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Some Contributions to the Theory of Abstract Volterra Equations

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Abstract

In the present paper, we investigate abstract Volterra equations and abstract time-fractional equations in the setting of sequentially complete locally convex spaces. We display the relationship between $(a, k * a^{*,l})$-regularized C-resolvent families and (a, k)-regularized $((z - A)^{-l}C)$-resolvent families, continuing in such a way the researches of I. Miyadera, N. Tanaka [21] and S. W. Wang, Z. Y. Huang [25]. In the final part of the paper, we analyze the well-posedness of corresponding abstract Cauchy problems.

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1 Introduction and preliminaries

The theory of abstract Volterra equations continues to be an active field of research due to its wide applications in modeling of various problems of mathematical physics. Recently there has also been a growing interest in so-called fractional calculus, and especially, in the theory of abstract time-fractional equations. In this paper, we continue our previous researches in the above areas and the work so far carried out by many other authors.

We shall work with Hausdorff, sequentially complete locally convex vector spaces over the field of complex numbers. Henceforth E denotes such a space, SCLCS for short, and the abbreviation $\circledast$ stands for the fundamental system of seminorms which defines the topology of E. By $L(E)$ is denoted the space of all continuous linear mappings from E into E. In the sequel, A denotes a closed linear operator acting on E and $L(E) \ni C$ denotes an injective operator. We will always assume that $CA \subseteq AC$. The domain, resolvent set and range of A are denoted by $D(A)$, $\rho(A)$ and $R(A)$, respectively. Since it makes no misunderstanding, we will identify A with its graph. Recall that the *C-resolvent set of A*, $\rho_C(A)$, is defined by

$$\rho_C(A) := \{\lambda \in \mathbb{C} : \lambda - A \text{ is injective and } (\lambda - A)^{-1}C \in L(E)\}.$$

Suppose F is a linear subspace of E. Then the *part of A in F*, denoted by $A_{|F}$, is a linear operator defined by $D(A_{|F}) := \{x \in D(A) \cap F : Ax \in F\}$ and $A_{|F}x := Ax$, $x \in D(A_{|F})$. Given $\alpha > 0$ in advance, set $g_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $t > 0$, where $\Gamma(\cdot)$ denotes the Gamma function. By $BV[0,T]$ ($AC[0,T]$) we denote the space of all scalar-valued functions that are of bounded variation on $[0,T]$ (the space of all scalar-valued absolutely continuous functions on $[0,T]$), where $T > 0$. If $t \in \mathbb{R}$, then $\lfloor t \rfloor$ and $\lceil t \rceil$ denote the largest integer $\leq t$ and the smallest integer $\geq t$, respectively. The Sobolev space $W^{1,1}([0,T] : E)$ is defined by $W^{1,1}([0,T] : E) = \{f \in L^1([0,T] : E) : f(s) = f(0) + \int_0^s g(\sigma)d\sigma$ for some $g \in L^1([0,T] : E)\}$.

The convolution like mapping $*$ is given by $f * g(t) := \int_0^t f(t-s)g(s)ds$. We always use the principal branch to take the powers; put, by common consent, $0^\alpha := 0$ and $g_0(t) := \delta(t)$ (the Dirac delta distribution). Denote by $a^{*,n}(t)$ the n-th convolution power of $a(t)$ and put $a^{*,0}(t) := \delta(t)$. We refer the reader to [17, pp. 99–102] for the basic material concerning integration in sequentially complete locally convex spaces.

Let $\alpha > 0$, let $\beta > 0$ and let the *Mittag-Leffler function* $E_{\alpha,\beta}(z)$ be defined by $E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}$, $z \in \mathbb{C}$. Set $E_\alpha(z) := E_{\alpha,1}(z)$, $z \in \mathbb{C}$. Then it is well known ([2], [26]) that, for every $\alpha > 0$, there exist $b_\alpha \geq 1$ and $c_\alpha \geq 1$ such that:

$$E_{\alpha,\alpha}(t) \leq b_\alpha\big(1 + t^{(1-\alpha)/\alpha}\big)e^{t^{1/\alpha}},\ t > 0 \text{ and } E_\alpha(t) \leq c_\alpha e^{t^{1/\alpha}},\ t \geq 0. \quad (1)$$

Henceforth $\mathbf{D}_t^\alpha$ denotes the *Caputo fractional derivative of order α* ([2]).

The following definition has been recently introduced in [9].

Definition 1.1 *Let $0 < \tau \leq \infty$, $k \in C([0,\tau))$, $k \neq 0$ and let $a \in L^1_{loc}([0,\tau))$, $a \neq 0$. A strongly continuous operator family $(R(t))_{t\in[0,\tau)}$ is called a (local, if $\tau < \infty$) (a,k)-regularized C-resolvent family having A as a subgenerator iff the following holds:*

(i) $R(t)A \subseteq AR(t),\ t \in [0,\tau),\ R(0) = k(0)C$ *and* $CA \subseteq AC$,

(ii) $R(t)C = CR(t),\ t \in [0,\tau)$ *and*

(iii) $R(t)x = k(t)Cx + \int_0^t a(t-s)AR(s)x\,ds,\ t \in [0,\tau),\ x \in D(A);$

$(R(t))_{t\in[0,\tau)}$ *is said to be non-degenerate if the condition* $R(t)x = 0,\ t \in [0,\tau)$ *implies* $x = 0$, *and* $(R(t))_{t\in[0,\tau)}$ *is said to be locally equicontinuous if, for every* $t \in (0,\tau)$, *the family* $\{R(s) : s \in [0,t]\}$ *is equicontinuous. In the case* $\tau = \infty$, $(R(t))_{t\geq 0}$ *is said to be exponentially equicontinuous (equicontinuous) if there exists* $\omega \in \mathbb{R}$ *(*$\omega = 0$*) such that the family* $\{e^{-\omega t}R(t) : t \geq 0\}$ *is equicontinuous.*

If $k(t) = \frac{t^\alpha}{\Gamma(\alpha+1)}$, where $\alpha \geq 0$, then it is also said that $(R(t))_{t\in[0,\tau)}$ is an α-*times integrated* (a,C)*-resolvent family*; in such a way, we obtain the unification concept for (local) α-times integrated C-semigroups ($a(t) \equiv 1$) and cosine functions ($a(t) \equiv t$) in locally convex spaces ([27]). If $k(t) = \int_0^t K(s)ds,\ t \in [0,\tau)$, where $K \in L^1_{loc}([0,\tau))$ and $K \neq 0$, we generalize the notions of (local) K-convoluted C-semigroups and cosine functions ([7]); 0-times integrated (a,C)-resolvent family is also said to be an (a,C)*-regularized resolvent family.* Henceforth we consider only non-degenerate (a,k)-regularized C-resolvent families and assume that $k(t)$ is a scalar-valued kernel.

The following condition will be used occasionally:

(P1): $k(t)$ is Laplace transformable, i.e., it is locally integrable on $[0,\infty)$ and there exists $\beta \in \mathbb{R}$ such that $\tilde{k}(r) := \mathcal{L}(k)(r) := \lim_{b\to\infty} \int_0^b e^{-rt}k(t)dt := \int_0^\infty e^{-rt}k(t)dt$ exists for all $r \in \mathbb{C}$ with $\Re r > \beta$. Put $abs(k)$:=inf$\{\Re r : \tilde{k}(r)$ exists$\}$ and denote by $\mathcal{L}^{-1}$ the inverse Laplace transform.

If $a(t)$ is a kernel, then one can define the integral generator $\hat{A}$ of $(R(t))_{t\in[0,\tau)}$ by setting

$$\hat{A} := \Big\{(x,y) \in E \times E : R(t)x - k(t)Cx = \int_0^t a(t-s)R(s)yds,\ t \in [0,\tau)\Big\}.$$

Here we would like to mention in passing that the integral generator of a global exponentially equicontinuous (a,k)-regularized C-resolvent family $(R(t))_{t\geq 0}$ can be introduced if $a(t)$ and $k(t)$ satisfy (P1). In case that $a(t)$ is a kernel, the definition of integral generator of $(R(t))_{t\geq 0}$ coincides with the corresponding one introduced above ([10]).

The main topics of our actual work can be described as follows. In the second section of the paper, our intention is to generalize the following theorem appearing in [7] (cf. also [12], [21] and [25] for some special cases).

