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Fixed Points on Two Generalized Metric Spaces

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Abstract. A fixed point theorem in generalized metric spaces is proved. The obtained result can be considered as an extension and generalization of some well-known fixed point theorems from metric spaces to generalized metric spaces.

Keywords: Cauchy sequence, fixed point, generalized metric spaces.

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1. Introduction and Preliminaries

Branciari [2] introduced the concept of generalized metric space where the triangular inequality of a metric space has been replaced by a tetrahedral inequality. Every metric space is a generalized metric space but the converse is not true (see [2]).

In [2] it was claimed that as in a metric space: a topology can be generated in a g.m.s X with the help of the neighborhood basis given by $\mathrm{B} = \{B(x,r) : x \in X, r > 0\}$ where $B(x,r) = \{y \in X : d(x,y) < r\}$ is the open ball with center x and radius r; the generalized metric d is continuous in each of coordinates and the g.m.s X is a Hausdorff space. These were taken also for granted in ([1], [3], [4]).

In 2009 Das et al [5] and Sarma et al [8] show that the above propositions are not true. In [8] they present a version and proof of Banach's Contraction principle in a g.m.s , with the Hausdorff condition added. When transferred a theorem from metric spaces to g.m.s, maybe some more conditions should be added, for example:

the continuity of generalized metric d, the Hausdorffity of (X,d), etc.

We conclude that the Hausdorff condition of g.m.s in theorem 1.3 [8] is not necessary, because the uniqueness of the limit of Cauchy sequence (a_n) comes as a result without the Hausdorff condition, in the same way as in our main theorem 2.1.

Under this situation, it is reasonable to be considered if other important fixed point theorems can be obtained in generalized metric space.

In this paper we will show that, some such fixed point theorems, are the theorems of Popa [7], Fisher [6] etc.

Firstly, we will give some known definitions and notations.

Definition 1.1 ([2]) Let X be a set and $d: X^2 \to R^+$ a mapping such that for all $x, y \in X$ and for all distinct points $z, w \in X$, each of them different from *x* and *y*, one has

(i) $d(x,y)=0$ if and only if $x=y$,

(ii) $d(x,y)=d(y,x)$,

(iii) $d(x,y) \le d(x,z)+d(z,w)+d(w,y)$ (Tetrahedral inequality)

Then *d is called a generalized metric* and (X,d) is a generalized metric space (or shortly g.m.s).

Let (x_n) be a sequence in X and $x \in X$.

If for every $\varepsilon > 0$ there is an $n_0 \in N$ such that $d(x_n, x) < \varepsilon$, for all $n > n_0$ then (x_n) is said to be convergent, (x_n) converges to x and x *is* the limit of (x_n). We denote this by $\lim_{n\to\infty} x_n = x$.

If for every $\varepsilon > 0$ there is an $n_0 \in N$ such that $d(x_n, x_{n+m}) < \varepsilon$, for all $n > n_0$, then (x_n) is called a Cauchy sequence in X. If every Cauchy sequence is convergent in X , then X is called a complete generalized metric space.

Example1.2[5] Let us define $X = \left\{ \frac{1}{n} : n = 1, 2, \ldots \right\} \cup \{0, 2\}$,

$$d: X \times X \to R^+ : \quad d(x,y) = \begin{cases} 0 & for\ x = y \\ \frac{1}{n} & for\ x \in \{0,2\},\ y = \frac{1}{n} \\ \frac{1}{n} & for\ x = \frac{1}{n},\ y \in \{0,2\} \\ 1 & otherwise \end{cases}$$

Then it is easy to see that (X,d) is a generalized metric space but (X,d) is not a standard metric space because it lacks the triangular property:

$$1 = d(\frac{1}{2},\frac{1}{3}) > d(\frac{1}{2},0) + d(0,\frac{1}{3}) = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

In this g.m.s the sequence $(\frac{1}{n})$ converges to both 0 and 2 and it is not a Cauchy sequence. (X,d) is not a Hausdorff space and d is not continuous distance in a sense presented in [2]: $1 = \lim_{n\to\infty} d(\frac{1}{n},\frac{1}{2}) \neq d(0,\frac{1}{2}) = \frac{1}{2}$

Let $T : X \to X$ be a mapping where *X* is a g.m.s. For each $x \in X$, let

$\mathrm{O(x)=\{x,Tx,T^2}x,...\}$

which will be called the orbit of *T* at *x*. $\mathrm{O(x)}$ is called T-orbitally complete if and only if every Cauchy sequence which is contained in $\mathrm{O(x)}$ converges to a point in *X*.

