Int. Journal of Math. Analysis, Vol. 5, 2011, no. 32, 1553 - 1567

Singularity in Optimal Control

Gerardo Sánchez Licea

Facultad de Ciencias, Universidad Nacional Autónoma de México
Departamento de Matemáticas, México DF 04510, México
gslicea@apolo.acatlan.unam.mx

Abstract

This paper concentrates on the application of the concept of a directionally convergent sequence of functions in the space L^1 which shows to be useful to solve several examples of optimal control for which the solutions have singularities. The technique used to attack these examples is strongly related to a generalization of the one given by Hestenes, in terms of a directionally convergent sequence of trajectories, originally posed for the classical isoperimetric calculus of variations problem.

Mathematics Subject Classification: 49K15

Keywords: Optimal control, singularity, strengthened Legendre-Clebsch condition, strong minima, isoperimetric problems, directionally convergent sequence of functions.

1 Introduction

In [3] the authors provide second order sufficient conditions for weak minima for optimal control problems which include equality and inequality functional constraints, in particular, that type of restraints can be modified in order to become isoperimetric restrictions. The method of sufficiency given in [3], allows to prove a weak quadratic growth of the cost functional at the reference point of consideration. The assumption that the second variation be positive definite on the cone of critical directions which is in turn equivalent to the positivity of the second variation on the set of nonzero directions together with the strengthened condition of Legendre-Clebsch is crucial. On the other hand, in the classical calculus of variations it is well-known that the nonnegativity of the second variation along an arc x_0 over the set of admissible variations becomes a second order necessary condition for optimality. The theory of Jacobi concerns with a characterization of the nonnegativity of the second variation over the set of admissible variations with the nonexistence of conjugate points on the underlying time interval under consideration. In concrete, a *smooth nonsingular* trajectory x_0 satisfies the fact that its second variation is nonnegative over the set of admissible variations if and only if x_0 satisfies the condition of Legendre and there are no conjugate points on x_0 in the underlying open interval. Moreover, the second variation is positive over the set of nonnull admissible variations if and only if x_0 satisfies the strengthened condition of Legendre and there are no conjugate points on x_0 in the half-open interval. One of the unfortunate features of Jacobi's theory as well as the the theory developed in [3] arises when the point under consideration is *singular*, that is, when it does not satisfy the *strengthened condition of Legendre*, in this case, the above theories are not applicable. Indeed, all the classical sufficiency theorems in the theory of calculus of variations assume the strengthened condition of Legendre, and therefore the extremals under consideration satisfying the classical sufficiency conditions must be nonsingular.

Because of this fact, Ewing mentions in [2], that in the theory of calculus of variations, there is a gap between the necessary and sufficient conditions for optimality. In fact, in [2], Ewing devotes an entire section to problems for which Legendre strengthened condition fails. There he shows that we can partially close this gap by an elementary device discussed in [1] of adding a *penalty term*. This procedure is illustrated by means of three examples where the trajectory being examined is singular, but one can obtain a solution directly from the properties of the particular examples. However, this technique may not hold in general. Ewing states that 'although the use of the penalty term sheds light on the theory, it provides no panacea for attacking particular examples. Indeed there are no panaceas!'

There has been a recent interest to develop a robust theory applicable to singular problems. Several attempts have been made not only for the theory of calculus of variations but also for optimal control problems. In particular, in [4], it was considered a class of optimal control problems with one fixed end-point and involving equality and inequality constraints in the control whose admissibility set is convex. A more general problem where both end-points vary and the control set is not necessarily convex is studied in [9]. The main results of [4] and [9] concentrate in deriving second order necessary conditions for optimality in terms of the variations and in terms of a generalized notion of conjugate points. It is worth to emphasize that in the above references the authors are able to obtain necessary conditions for optimality without making any assumptions of nonsingularity, that is, the extremals under consideration may not satisfy the strengthened condition of Legendre-Clebsch. In [6] the author studies a class of calculus of variations problems involving a finite set of equality isoperimetric restrictions. There the original problem is transformed into an optimal control problem by means of adding a system of differential equations. A novelty of that paper is the introduction of a new set of points, which characterizes the classical second order necessary condition in terms of the variations without imposing hypotheses of smoothness or nonsingularity. Thus the nonexistence of such extended conjugate points becomes a necessary condition for optimality even when the arc under consideration is singular. Various extensions of this theory to optimal control problems which seek to unify and improve the approaches of conjugacy obtained in [4] and [9] can be found in [7]. It is of importance to observe that since the theories developed in [4], [6], [7], and [9], provide second order necessary conditions for optimality, they are successful to obtain or to discard candidates for optimality, however, in the general case these theories do not furnish a solution of the problem under consideration.

In this paper we present the concept of a directionally convergent sequence of functions in the space L^1 and show how this concept can be successfully applied to solve some particular examples whose solutions contain singularities. In particular, we show how in several situations the fulfilment of the condition of Legendre but not its strengthened form, the positivity of the second variation over the set of nonnull admissible variations, and the nonnegativity of the Weierstrass excess function evaluated on an appropriately selected set of points, become sufficient conditions for a strict strong minimum. The above method is strongly based on some auxiliary results which are in turn a consequence of a generalization of the concept of a directionally convergent sequence of trajectories (absolutely continuous functions) which was firstly introduced in a calculus of variations context by Hestenes in [5].