Theorem 1.2 *Let E be a Banach space.*

(i) Suppose A is a closed operator, $k \in \mathbb{N}_0$, $\lambda_0^2 \in \rho_C(A)$ *and* $R(C) \subseteq D\big((\lambda_0^2 - A)^{-(k+1)}\big)$. *Then A is a subgenerator of a* $(2k)$*-times (resp.* $(2k+1)$*-times) integrated C-cosine function* $(C_{2k}(t))_{t\in[0,\tau)}$ *(resp.* $(C_{2k+1}(t))_{t\in[0,\tau)}$*)*

iff A *is a subgenerator of a* $\big((\lambda_0^2-A)^{-k}C\big)$*-regularized cosine (resp.* $\big((\lambda_0^2-A)^{-(k+1)}C\big)$*-regularized cosine function)* $(C_0(t))_{t\in[0,\tau)}$*, and moreover, the following formulae hold:*

$$C_{2k}(t)x=(\lambda_0^2-A)^k\int_0^t\frac{(t-s)^{2k-1}}{(2k-1)!}C_0(s)x\,ds,\ \ t\in[0,\tau),\ x\in E,$$

$$\begin{aligned}C_0(t)x&=\Bigg\{\bigg[(-1)^k\sum_{i=1}^k\binom{k}{i}\lambda_0^{2i}(P_{i-1}h_{\lambda_0})*_0(P_{i-1}h_{-\lambda_0})\bigg]*_0C_{2k}\Bigg\}(t)x\\&+(-1)^kC_{2k}(t)x\\&+\sum_{i=1}^k(-1)^{k-i}\frac{d}{dt}\big[(P_{k-i}h_{\lambda_0})*_0(P_{k-i}h_{-\lambda_0})\big](t)(\lambda_0^2-A)^{-k}Cx,\end{aligned}$$

for any $t\in[0,\tau)$ *and* $x\in E$*, where* $P_i(t)=\frac{t^i}{i!}$*,* $t\in[0,\tau)$*,* $0\le i\le k$ *and* $h_{\pm\lambda_0}(t)=e^{\pm\lambda_0 t}$*,* $t\in[0,\tau)$*.*

(ii) Suppose $k\in\mathbb{N}$*,* $\lambda_0\in\rho_C(A)$ *and* $R(C)\subseteq D\big((\lambda_0-A)^{-k}\big)$*. Then* A *is a subgenerator of a* k*-times integrated* C*-semigroup* $(S_k(t))_{t\in[0,\tau)}$ *iff* A *is a subgenerator of a* $\big((\lambda_0-A)^{-k}C\big)$*-regularized semigroup* $(S_0(t))_{t\in[0,\tau)}$*. Furthermore, the following formulae hold:*

$$\begin{aligned}S_0(t)x&=(-1)^k\Bigg[S_k(t)x+\sum_{i=1}^k\binom{k}{i}\lambda_0^i\int_0^te^{\lambda_0(t-s)}\frac{(t-s)^{i-1}}{(i-1)!}S_k(s)x\,ds\\&+\sum_{i=0}^{k-1}\frac{e^{\lambda_0t}t^{k-1-i}}{(k-1-i)!}(A-\lambda_0)^{-(i+1)}Cx\Bigg],\ t\in[0,\tau),\ x\in E\end{aligned}$$

$$S_k(t)x=(\lambda_0-A)^k\int_0^t\frac{(t-s)^{k-1}}{(k-1)!}S_0(s)x\,ds,\ \ t\in[0,\tau),\ x\in E.$$

The third section of the paper is devoted to the study of various types of abstract Cauchy problems linked with Volterra equations and time-fractional equations with Caputo fractional derivatives. Of concern is the following abstract Volterra equation:

$$u(t)=f(t)+\int_0^ta(t-s)Au(s)\,ds,\ t\in[0,\tau),\tag{2}$$

where $f\in C([0,\tau):E)$, $0<\tau\le\infty$, A is a closed linear operator on E and $a\in L^1_{loc}([0,\tau))$, $a\neq 0$. We introduce the *solution space* $Z_a(A)$ as the set which consists of those elements $x \in E$ for which there exists a unique solution of (2) with $\tau = \infty$ and $f(t) \equiv x$. The topology on $Z_a(A)$ is introduced and the precise characterization of the space $Z_a(A)$ is given provided that, for every $\tau > 0$, there exists $n_\tau \in \mathbb{N}_0$ such that A is a subgenerator of a locally equicontinuous n_τ-times integrated (a, C)-resolvent family $(R_{n_\tau}(t))_{t\in[0,\tau)}$ satisfying the condition (13) stated below (cf. Proposition 3.1 for full details). It is well known that the operator $A_{|Z_a(A)}$ is the generator of a locally equicontinuous semigroup in $Z_a(A)$ provided that $a(t) = 1$ (cf. [4, Theorem 4.8] for the corresponding result in which, additionally, it has been assumed that E is a Fréchet space). In case $a(t) = t$, it is shown that the operator $A_{|Z_a(A)}$ is the generator of a locally equicontinuous cosine function in $Z_a(A)$. Unfortunately, it is not clear whether the operator $A_{|Z_a(A)}$ is the generator of a locally equicontinuous (g_α, I)-regularized resolvent family in $Z_a(A)$ provided $\alpha \in (0, \infty) \setminus \{1, 2\}$ and $a(t) = g_\alpha(t)$, $t > 0$. In Proposition 3.3, we establish the necessary and sufficient conditions for the generation of locally equicontinuous (a, k)-regularized C-resolvent families in terms of the existence and uniqueness of solutions to associated Volterra equations. The main result of third section is Theorem 3.6 in which we generalize the Ljubich uniqueness theorem to abstract time-fractional equations. In the remaining part of the paper, we consider the C-wellposedness of (inhomogeneous) abstract time-fractional equations with Caputo fractional derivatives (cf. (2) with $a(t) = g_\alpha(t)$, $t \in [0, \tau)$; $\alpha > 0$) and slightly improve several results from [11], [14] and [23].

2 On (a, k)-regularized C-resolvent families

We start this section by stating the following theorem.

Theorem 2.1 *Suppose $l \in \mathbb{N}$, $z \in \rho_C(A)$, E is a SCLCS, A is a subgenerator of a (local) (a, k)-regularized $((z - A)^{-l}C)$-resolvent family $(S_{a,k}(t))_{t\in[0,\tau)}$ on E, and:*

$$\big(z - A\big)^{-1}C \in L(E), \cdots, \big(z - A\big)^{-l}C \in L(E). \tag{3}$$

Set, for every $x \in E$ and $t \in [0, \tau)$,

$$\begin{aligned} &S_{a,k,l}(t)x := z^l\big(a^{*,l} * S_{a,k}(\cdot)x\big)(t) + \sum_{j=1}^{l}(-1)^j\binom{l}{j}z^{(l-j)} \\ &\times \Big[\big(a^{*,l-j} * S_{a,k}(\cdot)x\big)(t) - \big(k * a^{*,l-j}\big)(t)\big(z - A\big)^{-l}Cx \\ &- \big(k * a^{*,l-j+1}\big)(t)A\big(z - A\big)^{-l}Cx - \cdots - \big(k * a^{*,l-1}\big)(t)A^{j-1}\big(z - A\big)^{-l}Cx\Big]. \end{aligned}$$

Then the following holds:

(i) $(S_{a,k,l}(t))_{t\in[0,\tau)}$ *is an* $(a, k*a^{*,l})$*-regularized* C*-resolvent family with a subgenerator* A.

(ii) If

$$A\int_0^t a(t-s)S_{a,k}(s)x\,ds = S_{a,k}(t)x - \big(z-A\big)^{-l}Cx,\ x\in E,\ t\in[0,\tau), \tag{4}$$

then

$$S_{a,k,l}(t)x = \big(z-A\big)^{l}\int_0^t a^{*,l}(t-s)S_{a,k,l}(s)x\,ds,\ x\in E,\ t\in[0,\tau)$$

and

$$A\int_0^t a(t-s)S_{a,k,l}(s)x\,ds = S_{a,k,l}(t)x - \big(k*a^{*,l}\big)(t)Cx,\ x\in E,\ t\in[0,\tau). \tag{5}$$

(iii) If $(S_{a,k}(t))_{t\in[0,\tau)}$ *is locally equicontinuous (globally exponentially equicontinuous), then* $(S_{a,k,l}(t))_{t\in[0,\tau)}$ *is likewise locally equicontinuous (globally exponentially equicontinuous).*

Proof. Using (3) and the binomial formula, it simply follows that $A^{j-1}(z-A)^{-1}C\in L(E)$, $1\le j\le l$, which implies that $S_{a,k,l}(t)\in L(E)$, $t\in[0,\tau)$. It is checked at once that $S_{a,k,l}(0)=0$ as well as that $S_{a,k,l}(t)A\subseteq AS_{a,k,l}(t)$, $t\in[0,\tau)$ and $S_{a,k,l}(t)C = CS_{a,k,l}(t)$, $t\in[0,\tau)$. Furthermore, $(S_{a,k,l}(t))_{t\in[0,\tau)}$ is strongly continuous and it suffices to show that, for every fixed $x\in D(A)$,

$$\int_0^t a(t-s)S_{a,k,l}(s)Ax\,ds = S_{a,k,l}(t)x - \big(k*a^{*,l}\big)(t)Cx,\ t\in[0,\tau).$$

Using the standard computation involving the functional equality $\int_0^t a(t-s)S_{a,k}(s)Ax\,ds = S_{a,k}(t)x-(z-A)^{-l}Cx$, $t\in[0,\tau)$, we will only have to prove that:

$$Cx = \sum_{j=1}^{l}(-1)^j\binom{l}{j}z^{l-j}A^{j-1}\big(z-A\big)^{-l}CAx + z^l\big(z-A\big)^{-l}Cx. \tag{6}$$

