lemma 2.2 is complete.
Lemma 2.3. (see[5])Suppose that (A_1) and (A_2) hold, then $T : P \to P$ is completely continuous.
Lemma 2.4.(see[6]) Let E be a Banach space and $P \subset E$ is a cone in E. Assume that Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$. Let $T : P\bigcap(\overline{\Omega_2}\backslash\Omega_1) \to P$ be a completely continuous operator. In addition suppose either
(1) $\| Tu \| \leq \| u \|, \forall u \in P\bigcap \partial\Omega_1$ and $\| Tu \| \geq \| u \|, \forall u \in P\bigcap \partial\Omega_2$ or
(2) $\| Tu \| \leq \| u \|, \forall u \in P\bigcap \partial\Omega_2$ and $\| Tu \| \geq \| u \|, \forall u \in P\bigcap \partial\Omega_1$
holds. Then T has a fixed point in $P\bigcap(\overline{\Omega_2}\backslash\Omega_1)$.

3 Existence of positive solutions

In this section, we study the existence of symmetric positive solutions for the equation (1) with the boundary conditions (2). First we give the following notations:

$$f_0 = lim\ inf_{t\to+0} min_{t\in[0,1]} \frac{f(t,u)}{u}, f^0 = lim\ sup_{t\to+0} max_{t\in[0,1]} \frac{f(t,u)}{u},$$
$$f_\infty = lim\ inf_{t\to+\infty} min_{t\in[0,1]} \frac{f(t,u)}{u}, f^\infty = lim\ sup_{t\to+\infty} max_{t\in[0,1]} \frac{f(t,u)}{u}.$$

$$Let\ A = \int_0^1 L^2 G(s,s)a(s)ds, \quad B = \int_0^1 G(s,s)a(s)ds \tag{5}$$

Here we assume that $\frac{1}{Af_\infty} = 0$ if $f_\infty = \infty$ and $\frac{1}{Bf^0} = \infty$ if $f^0 = 0$ and $\frac{1}{Af_0} = 0$ if $f_0 = \infty$ and $\frac{1}{Bf^\infty} = \infty$ if $f^\infty = 0$.
Theorem 3.1. Suppose that $Af_\infty > Bf^0$, (A_1) and (A_2) hold. Then for each $\lambda \in (\frac{1}{Af_\infty}, \frac{1}{Bf^0})$, the equation (1) with the boundary conditions (2) has at least one symmetric positive solution.
Proof. From the definition of f^0, there exists a number $l_1 > 0$ such that $f(s,u) \leq (f^0 + \varepsilon)u$ for $u \in (0, l_1]$, then for $u \in P$ and $\| u \| = l_1$, we have

$$\begin{aligned} Tu &= \lambda \int_0^1 G(t,s)a(s)f(s,u(s))ds \\ &\leq \lambda \int_0^1 G(s,s)a(s)f(s,u(s))ds \\ &\leq \lambda \int_0^1 G(s,s)a(s)(f^0+\varepsilon)uds \\ &\leq \lambda B(f^0+\varepsilon) \| u \| . \end{aligned}$$

We can choose $\varepsilon > 0$ small enough such that $\lambda B(f^0 + \varepsilon) \leq 1$. Then we have $\| Tu \| \leq \| u \|$, Let $\Omega_1 = \{u \in E \,|\, \| u \| < l_1\}$ then

$$\| Tu \| \leq \| u \| \quad for\ u \in P \bigcap \partial\Omega_1. \tag{6}$$

For another hand, from the definition of f_∞, there exists a number $l_2 > l_1 > 0$ such that $f(s, u) \geq (f_\infty - \varepsilon)u$ for $u \in [l_2, +\infty)$, $0 \leq s \leq 1$. Then for $u \in P$ and $\| u \| = l_2$, we have

$$\begin{aligned} Tu &= \lambda \int_0^1 G(t, s)a(s)f(s, u(s))ds \\ &\geq \lambda \int_0^1 LG(s, s)a(s)f(s, u(s))ds \\ &\geq \lambda \int_0^1 L^2G(s, s)a(s)(f_\infty - \varepsilon) \| u \| ds \\ &\geq \lambda A(f_\infty - \varepsilon) \| u \| . \end{aligned}$$

We can choose $\varepsilon > 0$ small enough such that $\lambda A(f_\infty - \varepsilon) \geq 1$. Then we have $\| Tu \| \geq \| u \|$. Let $\Omega_2 = \{u \in E \,|\, \| u \| < l_2\}$. Then $\Omega_1 \subset \overline{\Omega_2}$, and

$$\| Tu \| \geq \| u \| \quad for\ u \in P \bigcap \partial\Omega_2. \tag{7}$$

From the Lemma 2.4, the equation (6) and (7), there exists a fixed point of the operator T in $P \bigcap (\overline{\Omega_2} \backslash \Omega_1)$, this completes the proof.

4 Examples

In this section, we give five examples to illustrate our results.
Examples 4.1. Consider the four-point BVP

$$u'' + \lambda min\{t, 1-t\} \frac{11ue^{2u}}{(1 + t(1-t))e^u + e^{2u}} = 0, \; 0 < t < 1, \tag{8}$$

$$u(t) = u(1-t), u'(0) - u'(1) = \frac{1}{2}u(\frac{1}{2}) + \frac{1}{4}u(\frac{1}{4}). \tag{9}$$

Proof. Set $a(t) = min\{t, 1-t\}$, $f(t, u) = \frac{11ue^{2u}}{1+t(1-t))e^u+e^{2u}}$, by calculating, we obtain $L = \frac{9}{11}$, $A = \frac{216756}{371712}$ $B = \frac{2676}{3072}$, $f^0 = 5.5$ and $f_\infty = 11$, if $\lambda \in (\frac{33792}{216756}, \frac{6144}{29436})$, then from the Theorem 3.1, the equation (8) with the boundary conditions (9) has at least one symmetric positive solution.

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