The paper is organized as follows. In Section 2 we state the isoperimetric optimal control problem we shall deal with, in which the strategies belong to the space of absolutely continuous functions and the controls lie on the Banach space L^1, moreover, we also state some notation and basic definitions. In Section 3 we present the definition of a directionally convergent sequence of functions in L^1 and we enunciate three auxiliary results on which the solution of the examples given in Section 4 is strongly based. In the final section we illustrate the usefulness of these auxiliary results by means of three examples for which the solutions under consideration are singular strict strong minima of the problems in hand.

2 Statement of the problem

The fixed-endpoint optimal control problem we shall study in this paper can be stated as follows. Suppose we are given an interval $T := [0, 1]$, and functions L, L_1, and f mapping $T \times \mathbf{R} \times \mathbf{R}$ to $\mathbf{R}$.

Let $X := AC(T; \mathbf{R})$ denote the space of absolutely continuous functions mapping T to $\mathbf{R}$, let $U := L^1(T;\mathbf{R})$, set $Z := X \times U$, and denote by Z_e the set of all $(x,u) \in Z$ satisfying

a. $L(t,x(t),u(t))$ and $L_1(t,x(t),u(t))$ are integrable on T.
b. $\dot{x}(t) = f(t,x(t),u(t))$ a.e. in T.
c. $x(0) = x(1) = 0$.
d. $I_1(x,u) := \int_0^1 L_1(t,x(t),u(t))dt \le 0$.

We shall consider the problem of minimizing I over Z_e, where

$$I(x,u) := \int_0^1 L(t,x(t),u(t))dt.$$

For this problem, denoted by (P), an *admissible process* is an element of Z_e. An admissible process (x,u) is called a *strong minimum* of problem (P) if there exists $\epsilon > 0$ such that $I(x,u) \le I(y,v)$ for all $(y,v) \in Z_e$, $(y,v) \ne (x,u)$, with $\|y-x\|_\infty < \epsilon$. If the inequality can be replaced by a strict inequality then (x,u) is said to be a *proper strong minimum* of (P).

An admissible process (x,u) is called a *relaxed strong minimum* of problem (P) if there exists $\epsilon > 0$ such that $I(x,u) \le I(y,v)$ for all $(y,v) \in Z_e$, $(y,v) \ne (x,u)$, with $\|y-x\|_\infty + \|v-u\|_1 < \epsilon$. If the inequality can be replaced by a strict inequality then (x,u) is said to be a *proper relaxed strong minimum* of (P).

We shall assume throughout the paper that the functions L, L_1, and f are continuous and of class C^2 with respect to x and u on $T \times \mathbf{R} \times \mathbf{R}$.

Consider the *first variation* of I along $(x,u) \in X \times L^\infty(T;\mathbf{R})$ over $(y,v) \in Z$ given by

$$I'((x,u);(y,v)) := \int_0^1 \{L_x(t,x(t),u(t))y(t) + L_u(t,x(t),u(t))v(t)\}dt,$$

and the *second variation of I* along $(x,u) \in X \times L^\infty(T;\mathbf{R})$ over $(y,v) \in X \times L^2(T;\mathbf{R})$ given by

$$I''((x,u);(y,v)) := \int_0^1 2\Omega(t,y(t),v(t))dt$$

where, for all $(t,y,v) \in T \times \mathbf{R} \times \mathbf{R}$,

$$2\Omega(t,y,v) := L_{xx}(t,x(t),u(t))y^2 + 2L_{xu}(t,x(t),u(t))yv + L_{uu}(t,x(t),u(t))v^2.$$

Also, denote by $\mathcal{E}$ the *Weierstrass excess function* which corresponds to

$$\mathcal{E}(t,x,u,v) := L(t,x,v) - L(t,x,u) - L_u(t,x,u)(v-u)$$

for all $(t, x, u, v) \in T \times \mathbf{R} \times \mathbf{R} \times \mathbf{R}$.

- For all $(t, x, u, p) \in T \times \mathbf{R} \times \mathbf{R} \times \mathbf{R}$ let

$$H(t, x, u, p) := pf(t, x, u) - L(t, x, u) - L_1(t, x, u).$$

- Given $p \in X$, a process (x, u) satisfies the *condition of Legendre* if

$$H_{uu}(t, x(t), u(t), p(t)) \leq 0 \quad \text{for all } t \in T.$$

If the above inequality can be replaced by a strict inequality the process (x, u) satisfies the *strengthened condition of Legendre.*

- Given $p \in X$, a process (x, u) is *singular* if

$$H_{uu}(t, x(t), u(t), p(t)) = 0 \quad \text{for some } t \in T.$$

- For all $(x, u) \in X \times L^\infty(T; \mathbf{R})$, denote by $Y(x, u)$ the class of all $(y, v) \in X \times L^2(T; \mathbf{R})$ satisfying

$$\dot{y}(t) = A(t)y(t) + B(t)v(t) \text{ a.e. in } T,\ \ y(0) = y(1) = 0,$$

where, $A(t) := f_x(t, x(t), u(t))$ and $B(t) := f_u(t, x(t), u(t))$ $(t \in T)$.