By the binomial formula, the above equality holds if x is replaced by $(z-A)^{-l}Cx$. Making use of this fact as well as the equalities $A^{j-1}(z-A)^{-l}CA(z-A)^{-l}Cx = (z-A)^{-l}C(A^{j-1}(z-A)^{-l}CAx)$ and $C(z-A)^{-l}Cx = (z-A)^{-l}C^2x$, we obtain (6). The proof of (i) is thereby completed. The remainder of the proof is straightforward and therefore omitted. □

Corollary 2.2 *Suppose $\alpha > 0$, $l \in \mathbb{N}$, $z \in \rho_C(A)$, E is a SCLCS, A is a subgenerator of a (local) $(g_\alpha, (z-A)^{-l}C)$-regularized resolvent family $(S_\alpha(t))_{t\in[0,\tau)}$ on E, and (3) holds. Set, for every $x \in E$ and $t \in [0,\tau)$,*

$$S_{l,\alpha}(t)x := z^l\big(g_{l\alpha} * S_\alpha(\cdot)x\big)(t) + \sum_{j=1}^{l}(-1)^j\binom{l}{j}z^{(l-j)}$$
$$\times\Big[\big(g_{(l-j)\alpha} * S_\alpha(\cdot)x\big)(t) - g_{(l-j)\alpha+1}(t)\big(z-A\big)^{-l}Cx$$
$$- g_{(l-j+1)\alpha+1}(t)A\big(z-A\big)^{-l}Cx - \cdots - g_{(l-1)\alpha+1}(t)A^{j-1}\big(z-A\big)^{-l}Cx\Big].$$

Then the following holds:

(i) $(S_{l,\alpha}(t))_{t\in[0,\tau)}$ is a $(g_\alpha, g_{l\alpha+1})$-regularized C-resolvent family with a subgenerator A.

(ii) If

$$A\int_0^t g_\alpha(t-s)S_\alpha(s)x\,ds = S_\alpha(t)x - \big(z-A\big)^{-l}Cx,\ x \in E,\ t \in [0,\tau), \tag{7}$$

then

$$S_{l,\alpha}(t)x = \big(z-A\big)^l\int_0^t g_{l\alpha}(t-s)S_\alpha(s)x\,ds,\ x \in E,\ t \in [0,\tau)$$

and

$$A\int_0^t g_\alpha(t-s)S_{l,\alpha}(s)x\,ds = S_{l,\alpha}(t)x - g_{l\alpha+1}(t)Cx,\ x \in E,\ t \in [0,\tau). \tag{8}$$

(iii) If $(S_\alpha(t))_{t\in[0,\tau)}$ is locally equicontinuous (globally exponentially equicontinuous), then $(S_{l,\alpha}(t))_{t\in[0,\tau)}$ is likewise locally equicontinuous (globally exponentially equicontinuous).

Now we consider the situation in which $(S_{a,k,l}(t))_{t\in[0,\tau)}$ is given in advance.

Theorem 2.3 *Suppose* $l\in\mathbb{N}$, $r_0>\max(0,abs(a),abs(k))$, $z\in\rho_C(A)$, E *is a SCLCS,* A *is a subgenerator of a (local)* $(a,k*a^{*,l})$*-regularized* C*-resolvent family* $(S_{a,k,l}(t))_{t\in[0,\tau)}$ *on* E*, and (3) holds. Let* $a(t)$ *and* $k(t)$ *satisfy (P1), let* $\lim_{r\to\infty}\tilde{a}(r)=0$, $|z\tilde{a}(r)|<1$, $r>r_0$ *and let the following conditions hold:*

(a) For every $j=1,\cdots,l$*, there exists a continuous function* $t\mapsto F_j(t)$, $t\geq 0$ *satisfying* $F_j(0)=\delta_{j,l}$ *and*

$$\widetilde{F_j}(r)=\frac{\tilde{k}(r)\tilde{a}(r)^{l-j}}{\big(1-z\tilde{a}(r)\big)^{l+1-j}},\ r>r_0,\ \tilde{a}(r)\neq 0, \tag{9}$$

where $\delta_{j,l}$ *denotes the Kronecker's delta.*

(b) There exist a function $G(t)$ *satisfying (P1) and a number* $b\in\mathbb{C}$ *such that:*

$$\tilde{G}(r)=(-1)^l\big(1-z\tilde{a}(r)\big)^{-l}-b,\ r>r_0,\ \tilde{a}(r)\neq 0. \tag{10}$$

Set, for every $x\in E$ *and* $t\in[0,\tau)$,

$$S_{a,k}(t)x:=bS_{a,k,l}(t)x+\big(G*S_{a,k,l}(\cdot)x\big)(t)+\sum_{j=1}^{l}(-1)^{l-j}F_j(t)\big(z-A\big)^{-j}Cx.$$

Then the following holds:

(i) $(S_{a,k}(t))_{t\in[0,\tau)}$ *is an* (a,k)*-regularized* $((z-A)^{-l}C)$*-resolvent family with a subgenerator* A.

(ii) Assume $R(C)\subseteq D((z-A)^{-(l+1)})$ *and (5). Then (4) holds.*

(iii) If $(S_{a,k,l}(t))_{t\in[0,\tau)}$ *is locally equicontinuous, then* $(S_{a,k}(t))_{t\in[0,\tau)}$ *is likewise locally equicontinuous.*

(iv) Suppose $\tau=\infty$, $M\geq 1$, $\omega\geq 0$,

$$\int_0^t|G(s)|ds+\sum_{j=1}^{l}\big|F_j(t)\big|\leq Me^{\omega t},\ t\geq 0,$$

and $(S_{a,k,l}(t))_{t\geq 0}$ *is exponentially equicontinuous. Then* $(S_{a,k}(t))_{t\geq 0}$ *is likewise exponentially equicontinuous.*

Proof. We will only prove the assertion (i). It is clear that the prescribed assumptions imply that $(S_{a,k}(t))_{t\in[0,\tau)}$ is a strongly continuous operator family as well as that $S_{a,k}(0) = (z-A)^{-l}C$, $S_{a,k}(t)A \subseteq AS_{a,k}(t)$, $t \in [0,\tau)$ and $S_{a,k}(t)((z-A)^{-l}C) = ((z-A)^{-l}C)S_{a,k}(t)$, $t \in [0,\tau)$. It remains to be proved that, for every fixed $x \in D(A)$,

$$S_{a,k}(t)x - k(t)\big(z-A\big)^{-l}Cx = \int_0^t a(t-s)S_{a,k}(s)Axds,\ t \in [0,\tau).$$

With the help of Laplace transform, a straightforward computation involving the functional equality of $(S_{a,k,l}(t))_{t\in[0,\tau)}$, the resolvent equation and (9)-(10) indicates that the above equality holds if, for any $r > r_0$ with $\tilde{a}(r) \neq 0$:

$$\begin{aligned}\sum_{j=1}^{l}(-1)^{l-j}\frac{\tilde{k}(r)\tilde{a}(r)^{l-j}}{\big(1-z\tilde{a}(r)\big)^{l+1-j}}\big(z-A\big)^{-j}Cx - \tilde{k}(r)\big(z-A\big)^{-l}Cx \\ = (-1)^{l+1}\big(1-z\tilde{a}(r)\big)^{-l}\tilde{k}(r)\tilde{a}(r)^{l}Cx \\ + \sum_{j=1}^{l}(-1)^{l-j}\frac{\tilde{k}(r)\tilde{a}(r)^{l+1-j}}{\big(1-z\tilde{a}(r)\big)^{l+1-j}}\Big[z\big(z-A\big)^{-j}Cx - \big(z-A\big)^{-(j-1)}Cx\Big].\end{aligned}$$

It can be easily seen that the coefficients of $\big(z-A\big)^{-j}Cx$ $(0 \leq j \leq l)$ on both sides of the previous equality are equal, which completes the proof. □

Consider now the situation of Theorem 2.1 with $a(t) = g_\alpha(t)$ and $k(t) = 1$ $(\alpha > 0)$. Noticing that

$$\frac{r^{\alpha j}}{(r^\alpha - z)^l} = \sum_{n=0}^{j}\binom{j}{n}z^{j-n}\big(r^\alpha - z\big)^{n-l},\ 0 \leq j \leq l-1,\ r > |z|^{1/\alpha},$$

the well-known formula [2, (1.26)] implies:

$$\mathcal{L}^{-1}\Big(\frac{r^{\alpha j}}{(r^\alpha - z)^l}\Big)(t) = \sum_{n=0}^{j}\binom{j}{n}z^{j-n}\big(\cdot^{\alpha-1}E_{\alpha,\alpha}\big(z\cdot^{\alpha}\big)\big)^{*,l-n}(t), \tag{11}$$

provided $t \geq 0$, $0 \leq j \leq l-1$. Using the same formula again, we reveal that:

$$\mathcal{L}^{-1}\Big(\frac{r^{\alpha-1}}{(r^\alpha - z)^{l+1-j}}\Big)(t) = \Big(E_\alpha\big(\cdot^{\alpha}\big) * \big(\cdot^{\alpha-1}E_{\alpha,\alpha}\big(z\cdot^{\alpha}\big)\big)^{*,l-j}\Big)(t), \tag{12}$$

provided $t \geq 0$, $1 \leq j \leq l$. Keeping in mind (1) and (12), we obtain that the mappings $t \mapsto \mathcal{L}^{-1}(\frac{r^{\alpha-1}}{(r^\alpha-z)^{l+1-j}})(t)$, $t \geq 0$ $(1 \leq j \leq l)$ are continuous and exponentially bounded. Using (1) and (11), we get that $|t^{\alpha-1}E_{\alpha,\alpha}(zt^\alpha)| \leq b_\alpha(t^{\alpha-1}+|z|^{(1-\alpha)/\alpha})e^{|z|^{1/\alpha}t}$, $t \geq 0$ and that the mappings $t \mapsto [\mathcal{L}^{-1}(\frac{r^{\alpha j}}{(r^\alpha-z)^k})(\cdot) * S_{k,\alpha}(\cdot)x](t)$, $t \in [0,\tau)$ $(0 \leq j \leq l-1)$ are continuous for every $x \in E$. With $b := (-1)^l$, $G(t) := \sum_{j=0}^{l-1}(-1)^{j+1}\binom{l}{j}z^{l-j}\mathcal{L}^{-1}(\frac{r^{\alpha j}}{(r^\alpha-z)^l})(t)$, $t > 0$ and $F_j(t) := \mathcal{L}^{-1}(\frac{r^{\alpha-1}}{(r^\alpha-z)^{l+1-j}})(t)$, $t \geq 0$ $(1 \leq j \leq l)$, we obtain the following important corollary.

Corollary 2.4 *Suppose $\alpha > 0$, $l \in \mathbb{N}$, $z \in \rho_C(A)$, E is a SCLCS, A is a subgenerator of a (local) $(g_\alpha, g_{l\alpha+1})$-regularized C-resolvent family $(S_{l,\alpha}(t))_{t\in[0,\tau)}$ on E, and (3) holds. Set, for every $x \in E$ and $t \in [0,\tau)$,*

$$\begin{aligned} S_\alpha(t)x :=&(-1)^l S_{l,\alpha}(t)x \\ &+ \sum_{j=0}^{l-1}(-1)^{j+1}\binom{l}{j}z^{l-j}\Big[\mathcal{L}^{-1}\Big(\frac{r^{\alpha j}}{(r^\alpha-z)^l}\Big) * S_{l,\alpha}(\cdot)x\Big](t) \\ &+ \sum_{j=1}^{l}(-1)^{l-j}\mathcal{L}^{-1}\Big(\frac{r^{\alpha-1}}{(r^\alpha-z)^{l+1-j}}\Big)(t)\big(z-A\big)^{-j}Cx. \end{aligned}$$

Then the following holds:

(i) $(S_\alpha(t))_{t\in[0,\tau)}$ is a $(g_\alpha, (z-A)^{-l}C)$-regularized resolvent family with a subgenerator A.

(ii) Assume $R(C) \subseteq D((z-A)^{-(l+1)})$ and (8). Then (7) holds.

(iii) If $(S_{l,\alpha}(t))_{t\in[0,\tau)}$ is locally equicontinuous (globally exponentially equicontinuous), then $(S_\alpha(t))_{t\in[0,\tau)}$ is likewise locally equicontinuous (globally exponentially equicontinuous).

Remark 2.5 *(i) Taken together, Corollary 2.2 and Corollary 2.4 provide a generalization of Theorem 1.2. It is worth noting that Corollary 2.2 is a proper extension of the necessity in Theorem 1.2, provided that (7) holds. If the condition (H5) introduced in [9] holds for $(S_\alpha(t))_{t\in[0,\tau)}$, then Corollary 2.4(i) produces a proper extension of the sufficiency in Theorem 1.2.*

(ii) Assuming that E is a webbed, bornological space (this, in particular, holds if E is a Fréchet space), an induction argument combined with the closed graph theorem indicates that (3) holds if $R(C) \subseteq D((z-A)^{-l})$. It is also clear that (3) holds provided that $C = I$ and that E is a general SCLCS.

(iii) If $z = 0$ and $k(0) = 1$, then the conditions (a) and (b) of Theorem 2.3 hold with $F_j(t) = (k * a^{*,l-j})(t)$, $t \geq 0$ $(1 \leq j \leq l)$, $b = (-1)^l$ and $G(t) \equiv 0$. This enables one to simply formulate an analogue of Theorem 2.3 provided that $z = 0$ and $k(0) \neq 0$.

We leave to the interested reader details concerning inheritance of differential and analytical properties from $(S_{a,k,l}(t))_{t\geq 0}$ to $(S_{a,k}(t))_{t\geq 0}$ (and vice versa).

3 On the well-posedness of related abstract Cauchy problems

The main purpose of this section is to present a brief survey of results about abstract fractional Cauchy problems and abstract Cauchy problems connected with Volterra integral equations. Recall [9], a function $u \in C([0,\tau) : E)$ is called a *(mild) solution of* (2) iff $(a * u)(t) \in D(A)$, $t \in [0,\tau)$ and $A(a*u)(t) = u(t) - f(t)$, $t \in [0,\tau)$. The concepts of strong and weak solutions of (2) can be also introduced ([24]). We will always assume that there exists a unique solution of (2) with $\tau = \infty$ and $f(t) \equiv 0$, so that 0 belongs to the solution space $Z_a(A)$; observe that this condition automatically holds provided that the operator A satisfies the assumptions stated in the formulation of Proposition 3.1 below. Therefore, $Z_a(A)$ is a linear subspace of E.

The spaces $Z_a(A)$ with $a(t) = 1$ or $a(t) = t$ have been analyzed in [4] and [7] (cf. [4, Section IV] and [7, Proposition 3.1.28(ii) and p. 259]). In the following proposition, we consider the general case.

Proposition 3.1 *Suppose that E is a SCLCS and that, for every $\tau > 0$, there exists $n_\tau \in \mathbb{N}_0$ such that A is a subgenerator of a locally equicontinuous n_τ-times integrated (a,C)-resolvent family $(R_{n_\tau}(t))_{t\in[0,\tau)}$ satisfying*

$$A\int_0^t a(t-s)R_{n_\tau}(s)x\,ds = R_{n_\tau}(t)x - \frac{t^{n_\tau}}{n_\tau!}Cx,\ t \in [0,\tau),\ x \in E. \tag{13}$$

Then $x \in Z_a(A)$ iff, for every $\tau > 0$, $R_{n_\tau}(t)x \in R(C)$, $t \in [0,\tau)$ and the mapping $t \mapsto C^{-1}R_{n_\tau}(t)x$, $t \in [0,\tau)$ is n_τ-times continuously differentiable. If this is the case, the unique solution $u(\cdot,x)$ of (2) on $[0,\tau)$ is given by $u(t,x) = \frac{d^{n_\tau}}{dt^{n_\tau}}C^{-1}R_{n_\tau}(t)x,\ t \in [0,\tau)$.

Proof. Assume first $x \in Z_a(A)$. By making use of Theorem 3.2(i) given below, we obtain that, for every $\tau > 0$, $\int_0^t R_{n_\tau}(s)x\,ds = (g_{n_\tau+1}(\cdot)C * u)(t)$, $t \in [0,\tau)$. This implies that, for every $\tau > 0$, $R_{n_\tau}(t)x \in R(C)$, $t \in [0,\tau)$ and that the mapping $t \mapsto C^{-1}R_{n_\tau}(t)x$, $t \in [0,\tau)$ is n_τ-times continuously differentiable, proving the necessity. In order to prove the sufficiency, notice that, for every $\tau > 0$, the mappings $t \mapsto (a * \frac{d^j}{d\cdot^j} R_{n_\tau}(\cdot)x)(t)$, $t \in [0, \tau)$ are continuous $(0 \le j \le n_\tau)$ and that $(a*\frac{d^j}{d\cdot^j} R_{n_\tau}(\cdot)x)(t) = (a*1*\frac{d^{j-1}}{d\cdot^{j-1}} R_{n_\tau}(\cdot)x)(t)$, $t \in [0, \tau)$ $(1 \le j \le n_\tau)$. Using induction and the closedness of A, we infer that $A(a * \frac{d^j}{d\cdot^j} R_{n_\tau}(\cdot)x)(t) = \frac{d^j}{dt^j} R_{n_\tau}(t)x - \frac{t^{n_\tau - j}}{(n_\tau - j)!} Cx$, provided $t \in [0, \tau)$ and $0 \le j \le n_\tau$. The remnant of the proof is a simple consequence of the above equality. □