- For all $u \in U$ let

$$D(u) := \int_0^1 \varphi(u(t))dt \quad \text{where} \quad \varphi(c) := (1 + c^2)^{1/2} - 1.$$

3 Definition and auxiliary results

In the following definition we shall present the concept of a directionally convergent sequence of functions in $U = L^1(T; \mathbf{R})$. This concept is a generalization of the one firstly introduced by Hestenes in [5], page 155. The former became an appropriate tool to obtain a sufficiency theorem for a strong minimum in the classical calculus of variations problem with equality isoperimetric constraints. It treats explicitly with a convergent sequence of trajectories in the space of absolutely continuous functions and it also turns out to be a generalization of that given for the finite dimensional case, see [5], page 25.

3.1 Definition: *Let $u_0 \in U$ and let $\{u_q\}$ be a sequence in U with $u_q \neq u_0$. For all $q \in \mathbf{N}$ and $t \in T$, set*

$$d_q := [2D(u_q - u_0)]^{1/2}, \quad v_q(t) := \frac{u_q(t) - u_0(t)}{d_q}.$$

The sequence $\{u_q\}$ *will be said to converge to* u_0 *in the direction* v_0 *if* $v_0 \in L^2(T;\mathbf{R})$,

$$\lim_{q\to\infty} D(u_q - u_0) = 0,$$

and $\{v_q\}$ *converges weakly to* v_0 *in* $L^1(T;\mathbf{R})$.

In the following three Lemmas, we are going to assume given $u_0 \in U$ and a sequence $\{u_q\}$ in U such that $\lim_{q\to\infty} D(u_q - u_0) = 0$ and $d_q > 0$ $(q \in \mathbf{N})$. Define

$$w_q(t) := \left[1 + \frac{1}{2}\varphi(u_q(t) - u_0(t))\right]^{1/2}.$$

The next auxiliary results play a crucial role in the solution of the singular examples which we shall present in Section 4. The formulation and proofs can be found in [8].

3.2 Lemma: *For some* v_0 *and some subsequence of* $\{u_q\}$*, again denoted by* $\{u_q\}$*,* $\{u_q\}$ *converges to* u_0 *in the direction* v_0*. Moreover,* $\{u_q\}$ *converges almost uniformly to* u_0 *on* T *and hence* $w_q(t) \to 1$ *almost uniformly on* T*.*

3.3 Lemma: *Let* $A_q, B_q \in L^\infty(T;\mathbf{R})$ *for which there exist constants* $m_0, m_1 > 0$ *such that* $\|A_q\|_\infty \le m_0$*,* $\|B_q\|_\infty \le m_1$ $(q \in \mathbf{N})$ *and denote by* y_q *the solution of the initial value problem*

$$\dot{y}(t) = A_q(t)y(t) + B_q(t)v_q(t) \text{ (a.e. in } T), \quad y(t_0) = 0.$$

Then there exist $\sigma_0 \in L^2(T;\mathbf{R})$ *and some subsequence of* $\{y_q\}$*, again denoted by* $\{y_q\}$*, such that* $\{\dot{y}_q\}$ *converges weakly in* $L^1(T;\mathbf{R})$ *to* σ_0*. Moreover, if we define*

$$y_0(t) := \int_0^t \sigma_0(s)ds \quad (t \in T),$$

then $y_q(t) \to y_0(t)$ *uniformly on* T*.*

3.4 Lemma: *Suppose* $S \subset T$ *is measurable and* $w_q(t) \to 1$ *uniformly on* S*. Let* $R_q(\cdot)$ *measurable on* S*,* $R_0(\cdot) \in L^\infty(S;\mathbf{R})$*,* $R_q(t) \to R_0(t)$ *uniformly on* S*, and* $R_0(t) \ge 0$ $(t \in S)$*. Then, for some subsequence of* $\{u_q\}$*, again denoted by* $\{u_q\}$*,*

$$\liminf_{q\to\infty} \int_S R_q(t)v_q^2(t)dt \ge \int_S R_0(t)v_0^2(t)dt.$$

4 Examples

In this section we shall solve three examples, two of them with an inequality isoperimetric constraint and for which the admissible processes under consideration are *singular*. The essential mechanism on which the solution of these examples is based arises from the proof of Theorem 14.1 of [5].

4.1 Example: *Consider the problem of minimizing*

$$I(x,u) = \int_0^1 \sqrt{t}(u^2(t) - x^2(t))dt$$

subject to

a. $\sqrt{t}(u^2(t) - x^2(t))$ *and* $u^4(t) - x^4(t)$ *are integrable on* T.
b. $\dot{x}(t) = \sqrt{t}x(t)\cos u(t) + u(t)$ *a.e. in* T.
c. $x(0) = x(1) = 0$.
d. $I_1(x,u) := \int_0^1 \{u^4(t) - x^4(t)\}dt \leq 0$.