Put $p_C(x) := p(C^{-1}x)$, $p \in \circledast$, $x \in R(C)$. Then $p_C(\cdot)$ is a seminorm on $R(C)$ and the calibration $(p_C)_{p\in\circledast}$ induces a locally convex topology on $R(C)$; we denote the above space by $[R(C)]_\circledast$. Notice that $[R(C)]_\circledast$ is a SCLCS, and that $[R(C)]_\circledast$ is a (Fréchet, Banach space) provided that E is. Assume now that A is a subgenerator of a locally equicontinuous (a, C)-regularized resolvent family $(R_0(t))_{t\ge0}$ satisfying (13). Then $[R(C)]_\circledast$ is continuously embedded into $Z_a(A)$. Let $K_C : C([0, \tau) : E) \to C([0, \tau) : E)$ be defined by $K_C u := k*Cu$, $u \in C([0, \tau) : E)$ and let (τ_n) be a strictly increasing sequence in $[0, \tau)$ with $\lim_{n\to\infty} \tau_n = \tau$. The totality of seminorms $(p_n(f) := \sup_{t\in[0,\tau_n]} p(f(t)))_{p\in\circledast,n\in\mathbb{N}}$ induces a Hausdorff locally convex topology on $C([0, \tau) : E)$. Certainly, $C([0, \tau) : E)$ is sequentially complete, $L(C([0, \tau) : E)) \ni K_C$ is injective, and $C([0, \infty) : E)$ is a Fréchet space provided that E is. The space $Z_a(A)$ is topologized as follows. If $p \in \circledast$ and $n \in \mathbb{N}$, then $p_n(\cdot) := \sup_{t\in[0,n]} p(u(t, \cdot))$ is a seminorm on $Z_a(A)$. The totality of these seminorms induces a Hausdorff locally convex topology on $Z_a(A)$. It is checked at once that $Z_a(A)$ is a SCLCS, and that $Z_a(A)$ a Fréchet space provided that E is. Notice that $Z_a(A)$ is topologically equivalent to a subspace of $C([0, \infty) : E)$, via the embedding $(\Lambda x)(t) := u(t, x)$, $t \ge 0$, $x \in Z_a(A)$, and that the inclusion mapping from $Z_a(A)$ into E is continuous. Define now, for every $t \ge 0$ and $x \in Z_a(A)$, $R(t)x := u(t, x)$. Then $R(0) = I_{|Z_a(A)}$ and $A \int_0^t a(t - s)R(s)xds = R(t)x - x$, $x \in Z_a(A)$, $t \ge 0$, where the last integral is taken with respect to the initial topology of E. Assuming $x \in D(A_{|Z_a(A)})$, it simply follows from the uniqueness of solution that $u(t, x) = \int_0^t a(t - s)u(s, Ax)ds + x$, $t \ge 0$ and $R(t)A_{|Z_a(A)} \subseteq A_{|Z_a(A)}R(t)$, $t \ge 0$. If $a(t) = 1$ $(a(t) = t)$, then $R(t)(Z_a(A)) \subseteq Z_a(A)$, $t \ge 0$, so that $(R(t))_{t\ge0}$ is a locally equicontinuous semigroup (cosine function) in $Z_a(A)$ with the generator $A_{|Z_a(A)}$. The above assertion is clear provided $a(t) = 1$ while, in the case $a(t) = t$, it follows from the obvious well-known equality $2R(t)R(s)x = R(t + s)x + R(|t - s|)x$, $t,\ s \ge 0$, $x \in Z_a(A)$. It is not clear whether, in general, $R(t)(Z_a(A)) \subseteq Z_a(A)$, $t \ge 0$ (which would immediately imply the continuity of mapping $t \mapsto R(t)x \in Z_a(A)$, $t \ge 0$ for every fixed $x \in Z_a(A)$; furthermore, $(R(t))_{t\ge0}$ would be an $(a, 1)$-regularized resolvent family in $Z_a(A)$ with a subgenerator $A_{|Z_a(A)}$ if $R(t) \in L(Z_a(A))$, $t \ge 0$). In a similar manner, one can consider exponentially equicontinuous solution spaces (cf. [4, Section V] for more details).

The analysis of interpolation and extrapolation spaces for (a, k)-regularized C-resolvent families is a non-trivial problem which will not be further discussed in the context of this paper (cf. [1], [5]-[6], [16] and [22] for the corresponding results in the case of semigroups).

In the following theorem which has been recently proved in [9], the spaces $C^{n,k}([0, \tau) : E)$ ($n \in \mathbb{N}$, $k \in \mathbb{N}_0$) and $C_0^n([0, \tau) : E)$ ($n \in \mathbb{N}$) have the same meaning as in [15] and [8].

Theorem 3.2 *(i) Assume $f \in C([0, \tau) : E)$, A is a subgenerator of a locally equicontinuous (a, k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$ and the following condition holds:*

$$A \int_0^t a(t-s)R(s)x\,ds = R(t)x - k(t)Cx,\ t \in [0, \tau),\ x \in E. \tag{14}$$

Then the solutions of (2) are unique, and the following holds:

(i.1) Let $u(t)$ be a solution of (2). Then

$$(R * f)(t) = (kC * u)(t),\ t \in [0, \tau) \text{ and } R * f \in R(K_C). \tag{15}$$

(i.2) Let (15) hold for some $u \in C([0, \tau) : E)$. Then $u(t)$ is a unique solution of (2).

(ii) Assume $n \in \mathbb{N}$, $f \in C([0, \tau) : E)$, A is a subgenerator of a locally equicontinuous n-times integrated (a, C)-resolvent family $(R(t))_{t\in[0,\tau)}$, and (14) holds. Then (2) has a unique solution iff $C^{-1}(R * f) \in C_0^{n+1}([0, \tau) : E)$.

(iii) Assume (14) holds, $n \in \mathbb{N}$, A is a subgenerator of a locally equicontinuous n-times integrated (a, C)-regularized resolvent and $a \in BV_{loc}([0, \tau))$, resp. A is a subgenerator of a locally equicontinuous (a, C)-regularized resolvent family. Assume, further, that $C^{-1}f \in C^{(n+1)}([0, \tau) : E)$, $f^{(k-1)}(0) \in D(A^{n+1-k})$ and $A^{n+1-k}f^{(k-1)}(0) \in R(C)$, $1 \le k \le n + 1$, resp. $C^{-1}f \in W_{loc}^{1,1}([0, \tau) : E)$. Then (2) has a unique solution.

The following proposition is a generalization of [23, Theorem 2.5] and some results given in [7, Subsection 2.3]. The proof is simple and therefore omitted.

Proposition 3.3 *Consider the following assertions.*

(i) A is a subgenerator of a locally equicontinuous (a, k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$ satisfying (14).

(ii) For every $x \in E$, there exists a unique solution of (2) with $f(t) = k(t)Cx$, $t \in [0, \tau)$.

Then (i) ⇒ (ii). If, in addition, E is a Fréchet space, then the above are equivalent.

Consider now the following abstract fractional Cauchy problem:

$$\mathbf{D}_t^{\alpha} u(t) = Au(t) + f(t),\ t \in (0,\tau);\ u^{(k)}(0) = x_k,\ k = 0, 1, \cdot \cdot \cdot, \lceil \alpha \rceil - 1, \quad (16)$$

where $x_k \in C(D(A))$, $k = 0, 1, \cdot \cdot \cdot, \lceil \alpha \rceil - 1$ and $f \in C([0,\tau) : E)$. Let $\alpha \in (0,\infty) \setminus \mathbb{N}$. Following M. Li, C. Chen and F.-B. Li [14], a function $u \in C^{\lceil \alpha \rceil - 1}([0,\infty) : E)$ is said to be:

(i) a (*strong*) *solution* of (16) if $u \in C^{\lceil \alpha \rceil - 1}([0,\tau) : E)$, $Au \in C([0,\tau) : E)$, $\int_0^{\cdot} g_{\lceil \alpha \rceil - \alpha}(\cdot - s)\big[u(s) - \sum_{k=0}^{\lceil \alpha \rceil - 1} \frac{s^k}{k!} x_k\big] ds \in C^{\lceil \alpha \rceil}([0,\tau) : E)$ and (16) holds.

(ii) a *mild solution* of (16) if $u \in C([0,\tau) : E)$, $(g_\alpha * f)(t) \in D(A)$, $t \in [0,\tau)$ and

$$A\big(g_\alpha * u\big)(t) = u(t) - \big(g_\alpha * f\big)(t) - \sum_{k=0}^{\lceil \alpha \rceil - 1} \frac{t^k}{k!} x_k,\ t \in [0,\tau).$$

Let A be a subgenerator of a locally equicontinuous (g_α, k)-regularized C-resolvent family $(R(t))_{t \in [0,\tau)}$ which satisfies (14) with $a(t) = g_\alpha(t)$. Setting $G(t) := \big(g_\alpha * f\big)(t) + \sum_{k=0}^{\lceil \alpha \rceil - 1} \frac{t^k}{k!} x_k$, $t \in [0,\tau)$, it readily follows from Theorem 3.2(i) that every mild solution of (16) satisfies $(R*G)(t) = (kC*u)(t)$, $t \in [0,\tau)$. Furthermore, if the above equality holds for some $u \in C([0,\tau) : E)$, then $u(t)$ is a mild solution of (16); in such a way, we have proved an extension of [14, Proposition 4.2].