For this case,

$$L(t,x,u) = \sqrt{t}(u^2 - x^2), \quad L_1(t,x,u) = u^4 - x^4, \quad \text{and} \quad f(t,x,u) = \sqrt{t}x\cos u + u.$$

Let $(x_0,u_0) \equiv (0,0)$. Clearly (x_0,u_0) is an admissible process, that is, $(x_0,u_0) \in Z_e$.

We have

$$H(t,x,u,p) = p\sqrt{t}x\cos u + pu - \sqrt{t}u^2 + \sqrt{t}x^2 - u^4 + x^4.$$

Thus for all $p \in X$, $H_{uu}(t,x_0(t),u_0(t),p(t)) = -2\sqrt{t}$ $(t \in T)$ so that (x_0,u_0) is singular and it satisfies the condition of Legendre but not its strengthened form. Also, note that $A(t) = \sqrt{t}$, $B(t) = 1$ $(t \in T)$ so that $(y,v) \in Y(x_0,u_0)$ implies that $\dot{y}(t) = \sqrt{t}y(t) + v(t)$ a.e. in T. It follows that

$$\begin{aligned} I''((x_0,u_0);(y,v)) &= 2\int_0^1 \sqrt{t}(v^2(t) - y^2(t))dt \\ &= 2\int_0^1 (\sqrt{t}\dot{y}^2(t) - 2ty(t)\dot{y}(t) + t\sqrt{t}y^2(t) - \sqrt{t}y^2(t))dt \\ &= 2\int_0^1 \sqrt{t}\dot{y}^2(t)dt + 2\int_0^1 \dot{y}^2(t)dt + 2\int_0^1 (t\sqrt{t} - \sqrt{t})y^2(t)dt \\ &\geq 2\int_0^1 \sqrt{t}\dot{y}^2(t)dt + 2\int_0^1 (\dot{y}^2(t) - y^2(t))dt. \end{aligned}$$

It is well-known in the theory of calculus of variations that $\int_0^1 (\dot{y}^2(t) - y^2(t))dt \geq 0$ for all $y \in X$ satisfying $y(0) = y(1) = 0$.

Therefore,

$$I''((x_0,u_0);(y,v)) > 0 \quad \text{for all } (y,v) \in Y(x_0,u_0),\ (y,v) \neq (0,0). \tag{1}$$

Let $z_0 := (x_0,u_0)$. Note that for all $z = (x,u) \in Z_e$,

$$I(z) = I(z_0) + I'(z_0; z - z_0) + K(z) + \tilde{\mathcal{E}}(z) \tag{2}$$

where

$$\tilde{\mathcal{E}}(x,u) := \int_0^1 \mathcal{E}(t,x(t),u_0(t),u(t))dt,$$

$$K(x,u) := \int_0^1 \{M(t,x(t)) + (u(t)-u_0(t))N(t,x(t))\}dt,$$

and the functions M and N are given by

$$M(t,y) := L(t,y,u_0(t)) - L(t,x_0(t),u_0(t)) - L_x(t,x_0(t),u_0(t))(y-x_0(t)),$$

$$N(t,y) := L_u(t,y,u_0(t)) - L_u(t,x_0(t),u_0(t)).$$

By Taylor's theorem we have

$$M(t,y) = \frac{1}{2}P(t,y)(y-x_0(t))^2, \quad N(t,y) = Q(t,y)(y-x_0(t)),$$

where

$$P(t,y) := 2\int_0^1 (1-\lambda)L_{xx}(t,x_0(t)+\lambda(y-x_0(t)),u_0(t))d\lambda,$$

$$Q(t,y) := \int_0^1 L_{ux}(t,x_0(t)+\lambda(y-x_0(t)),u_0(t))d\lambda.$$

To obtain a contradiction, let us assume that (x_0,u_0) is not a proper strong minimum of problem (P). Then for all $q \in \mathbf{N}$, there exists $z_q := (x_q,u_q) \in Z_e$, with $z_q \neq z_0$ and $\|x_q\|_\infty < 1/q$, such that

$$I(z_q) \leq I(z_0) = 0. \tag{3}$$

Since $\|x_q\|_\infty < 1/q$, by (3),

$$\lim_{q\to\infty} \int_0^1 \sqrt{t}u_q^2(t)dt = 0$$

and hence $\sqrt[4]{t}|u_q|$ converges weakly to zero in $L^2(T;\mathbf{R})$, that is,

$$\lim_{q\to\infty} \int_0^1 \sqrt[4]{t}|u_q(t)|g(t)dt = 0 \quad \text{for all } g \in L^2(T;\mathbf{R}).$$

By taking $g(t) := 1/\sqrt[4]{t}$ if $0 < t \leq 1$, and $g(0) := 0$, we obtain

$$\lim_{q\to\infty} \int_0^1 |u_q(t)|dt = 0$$

implying that $D(u_q) \to 0$ as $q \to \infty$.