It is predictable that every strong solution of (16) is also a mild solution of (16). We will prove this fact. Notice first that the assumption $F \in C^1([0,\tau) : E)$ implies $(g_\alpha * F)'(t) = (g_\alpha * F')(t) + F(0) g_\alpha(t)$, $t \in (0,\tau)$, and $(g_\alpha * F)'(t) = (g_\alpha * F')(t) + F(0) g_\alpha(t)$, $t \in [0,\tau)$, under the additional condition $F(0) = 0$. Assume that $u(t)$ is a strong solution of (16). Then $u \in C^{\lceil \alpha \rceil - 1}([0,\tau) : E)$, $(g_\alpha * u)(t) \in D(A)$ and $A(g_\alpha * u)(t) = (g_\alpha * Au)(t)$, $t \in [0,\tau)$. Set $F(t) := \int_0^t g_{\lceil \alpha \rceil - \alpha}(t-s)\big[u(s) - \sum_{k=0}^{\lceil \alpha \rceil - 1} \frac{s^k}{k!} x_k\big] ds$, $t \in [0,\tau)$. Then $F^{(j)}(t) = \int_0^t g_{\lceil \alpha \rceil - \alpha}(t - s)\big[u^{(j)}(s) - \sum_{k=0}^{\lceil \alpha \rceil - 1} \frac{s^{k-j}}{(k-j)!} x_k\big] ds$, provided $t \in [0,\tau)$ and $0 \leq j \leq \lceil \alpha \rceil - 1$. This implies $(g_\alpha * F)^{(m)}(t) = (g_\alpha * F^{(m)})(t)$, $t \in [0,\tau)$ and

$$\begin{aligned}
&A\big(g_\alpha * u\big)(t) = \big(g_\alpha * Au\big)(t) \\
&= -\big(g_\alpha * f\big)(t) + \Big(g_\alpha * \tfrac{d^{\lceil \alpha \rceil}}{d \cdot^{\lceil \alpha \rceil}} \int_0^{\cdot} g_{\lceil \alpha \rceil - \alpha}(\cdot - s)\big[u(s) - \sum_{k=0}^{\lceil \alpha \rceil - 1} \tfrac{s^k}{k!} x_k\big] ds\Big)(t) \\
&= -\big(g_\alpha * f\big)(t) + \tfrac{d^{\lceil \alpha \rceil}}{dt^{\lceil \alpha \rceil}} \Big(g_{\lceil \alpha \rceil} * \big[u(\cdot) - \sum_{k=0}^{\lceil \alpha \rceil - 1} \tfrac{\cdot^k}{k!} x_k\big]\Big)(t)
\end{aligned}$$

$= -\big(g_\alpha * f\big)(t) + u(t) - \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{t^k}{k!}x_k,$

for any $t \in [0,\tau)$. Therefore, $u(t)$ is a mild solution of (16). The converse statement is not true, in general.

Assume now that $u(t)$ is a mild solution of (16) and that, additionally, $Au \in C([0,\tau) : E)$. Then $u(t) = (g_\alpha * Au)(t) + (g_\alpha * f)(t) + \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{t^k}{k!}x_k,\ t \in [0,\tau)$, and consequently, $u \in C^{\lceil\alpha\rceil-1}([0,\tau) : E)$. Furthermore,

$$\big(g_{\lceil\alpha\rceil} * Au\big)(t) = \Big(g_{\lceil\alpha\rceil-\alpha} * \Big[u(\cdot) - \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{\cdot^k}{k!}x_k\Big]\Big)(t) - \big(g_{\lceil\alpha\rceil} * f\big)(t),\ t \in [0,\tau),$$

which implies that $(g_{\lceil\alpha\rceil-\alpha} * [u(\cdot) - \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{\cdot^k}{k!}x_k]) \in C^{\lceil\alpha\rceil}([0,\tau) : E)$ and that (16) holds. Therefore, $u(t)$ is a strong solution of (16).

Suppose A is a subgenerator of an (a,k)-regularized C-resolvent family and $n \in \mathbb{N}$. Then one obtains inductively that, for every $t \in [0,\tau)$ and $x \in D(A^n)$:

$$R(t)x = k(t)Cx + \sum_{j=1}^{n-1}\big(a^{*,j} * k\big)(t)CA^jx + \big(a^{*,n} * R(\cdot)A^nx\big)(t). \tag{17}$$

The following is a strengthening of [14, Proposition 4.7, Proposition 4.8].

Proposition 3.4 *(i) Suppose* $\alpha \in (0,\infty)\setminus\mathbb{N}$, $x \in D(A)$, $C^{-1}f,\ AC^{-1}f \in C([0,\tau) : E)$ *and* A *is a subgenerator of a* (g_α, C)*-regularized resolvent family* $(R(t))_{t\in[0,\tau)}$. *Set* $v(t) := (g_{\lceil\alpha\rceil-\alpha} * f)(t),\ t \in [0,\tau)$. *If* $v \in C^{\lceil\alpha\rceil-1}([0,\tau) : E)$ *and* $v^{(k)}(0) = 0$ *for* $1 \le k \le \lceil\alpha\rceil - 2$, *then the function* $u(t) := R(t)x + (R * C^{-1}f)(t)x,\ t \in [0,\tau)$ *is a unique solution of the initial value problem:*

$$\begin{cases} u \in C^{\lceil\alpha\rceil}((0,\tau) : E) \cap C^{\lceil\alpha\rceil-1}([0,\tau) : E), \\ \mathbf{D}_t^\alpha u(t) = Au(t) + \frac{d^{\lceil\alpha\rceil-1}}{dt^{\lceil\alpha\rceil-1}}\big(g_{\lceil\alpha\rceil-\alpha} * f\big)(t),\ t \in (0,\tau), \\ u(0) = Cx,\ u^{(k)}(0) = 0,\ 1 \le k \le \lceil\alpha\rceil - 1. \end{cases}$$

(ii) Suppose $\alpha \in (0,1)$, $x \in D(A)$, $C^{-1}f,\ AC^{-1}f \in C([0,\tau) : E)$ *and* A *is a subgenerator of a* (g_α, C)*-regularized resolvent family* $(R(t))_{t\in[0,\tau)}$. *Then the function* $u(t) := R(t)x + (R*C^{-1}f)(t)x$, $t \in [0,\tau)$ *is a unique solution of the initial value problem:*

$$\begin{cases} u \in C^1((0,\tau) : E) \cap C([0,\tau) : E), \\ \mathbf{D}_t^\alpha u(t) = Au(t) + \big(g_{1-\alpha} * f\big)(t),\ t \in (0,\tau), \\ u(0) = Cx. \end{cases}$$

(iii) *Suppose* $r \geq 0$, $n \in \mathbb{N} \setminus \{1\}$, $x \in D(A^n)$, $A^j C^{-1} f \in C([0,\tau) : E)$ *for* $0 \leq j \leq n$, *and* A *is a subgenerator of a* $(g_{1/n}, g_{r+1})$*-regularized* C*-resolvent family* $(R(t))_{t\in[0,\tau)}$. *Then the function* $v(t) := R(t)x + (R * C^{-1}f)(t)x$, $t \in [0,\tau)$ *is a unique solution of the initial value problem:*

$$\begin{cases} v \in C^1((0,\tau) : E) \cap C([0,\tau) : E), \\ v'(t) = Av(t) + \sum_{j=1}^{n-1} g_{(j/n)+r}(t) CA^j x \\ \qquad + \sum_{j=0}^{n-1} \big(g_{(j/n)+r} * A^j f\big)(t) + \frac{d}{dt} g_{r+1}(t) Cx, \ t \in (0,\tau), \\ v(0) = g_{r+1}(0)Cx. \end{cases}$$

Furthermore, $v \in C^1([0,\tau) : E)$ *provided that* $r \geq 1$ *or that* $x = 0$ *and* $r \geq 0$.

Proof. The proof of the assertion (iii) follows from the equality (17) and a straightforward computation. In order to prove (i), set $u_1(t) := (R * C^{-1}f)(t)$, $t \in [0,\tau)$ and $u_2(t) := \big(g_{\lceil\alpha\rceil-\alpha} * u_1\big)(t)$, $t \in [0,\tau)$. The proof follows very easily once we show that the function $u_1(t)$ satisfies $u_1 \in C^{\lceil\alpha\rceil-1}([0,\tau) : E)$, $u_1^{(k)}(0) = 0, \ 1 \leq k \leq \lceil\alpha\rceil - 1$ and the following equation:

$$\mathbf{D}_t^\alpha u_1(t) = Au_1(t) + \frac{d^{\lceil\alpha\rceil-1}}{dt^{\lceil\alpha\rceil-1}} \big(g_{\lceil\alpha\rceil-\alpha} * f\big)(t), \ t \in (0,\tau). \tag{18}$$

Clearly,

$$\begin{aligned} u_2(t) &= \big(g_{\lceil\alpha\rceil-\alpha} * R * C^{-1} f\big)(t) \\ &= \big(g_{\lceil\alpha\rceil-\alpha+1} * f\big)(t) + \big(g_{\lceil\alpha\rceil} * Au_1\big)(t), \ t \in [0,\tau) \end{aligned}$$

and

$$u_1(t) = \big(g_{1-(\lceil\alpha\rceil-\alpha)} * \big(g_{\lceil\alpha\rceil-\alpha} * f\big)\big)(t) + \big(g_{1-(\lceil\alpha\rceil-\alpha)} * g_{\lceil\alpha\rceil-1} * Au_1\big)(t),$$

for any $t \in [0,\tau)$. Since $v \in C^{\lceil\alpha\rceil-1}([0,\tau) : E)$ and $v^{(k)}(0) = 0$ for $1 \leq k \leq \lceil\alpha\rceil - 2$, it follows that $u_1 \in C^{\lceil\alpha\rceil-1}([0,\tau) : E)$, $u_1^{(k)}(0) = 0$ for $0 \leq k \leq \lceil\alpha\rceil - 1$, as claimed, and $u_2 \in C^{\lceil\alpha\rceil}([0,\tau) : E)$. Furthermore,

$$\mathbf{D}_t^\alpha u_1(t) = \frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}} u_2(t) = Au_1(t) + \frac{d^{\lceil\alpha\rceil-1}}{dt^{\lceil\alpha\rceil-1}} \big(g_{\lceil\alpha\rceil-\alpha} * f\big)(t), \ t \in (0,\tau).$$

This yields (18) and completes the proof of (i). It is clear that the assertion (ii) is a simple consequence of (i). □

The abstract fractional Cauchy problem (16) is said to be C*-wellposed* (cf. [2] and [11] for some special cases) if:

(i) For every $x_0, \cdot\cdot\cdot, x_{\lceil\alpha\rceil-1} \in C(D(A))$, there exists a unique solution $u_f(t; x_0, \cdot\cdot\cdot, x_{\lceil\alpha\rceil-1})$ of (16).