Since $z_q \in Z_e$, the fact that $z_q \neq z_0$ implies that $u_q \neq u_0$, and then $d_q := [2D(u_q)]^{1/2} > 0$ $(q \in \mathbf{N})$. For all $q \in \mathbf{N}$ and $t \in T$, let

$$w_q(t) := \left[1 + \frac{1}{2}\varphi(u_q(t))\right]^{1/2} \quad \text{and} \quad v_q(t) := \frac{u_q(t)}{d_q^2}.$$

According to Definition 3.1, by Lemma 3.2, for some v_0 and some subsequence of $\{z_q\}$, again denoted by $\{z_q\}$, $\{u_q\}$ converges to u_0 in the direction v_0. For all $q \in \mathbf{N}$ and $t \in T$, define

$$y_q(t) := \frac{x_q(t)}{d_q}.$$

By Taylor's theorem, for all $q \in \mathbf{N}$

$$\dot{y}_q(t) = A_q(t)y_q(t) + B_q(t)v_q(t) \text{ (a.e. in } T)$$

where

$$A_q(t) = \int_0^1 f_x(t, x_0(t) + \lambda[x_q(t) - x_0(t)], u_0(t))d\lambda,$$

$$B_q(t) = \int_0^1 f_u(t, x_q(t), u_q(t) + \lambda[u_0(t) - u_q(t)])d\lambda.$$

Observe that $f_x(t, x, u) = \sqrt{t}\cos u$ and $f_u(t, x, u) = -\sqrt{t}x\sin u + 1$. Thus there exist $m_0, m_1 > 0$ such that $\|A_q\|_\infty \le m_0$, $\|B_q\|_\infty \le m_1$ ($q \in \mathbf{N}$). By Lemma 3.3, there exist $\sigma_0 \in L^2(T; \mathbf{R})$ and a subsequence of $\{z_q\}$, again denoted by $\{z_q\}$, such that, if

$$y_0(t) := \int_0^t \sigma_0(s)ds \quad (t \in T),$$

then $y_q(t) \to y_0(t)$ uniformly on T.

Since $z_q \in Z_e$, by Lemma 3.3, $y_0(0) = y_0(1) = 0$. Now, by definition of the functional K, for all $q \in \mathbf{N}$

$$\frac{K(z_q)}{d_q^2} = \int_0^1 \Big\{ \frac{M(t, x_q(t))}{d_q^2} + v_q(t)\frac{N(t, x_q(t))}{d_q} \Big\} dt.$$

In view of Lemma 3.3,

$$\lim_{q\to\infty} \frac{M(t, x_q(t))}{d_q^2} = \frac{1}{2}L_{xx}(t, x_0(t), u_0(t))y_0^2(t),$$

$$\lim_{q\to\infty} \frac{N(t, x_q(t))}{d_q} = L_{ux}(t, x_0(t), u_0(t))y_0(t)$$

both uniformly on T, and since $\{v_q\}$ converges weakly to v_0 in $L^1(T; \mathbf{R})$,

$$\frac{1}{2}I''(z_0; (y_0, v_0)) = \lim_{q\to\infty} \frac{K(z_q)}{d_q^2} + \frac{1}{2}\int_0^1 L_{uu}(t, x_0(t), u_0(t))v_0^2(t)dt. \qquad (4)$$

Let us now show that, for some subsequence of $\{z_q\}$, again denoted by $\{z_q\}$,

$$\liminf_{q\to\infty} \frac{\tilde{\mathcal{E}}(z_q)}{d_q^2} \ge \frac{1}{2}\int_0^1 L_{uu}(t, x_0(t), u_0(t))v_0^2(t)dt. \qquad (5)$$

By Lemma 3.2, we may choose $S \subset T$ measurable such that $u_q(t) \to u_0(t)$ uniformly on S. By Taylor's theorem, for all $t \in S$ and $q \in \mathbf{N}$

$$\frac{1}{d_q^2}\mathcal{E}(t, x_q(t), u_0(t), u_q(t)) = \frac{1}{2}R_q(t)v_q^2(t)$$

where

$$R_q(t) := 2\int_0^1 (1-\lambda)L_{uu}(t, x_q(t), u_0(t) + \lambda[u_q(t) - u_0(t)])d\lambda.$$

Clearly,

$$\lim_{q\to\infty} R_q(t) = R_0(t) := L_{uu}(t, x_0(t), u_0(t)) \quad \text{uniformly on } S.$$

Since $w_q(t) \to 1$ uniformly on S, $R_0(t) = 2\sqrt{t} \geq 0$ ($t \in T$) and $\mathcal{E}(t, x(t), u_0(t), u(t)) = \sqrt{t}u^2(t) \geq 0$ for all $t \in T$ and $(x, u) \in Z_e$, it follows by Lemma 3.4, that there is some subsequence of $\{z_q\}$, again denoted by $\{z_q\}$, such that

$$\liminf_{q\to\infty} \frac{\tilde{\mathcal{E}}(z_q)}{d_q^2} \geq \frac{1}{2}\int_S L_{uu}(t, x_0(t), u_0(t))v_0^2(t)dt.$$