(ii) For every $T \in (0,\tau)$ and $q \in ⊛$, there exist $c > 0$ and $r \in ⊛$ such that, for every $x_0, \cdot\cdot\cdot, x_{\lceil\alpha\rceil-1} \in C(D(A))$, the following holds:

$$q\big(u_f(t; x_0, \cdot\cdot\cdot, x_{\lceil\alpha\rceil-1})\big) \leq c \sum_{k=0}^{\lceil\alpha\rceil-1} r\big(C^{-1}x_k\big),\ t \in [0,T]. \tag{19}$$

Fix, for the time being, $f \in C([0,\tau) : E)$. Assume that there exists a unique solution of (16) in case $x_0 \in C(D(A))$ and $x_j = 0$, $1 \leq j \leq \lceil\alpha\rceil - 1$. Then $u_f(t; x_0) \equiv u_f(t; x_0, 0, \cdots, 0)$, $t \in [0,\tau)$ is a mild solution of (16) and $Au_f(\cdot; x_0) \in C([0,\tau) : E)$. By the foregoing, we get that $u_f(\cdot; x_0)$ is a unique function satisfying $u_f(\cdot; x_0),\ Au_f(\cdot; x_0) \in C([0,\tau) : E)$ and

$$u_f(t; x_0) = x_0 + \big(g_\alpha * f\big)(t) + \int_0^t g_\alpha(t-s)Au_f(s; x_0)ds,\ t \in [0,\tau). \tag{20}$$

If, additionally, A is densely defined, E is complete and (19) holds provided $x_0 \in C(D(A))$, $x_j = 0$, $1 \leq j \leq \lceil\alpha\rceil - 1$ and $f \equiv 0$, then A is a subgenerator of a locally equicontinuous (g_α, C)-regularized resolvent family on $[0,\tau)$ ([11]). Assume now A is densely defined, E is complete, $g \in C([0,\tau))$ and, for every $x_0 \in C(D(A))$, there exists a unique function $u(\cdot; x_0) \in C([0,\tau) : E)$ satisfying $Au(\cdot; x_0) \in C([0,\tau) : E)$,

$$u(t; x_0) = x_0 + \big(g_\alpha * g\big)(t)x_0 + \int_0^t g_\alpha(t-s)Au(s; x_0)ds,\ t \in [0,\tau), \tag{21}$$

as well as (19) with $x_j = 0$, $1 \leq j \leq \lceil\alpha\rceil - 1$, and $u_f(\cdot; x_0, \cdot\cdot\cdot, x_{\lceil\alpha\rceil-1})$ replaced by $u(\cdot; x_0)$ therein (cf. (20) with $f(t) = g(t)x_0$, $t \in [0,\tau)$). Put $k(t) := 1 + (g_\alpha * g)(t)$, $t \in [0,\tau)$. Arguing as in the proof of [24, Proposition 1.1, p. 32], we obtain that A is a subgenerator of a locally equicontinuous (g_α, k)-regularized C-resolvent family $(S(t))_{t\in[0,\tau)}$. Since $k \in AC_{loc}([0,\tau))$ and $k(0) = 1 \neq 0$, we infer from [15, Proposition 2.5] and its proof (cf. also [9, Proposition 2.4(ii)]) that there exists $b \in L^1_{loc}([0,\tau))$ such that $(R(t) \equiv S(t) + (b * S)(t))_{t\in[0,\tau)}$ is a locally equicontinuous (g_α, C)-regularized resolvent family with a subgenerator A. In other words, the above conclusion does not depend on the choice of continuous function $g(t)$ appearing in (21).

Assume now that, for every $x_0 \in C(D(A))$, there exists a unique function $u_f(t) \equiv u_f(t; x_0)$, $t \in [0,\tau)$ satisfying u_f, $Au_f \in C([0,\tau) : E)$ and (20). Obviously, $u_f(t)$ is a unique solution of (16) with $x_j = 0$, $1 \leq j \leq \lceil\alpha\rceil - 1$. If

A is a subgenerator of a (g_α, C)-regularized resolvent family $(S_\alpha(t))_{t\in[0,\tau)}$, then the unique solution of (16) with $f \equiv 0$ is given by:

$$u(t) = S_\alpha(t)C^{-1}x_0 + \sum_{j=1}^{\lceil\alpha\rceil} \int_0^t \frac{(t-s)^{j-1}}{(j-1)!} S_\alpha(s)C^{-1}x_{j-1}ds,\ t \in [0,\tau);$$

in this case, the abstract Cauchy problem (16) is C-wellposed if, additionally, $(S_\alpha(t))_{t\in[0,\tau)}$ is locally equicontinuous.

In the subsequent theorem, we will generalize the Ljubich uniqueness theorem. For the equations of integer order, we may pass to the theory of first order equations by means of the following auxiliary lemma of independent interest.

Lemma 3.5 *(cf. [12, Lemma 1.10], [7, Lemma 2.3.22] and [7, Theorem 2.3.23] for the Banach space case)*

(i) *Let $\lambda \in \mathbb{C}$, let $k \in \mathbb{N}\setminus\{1\}$ and let A be a closed linear operator on E. Put $\mathcal{C}_k(x_1, \cdots, x_k) := (Cx_1, \cdots, Cx_k)$, $(x_1, \cdots, x_k) \in E^k$, $D(\mathcal{A}_k) := D(A)\times E^{k-1}$ and $\mathcal{A}_k(x_1, \cdot\cdot\cdot, x_k) := (x_2, \cdot\cdot\cdot, x_k, Ax_1)$, $(x_1, \cdot\cdot\cdot, x_k) \in D(\mathcal{A}_k)$. Then $\lambda \in \rho_{\mathcal{C}_k}(\mathcal{A}_k)$ iff $\lambda^k \in \rho_C(A)$. If this is the case, then we have the following:*

$$\big(\lambda^k - A\big)^{-1}Cx = \pi_1\big(\big(\lambda - \mathcal{A}_k\big)^{-1}\mathcal{C}_k(0, \cdot\cdot\cdot, x)\big),\ x \in E,$$

where $\pi_1 : E^k \to E$ denotes the first projection and, for every $x_1, \cdots, x_k \in E$,

$$\big(\lambda - \mathcal{A}_k\big)^{-1}\mathcal{C}_k\big(x_1, \cdot\cdot\cdot, x_k\big) = \big(y_1, \cdot\cdot\cdot, y_k\big), \tag{22}$$

where

$$\begin{aligned} y_j = {} & \lambda^{j-1}\big(\lambda^k - A\big)^{-1}C\big(\lambda^{k-1}x_1 + \lambda^{k-2}x_2 + \cdot\cdot\cdot + x_k\big) \\ & - \big(Cx_{j-1} + \lambda Cx_{j-2} + \cdot\cdot\cdot + \lambda^{j-2}Cx_1\big),\ 1 \le j \le k. \end{aligned} \tag{23}$$

(ii) *Suppose $\lambda > 0$, $\{n\lambda : n \in \mathbb{N}\} \subseteq \rho_C(A)$ and, for every $\sigma > 0$ and $x \in E$, $\lim_{n\to\infty} \frac{(n\lambda-A)^{-1}Cx}{e^{n\lambda\sigma}} = 0$. Then, for every $x \in E$, there exists at most one solution of the initial value problem*

$$\begin{cases} u \in C^1((0,\infty) : E) \cap C([0,\infty) : E), \\ u'(t) = Au(t),\ t > 0, \\ u(0) = x. \end{cases}$$

(iii) *Suppose $T > 0$, $u \in C([0,T] : E)$, $\lambda > 0$ and the set $\{\int_0^T e^{n\lambda s}u(s)ds : n \in \mathbb{N}\}$ is bounded. Then $u(t) = 0$, $t \in [0,T]$.*