Since S can be chosen to differ from T by a set of an arbitrary small measure, and the the function

$$t \mapsto L_{uu}(t, x_0(t), u_0(t))v_0^2(t)$$

belongs to $L^1(T; \mathbf{R})$, this inequality holds when $S = T$ and this establishes (5). Now, since $L_x(t, x_0(t), u_0(t)) = -2\sqrt{t}x_0(t) = 0$ and $L_u(t, x_0(t), u_0(t)) = 2\sqrt{t}u_0(t) = 0$ ($t \in T$), then $I'(z_0; (y, v)) = 0$ for all $(y, v) \in Z$. With this in mind, (2), (3), (4), and (5),

$$\frac{1}{2}I''(z_0; (y_0, v_0)) \leq \lim_{q\to\infty} \frac{K(z_q)}{d_q^2} + \liminf_{q\to\infty} \frac{\tilde{\mathcal{E}}(z_q)}{d_q^2} = \liminf_{q\to\infty} \frac{I(z_q) - I(z_0)}{d_q^2} \leq 0.$$

Let us show that $(y_0, v_0) \in Y(z_0)$, we know by Lemma 3.2 that there exists $S \subset T$ measurable such that

$$A_q(t) \to A_0(t) := f_x(t, x_0(t), u_0(t)), \quad B_q(t) \to B_0(t) := f_u(t, x_0(t), u_0(t))$$

both uniformly on S. Since $y_q(t) \to y_0(t)$ uniformly on S and $\{v_q\}$ converges weakly to v_0 in $L^1(S; \mathbf{R})$, it follows that $\{\dot{y}_q\}$ converges weakly in $L^1(S; \mathbf{R})$ to $A_0y_0 + B_0v_0$. By Lemma 3.3, $\{\dot{y}_q\}$ converges weakly in $L^1(S; \mathbf{R})$ to $\sigma_0 = \dot{y}_0$. Hence,

$$\dot{y}_0(t) = A_0(t)y_0(t) + B_0(t)v_0(t) \quad (t \in S).$$

Since S can be chosen to differ from T by a set of an arbitrary small measure, there cannot exist a subset of T of positive measure on which the functions y_0

and v_0 do not satisfy the differential equation $\dot{y}_0(t) = A_0(t)y_0(t) + B_0(t)v_0(t)$. Consequently,

$$\dot{y}_0(t) = A_0(t)y_0(t) + B_0(t)v_0(t) \text{ (a.e. in } T)$$

and so $(y_0, v_0) \in Y(z_0)$.

By (1), $(y_0, v_0) = (0, 0)$. Let us show that this is not the case. Since $y_q(t) \to y_0(t)$ uniformly on T, and $I_1(z_q) \leq 0$ ($q \in \mathbf{N}$),

$$\int_0^1 \frac{v_q^4(t)}{w_q^4(t)} dt \leq \int_0^1 y_q^4(t) dt$$

which tends to zero as $q \to \infty$. But this contradicts the fact that for all $q \in \mathbf{N}$

$$\int_0^1 \frac{v_q^2(t)}{w_q^2(t)} dt = 1.$$

Thus (3) cannot be true, in other words, there exists $\epsilon > 0$ such that if $z = (x, u)$, then

$$0 = I(z_0) < I(z) \text{ for all } z \in Z_e,\ z \neq z_0,\ \|x - x_0\|_\infty < \epsilon.$$

Therefore, (x_0, u_0) is a proper strong minimum of problem (P).

4.2 Example: *Consider the problem of minimizing*

$$I(x, u) = \int_0^1 \{u^4(t) + tu^2(t) - x^4(t)\} dt$$

subject to

a. $u^4(t) + tu^2(t) - x^4(t)$ *is integrable on* T.

b. $\dot{x}(t) = \sin(2u(t)) + \sin u(t)$ *a.e. in* T.

c. $x(0) = x(1) = 0$.

For this case,

$$L(t, x, u) = u^4 + tu^2 - x^4, \quad L_1(t, x, u) = 0, \quad \text{and} \quad f(t, x, u) = \sin(2u) + \sin u.$$

Let $(x_0, u_0) \equiv (0, 0)$. Clearly (x_0, u_0) is an admissible process, that is, $(x_0, u_0) \in Z_e$.

We have

$$H(t, x, u, p) = p\sin(2u) + p\sin u - u^4 - tu^2 + x^4.$$

Thus for all $p \in X$, $H_{uu}(t, x_0(t), u_0(t), p(t)) = -2t$ ($t \in T$) so that (x_0, u_0) is singular and it satisfies the condition of Legendre but not its strengthened form. Also, note that $A(t) = 0$, $B(t) = 3$ $(t \in T)$ so that $(y, v) \in Y(x_0, u_0)$ implies that $\dot{y}(t) = 3v(t)$ a.e. in T. It follows that

$$I''((x_0, u_0); (y, v)) = 2\int_0^1 tv^2(t)dt > 0 \tag{6}$$

for all $(y, v) \in Y(x_0, u_0)$, $(y, v) \neq (0, 0)$.

To obtain a contradiction, let us assume that (x_0, u_0) is not a strict strong minimum of problem (P). Let $z_0 := (x_0, u_0)$. Then for all $q \in \mathbf{N}$ there exists $z_q := (x_q, u_q) \in Z_e$, with $z_q \neq z_0$ and $\|x_q\|_\infty < 1/q$, such that

$$I(z_q) \leq I(z_0) = 0. \tag{7}$$

Since $\|x_q\|_\infty < 1/q$, by (7), $\|u_q\|_4 \to 0$ as $q \to \infty$. Consequently, $D(u_q) \to 0$ as $q \to \infty$.

Since $z_q \in Z_e$, the fact that $z_q \neq z_0$ implies that $u_q \neq u_0$, and then $d_q := [2D(u_q)]^{1/2} > 0$ $(q \in \mathbf{N})$. For all $q \in \mathbf{N}$ and $t \in T$, let

$$w_q(t) := \left[1 + \frac{1}{2}\varphi(u_q(t))\right]^{1/2} \quad \text{and} \quad v_q(t) := \frac{u_q(t)}{d_q^2}.$$

Observe that $f_x(t, x, u) = 0$ and $f_u(t, x, u) = 2\cos(2u) + \cos u$, and hence if A_q and B_q are given as in Example 4.1, there exist $m_0, m_1 > 0$ such that $\|A_q\|_\infty \leq m_0$ and $\|B_q\|_\infty \leq m_1$ $(q \in \mathbf{N})$. Moreover, since $L_{uu}(t, x_0(t), u_0(t)) = 2t \geq 0$ $(t \in T)$, $\mathcal{E}(t, x(t), u_0(t), u(t)) = u^4(t) + tu^2(t) \geq 0$ for all $t \in T$ and $(x, u) \in Z_e$, $L_x(t, x_0(t), u_0(t)) = -4x_0^3(t) = 0$ and $L_u(t, x_0(t), u_0(t)) = 4u_0^3(t) + 2tu_0(t) = 0$ $(t \in T)$, by using the technique of Example 4.1, one easily sees that there exists $(y_0, v_0) \in Y(x_0, u_0)$ such that if we define

$$y_q(t) := \frac{x_q(t)}{d_q} \quad (t \in T, q \in \mathbf{N}),$$

then $y_q(t) \to y_0(t)$ uniformly on T, $\{v_q\}$ converges weakly to v_0 in $L^1(T; \mathbf{R})$, and $I''(z_0; (y_0, v_0)) \leq 0$. By (6), $(y_0, v_0) = (0, 0)$. Let us show that this is not the case. Since $y_q(t) \to y_0(t)$ uniformly on T, by (7)

$$\int_0^1 \frac{v_q^4(t)}{w_q^4(t)}dt \leq \int_0^1 y_q^4(t)dt$$

which tends to zero as $q \to \infty$. Once again, this contradicts the fact that for all $q \in \mathbf{N}$

$$\int_0^1 \frac{v_q^2(t)}{w_q^2(t)}dt = 1.$$

Thus (7) cannot be true, in other words, there exists $\epsilon > 0$ such that if $z = (x, u)$, then

$$0 = I(z_0) < I(z) \text{ for all } z \in Z_e,\ z \neq z_0,\ \|x - x_0\|_\infty < \epsilon.$$

Therefore, (x_0, u_0) is a strict strong minimum of problem (P).

4.3 Example: *Consider the problem of minimizing*

$$I(x,u) = \int_0^1 \{tu^2(t) - tx(t)u(t) - x^3(t)\}dt$$

subject to

a. $tu^2(t) - tx(t)u(t) - x^3(t)$ *and* $u^8(t) - x^7(t)u(t)$ *are integrable on* T.

b. $\dot{x}(t) = tx(t)\cos u(t) + \sin u(t)$ *a.e. in* T.

c. $x(0) = x(1) = 0$.

d. $I_1(x,u) := \int_0^1 \{u^8(t) - x^7(t)u(t)\}dt \le 0$.

For this case,

$$L(t,x,u) = tu^2 - txu - x^3, \quad L_1(t,x,u) = u^8 - x^7 u, \quad \text{and} \quad f(t,x,u) = tx\cos u + \sin u.$$

Let $(x_0, u_0) \equiv (0,0)$. Clearly (x_0, u_0) is an admissible process, that is, $(x_0, u_0) \in Z_e$.

We have

$$H(t,x,u,p) = ptx\cos u + p\sin u - tu^2 + txu + x^3 - u^8 + x^7 u.$$

Thus for all $p \in X$, $H_{uu}(t, x_0(t), u_0(t), p(t)) = -2t$ $(t \in T)$ so that (x_0, u_0) is singular and it satisfies the condition of Legendre but not its strengthened form. Also, note that $A(t) = t$, $B(t) = 1$ $(t \in T)$ so that $(y,v) \in Y(x_0,u_0)$ implies that $\dot{y}(t) = ty(t) + v(t)$ a.e. in T. It follows that

$$\begin{aligned} I''((x_0,u_0);(y,v)) &= \int_0^1 \{2tv^2(t) - 2ty(t)v(t)\}dt \\ &= \int_0^1 \{2tv^2(t) - 2ty(t)[\dot{y}(t) - ty(t)]\}dt \\ &\ge \int_0^1 \{2tv^2(t) - 2ty(t)\dot{y}(t)\}dt \\ &= \int_0^1 \{2tv^2(t) + y^2(t)\}dt. \end{aligned}$$

Thus,

$$I''((x_0,u_0);(y,v)) > 0 \quad \text{for all } (y,v) \in Y(x_0,u_0),\ (y,v) \ne (0,0). \tag{8}$$

To obtain a contradiction, let us assume that (x_0, u_0) is not a proper relaxed strong minimum of problem (P). Let $z_0 := (x_0, u_0)$. Then for all $q \in \mathbf{N}$ there exists $z_q := (x_q, u_q) \in Z_e$, with $z_q \ne z_0$ and $\|x_q\|_\infty + \|u_q\|_1 < 1/q$, such that

$$I(z_q) \le I(z_0) = 0. \tag{9}$$

Since $\|u_q\|_1 < 1/q$, then $D(u_q) \to 0$ as $q \to \infty$.