Theorem 3.6 *Suppose $\alpha > 0$, $\lambda > 0$, $\{(n\lambda)^\alpha : n \in \mathbb{N}\} \subseteq \rho_C(A)$ and, for every $\sigma > 0$ and $x \in E$, $\lim_{n\to\infty} \frac{((n\lambda)^\alpha - A)^{-1}Cx}{e^{n\lambda\sigma}} = 0$. Then, for every $x_0, \cdots, x_{\lceil\alpha\rceil-1} \in E$, there exists at most one solution of the initial value problem*

$$\begin{cases} u \in C^{(\lceil\alpha\rceil)}((0,\infty) : E) \cap C^{(\lceil\alpha\rceil-1)}([0,\infty) : E), \\ \mathbf{D}_t^\alpha u(t) = Au(t), \ t > 0, \\ u^{(j)}(0) = x_j, \ 0 \le j \le \lceil\alpha\rceil - 1. \end{cases} \tag{24}$$

Proof. Suppose first $\alpha = k \in \mathbb{N} \setminus \{1\}$. Denote $v(t) := (u(t), u'(t), \cdots, u^{(k-1)}(t))$, $t \ge 0$. Then

$$\begin{cases} v \in C^1((0,\infty) : E^k) \cap C([0,\infty) : E^k), \\ v'(t) = \mathcal{A}_k v(t), \ t > 0, \\ v(0) = (x_0, \cdots, x_{k-1}). \end{cases}$$

By Lemma 3.5(i), we get that $\{n\lambda : n \in \mathbb{N}\} \subseteq \rho_{\mathcal{C}_k}(\mathcal{A}_k)$. Furthermore, the representation formulae (22)-(23) imply that, for every $\sigma > 0$ and $\mathbf{x} \in E^k$, $\lim_{n\to\infty} \frac{(n\lambda - \mathcal{A}_k)^{-1}\mathcal{C}_k\mathbf{x}}{e^{n\lambda\sigma}} = 0$, and the assertion follows from an application of Lemma 3.5(ii). Suppose now $\alpha \in (0,\infty)\setminus\mathbb{N}$ and $u(t)$ is a solution of initial value problem (24) with $x_j = 0$, $0 \le j \le \lceil\alpha\rceil - 1$. Set $z_n(t) := ((n\lambda)^\alpha - A)^{-1}Cu(t)$, $t \ge 0$, $n \in \mathbb{N}$. Then it is straightforward to see that $z_n(t)$ is a solution of the initial value problem:

$$\begin{cases} z_n \in C^{(\lceil\alpha\rceil)}((0,\infty) : E) \cap C^{(\lceil\alpha\rceil-1)}([0,\infty) : E), \\ \mathbf{D}_t^\alpha z_n(t) = (n\lambda)^\alpha z_n(t) - Cu(t), \ t > 0, \\ z_n^{(j)}(0) = 0, \ 0 \le j \le \lceil\alpha\rceil - 1. \end{cases}$$

This implies $z_n(t) = -(u * \cdot^{\alpha-1}E_{\alpha,\alpha}((n\lambda)^\alpha \cdot^{\alpha-1}))(t)$, $t \ge 0$, $n \in \mathbb{N}$. By the given assumptions, it follows that, for every $t > 0$ and $\sigma > 0$,

$$\lim_{n\to\infty} e^{-n\lambda\sigma} \int_0^t s^{\alpha-1} E_{\alpha,\alpha}\big((n\lambda)^\alpha s^\alpha\big) Cu(t-s)\,ds = 0. \tag{25}$$

We consider separately two possible cases: $0 < \alpha < 4$ and $\alpha > 4$. Let $p \in \circledast$, $t > 0$ and $0 < \sigma_0 < \sigma < t$. In the first case, one can apply the identity $E_{\alpha,\alpha}(z) = zE_{\alpha,2\alpha}(z) + \frac{1}{\Gamma(\alpha)}$, $z \in \mathbb{C}$, (1) and the asymptotic formulae [2, (1.27)-(1.28)] (cf. also [26, Theorem 1.1] and [19, (2.10), p. 286]) with $N = 2$ to obtain the existence of a positive real polynomial $P(x)$ and positive real numbers $T > 0$ and $M \ge 1$ such that $n\lambda\sigma_0 \ge T$,

$$\Big| s^{\alpha-1}(n\lambda s)^\alpha E_{\alpha,2\alpha}\big((n\lambda s)^\alpha\big) - \frac{1}{\alpha}(n\lambda)^{1-\alpha}e^{n\lambda s} \Big| \le g_\alpha(s) + MT^{-\alpha}, \text{ if } n\lambda s \ge T,$$

and

$$e^{-n\lambda\sigma}\int\limits_0^t \Big|s^{\alpha-1}E_{\alpha,\alpha}\big((n\lambda)^\alpha s^\alpha\big)-\frac{1}{\alpha}(n\lambda)^{1-\alpha}e^{n\lambda s}\Big|p(Cu(t-s))ds$$
$$\leq P(n)e^{n\lambda(\sigma_0-\sigma)}\int\limits_0^{\sigma_0}\big(1+s^{\alpha-1}\big)p(Cu(t-s))ds$$
$$+e^{-n\lambda\sigma}\int\limits_{\sigma_0}^t\big(2g_\alpha(s)+MT^{-\alpha}\big)p(C(t-s))ds.$$

In combination with (25) and the arbitrariness of p, the above implies $\lim_{n\to\infty}\int_0^t e^{n\lambda(t-s-\sigma)}Cu(s)ds=0$. Since $\lim_{n\to\infty}\int_{t-\sigma}^t e^{n\lambda(t-s-\sigma)}Cu(s)ds=0$, we obtain that $\lim_{n\to\infty}\int_0^{t-\sigma} e^{n\lambda(t-s-\sigma)}Cu(s)ds=0$. Using the injectiveness of C and Lemma 3.5(iii), we obtain that $u(s)=0$, $s\in[0,t-\sigma]$, which completes the proof. In the second case, we assume additionally $\sigma>t\cos(\frac{2\pi}{\alpha})$. Then a similar reasoning yields the existence of a positive real polynomial $P(x)$ and positive real numbers $T>0$ and $M\geq 1$ such that $n\lambda\sigma_0\geq T$,

$$\Big|s^{\alpha-1}(n\lambda s)^\alpha E_{\alpha,2\alpha}\big((n\lambda s)^\alpha\big)-\frac{1}{\alpha}(n\lambda)^{1-\alpha}e^{n\lambda s}$$
$$-\frac{1}{\alpha}(n\lambda)^{1-\alpha}\sum_{k\in\mathbb{Z}\setminus\{0\},\ |k|\leq\lfloor\alpha/4\rfloor}e^{n\lambda s e^{2\pi ik/\alpha}}\Big|\leq g_\alpha(s)+MT^{-\alpha},\ \text{ if } n\lambda s\geq T,$$

and

$$e^{-n\lambda\sigma}\int\limits_0^t\Big|s^{\alpha-1}E_{\alpha,\alpha}\big((n\lambda)^\alpha s^\alpha\big)-\frac{1}{\alpha}(n\lambda)^{1-\alpha}e^{n\lambda s}$$
$$-\frac{1}{\alpha}(n\lambda)^{1-\alpha}\sum_{k\in\mathbb{Z}\setminus\{0\},\ |k|\leq\lfloor\alpha/4\rfloor}e^{n\lambda s e^{2\pi ik/\alpha}}\Big|p(Cu(t-s))ds$$
$$\leq P(n)e^{n\lambda(\sigma_0-\sigma)}\int\limits_0^{\sigma_0}\big(1+s^{\alpha-1}\big)p(Cu(t-s))ds$$
$$+e^{-n\lambda\sigma}\int\limits_{\sigma_0}^t\big(2g_\alpha(s)+MT^{-\alpha}\big)p(C(t-s))ds.$$

Therefore,

$$\lim_{n\to\infty}e^{-n\lambda\sigma}\int\limits_0^t\Big[e^{n\lambda s}+\sum_{k\in\mathbb{Z}\setminus\{0\},\ |k|\leq\lfloor\alpha/4\rfloor}e^{n\lambda s e^{2\pi ik/\alpha}}\Big]Cu(t-s)ds=0.$$

Keeping in mind that $\sigma > t\cos(\frac{2\pi}{\alpha})$, we obtain

$$\lim_{n\to\infty} e^{-n\lambda\sigma} \int_0^t \sum_{k\in\mathbb{Z}\setminus\{0\},\ |k|\leq\lfloor\alpha/4\rfloor} e^{n\lambda s e^{2\pi ik/\alpha}} Cu(t-s)ds = 0,$$

so that $\lim_{n\to\infty}\int_0^t e^{n\lambda(t-s-\sigma)}Cu(s)ds = 0$ and $\lim_{n\to\infty}\int_0^{t-\sigma} e^{n\lambda(t-s-\sigma)}Cu(s)ds = 0$. Since C is injective, Lemma 3.5(iii) implies by letting $\sigma \to t\cos(\frac{2\pi}{\alpha})$ that $u(s) = 0$, $s \in [0, t(1-\cos(\frac{2\pi}{\alpha}))]$. The proof of theorem is completed. □

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