Since $z_q \in Z_e$, the fact that $z_q \neq z_0$ implies that $u_q \neq u_0$, and hence $d_q := [2D(u_q)]^{1/2} > 0$ $(q \in \mathbf{N})$. For all $q \in \mathbf{N}$ and $t \in T$, let

$$w_q(t) := \left[1 + \frac{1}{2}\varphi(u_q(t))\right]^{1/2} \quad \text{and} \quad v_q(t) := \frac{u_q(t)}{d_q^2}.$$

Observe that $f_x(t,x,u) = t\cos u$ and $f_u(t,x,u) = -tx\sin u + \cos u$, and hence if A_q and B_q are given as in Example 4.1, there exist $m_0, m_1 > 0$ such that $\|A_q\|_\infty \leq m_0$ and $\|B_q\|_\infty \leq m_1$ $(q \in \mathbf{N})$. Moreover, since $L_{uu}(t, x_0(t), u_0(t)) = 2t \geq 0$ $(t \in T)$, $\mathcal{E}(t, x(t), u_0(t), u(t)) = tu^2(t) \geq 0$ for all $t \in T$ and $(x,u) \in Z_e$, $L_x(t, x_0(t), u_0(t)) = -3x_0^2(t) = 0$ and $L_u(t, x_0(t), u_0(t)) = 2tu_0(t) - tx_0(t) = 0$ $(t \in T)$, by using the technique of Example 4.1, one easily sees that there exists $(y_0, v_0) \in Y(x_0, u_0)$ such that if we define

$$y_q(t) := \frac{x_q(t)}{d_q} \quad (t \in T, q \in \mathbf{N}),$$

then $y_q(t) \to y_0(t)$ uniformly on T, $\{v_q\}$ converges weakly to v_0 in $L^1(T;\mathbf{R})$, and $I''(z_0; (y_0, v_0)) \leq 0$. By (8), $(y_0, v_0) = (0,0)$. Let us show that this is not the case. Since $y_q(t) \to y_0(t)$ uniformly on T and $\{v_q\}$ converges weakly to v_0 in $L^1(T;\mathbf{R})$, the fact that $I_1(z_q) \leq 0$ $(q \in \mathbf{N})$, implies that

$$\int_0^1 \frac{v_q^8(t)}{w_q^8(t)} dt \leq \int_0^1 y_q^7(t) v_q(t) dt$$

which tends to zero as $q \to \infty$, contradicting that for all $q \in \mathbf{N}$

$$\int_0^1 \frac{v_q^2(t)}{w_q^2(t)} dt = 1.$$

Thus (9) cannot be true, in other words, there exists $\epsilon > 0$ such that if $z = (x,u)$, then

$$0 = I(z_0) < I(z) \text{ for all } z \in Z_e,\ z \neq z_0,\ \|x - x_0\|_\infty + \|u - u_0\|_1 < \epsilon.$$

Therefore, (x_0, u_0) is a proper relaxed strong minimum of problem (P).

References

[1] Ewing, G. M., Sufficient conditions for a non-regular problem in the calculus of variations, *Bull. Am. Math. Soc.,* **43** (1937), 371 - 376.

[2] Ewing, G. M., *Calculus of Variations with Applications,* Dover, New York, 1985

[3] E.S. Levitin, A.A. Milyutin, and N.P. Osmolovskii, Conditions of high order for a local minimum in problems with constraints, *Uspekhi Mat. Nauk.,* **33** (1978), 85-148; English transl., Russian Math. Surveys **33**, No. 6 (1978), 97-168.

[4] Loewen, P. D., Zheng H., Generalized conjugate points for optimal control problems, *Nonlinear Anal., Theor., Methods Appl.,* **22** (1994), 771 - 791.

[5] Hestenes, M. R., *Calculus of Variations and Optimal Control Theory,* John Wiley, New York, 1966.

[6] Rosenblueth, J. F., A new notion of conjugacy for isoperimetric problems, *Applied Mathematics and Optimization.,* **50** (2004), 209-228.

[7] Rosenblueth, J. F., Conjugate journey in optimal control, *International Mathematical Forum.,* **2** (2007), 633-674.

[8] Rosenblueth, J. F., Sánchez Licea G., Sufficiency and singularity in optimal control., *IMA Journal of Mathematical Control and Information.,* (submitted)

[9] Zeidan, V., Admissible directions and generalized coupled points for optimal control problems., *Nonlinear Anal., Theor., Methods Appl.,* **26** (1996), 479-507.

Received: March, 2011