Popa's theorem [7]: *Let (X,d) and (Y,ρ) be two complete metric spaces. If T is a mapping of X into Y and S is a mapping of Y into X satisfying the inequalities:*

$$d^2(Sy,STx) \le c_2 \max\{\rho(y,Tx)d(x,Sy), \rho(y,Tx)d(x,STx), d(x,Sy)d(x,STx)\} \quad (1)$$

$$\rho^2(Tx,TSy) \le c_1 \max\{d(x,Sy)\rho(y,Tx), d(x,Sy)\rho(y,TSy), \rho(y,Tx)\rho(y,TSy)\} \quad (2)$$

for all x in X and y in Y, where $0 \le c_1, c_2 < 1$, then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

First, we introduce and consider a class of implicit relations which will give a general character to the main theorem 2.1.

Definition 1.3 The set of all upper semi-continuous functions with 3 variables $f : R_+^3 \to R$ satisfying the properties:

(a). f is non decreasing in respect with each variable.

(b). $f(t,t,t) \le t,\ t \in R_+$

will be noted F_3 and every such function will be called a F_3-function

Some examples of F_3-function are as follows:

1. $f(t_1,t_2,t_3) = \max\{t_1,t_2,t_3\}$
2. $f(t_1,t_2,t_3) = [\max\{t_1t_2, t_2t_3, t_3t_1\}]^{1/2}$
3. $f(t_1,t_2,t_3) = [\max\{t_1^p, t_2^p, t_3^p\}]^{1/p}$, $p > 0$
4. $f(t_1,t_2,t_3) = (at_1t_2 + bt_2t_3 + ct_3t_1)^{1/2}$, where $a,b,c \ge 0$ *and* $a+b+c < 1$

2. Main Result

We now prove the following fixed point theorem that generalizes and extends some theorems of Popa [7] and of Fisher [6] from metric spaces to generalized metric spaces.

Theorem 2.1 Let $(X,d),(Y,\rho)$ *be two generalized metric spaces. Let* T *is a mapping of* X *into* Y *and* S *is a mapping of* Y *into* X *satisfying the inequalities:*

$$d(Sy,STx) \le cf_1\{d(x,Sy),d(x,STx),\rho(y,Tx)\} \quad (1)$$

$$\rho(Tx,TSy) \le cf_2\{\rho(y,Tx),\rho(y,TSy),d(x,Sy)\} \quad (2)$$

for all $x \in X$ *and* $y \in Y$, *where* $0 < c < 1$, $f_1, f_2 \in \mathrm{F}_3$ *If there exists* $x_0 \in X$ *such that* $O(x_0)$ *is ST-orbitally complete in X and* $O(Tx_0)$ *is TS-orbitally complete in Y, then ST has a unique fixed point* α *in* X *and TS has a unique fixed point* β *in* Y. *Further,* $T\alpha = \beta$ *and* $S\beta = \alpha$.

Prof. Let x_0 be an arbitrary point in X. Define the sequences (x_n) and (y_n) inductively as follows:

$$x_n = Sy_n = (ST)^n x_0 \quad and \quad y_1 = Tx_0, y_{n+1} = Tx_n = (TS)^n y_1,\ n \ge 1$$

Denote

$$d_n = d(x_n, x_{n+1}) \ and \ \rho_n = \rho(y_n, y_{n+1}), n = 1,2,...$$

Using the inequality (2) we get:

$$\rho_n = \rho(y_n, y_{n+1}) = \rho(Tx_{n-1}, TSy_n) \le$$

$$\le cf_2(\rho(y_n, y_n), \rho(y_n, y_{n+1}), d(x_{n-1}, x_n)) = cf_2(0, \rho_n, d_{n-1})$$

By this inequality and properties of f_2, it follows:

$$\rho_n \le cd_{n-1} \quad (3)$$

Using the inequality (1) we have

$$d_n = d(x_n, x_{n+1}) = d(Sy_n, STx_n) \le$$

$$\le cf_1(d(x_n, x_n), d(x_n, x_{n+1}), \rho(y_n, y_{n+1})) = cf_1(0, d_n, \rho_n)$$

and so $d_n \le c\rho_n$. By this inequality and (3) we obtain

$$d_n \le c^2 d_{n-1} \le cd_{n-1} \quad (4)$$

Using the mathematical induction, by the inequalities (3) and (4), we get:

$$d_n \le c^n d(x_0, x_1) \ and \ \rho_n \le c^n d(x_0, x_1) \quad (5)$$

So

$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} \rho(y_n, y_{n+1}) = 0$

Applying the inequality (2), we get

$$\rho(y_n, y_{n+2}) = \rho(Tx_{n-1}, TSy_{n+1}) \le$$
$$\le cf_2(\rho(y_{n+1}, y_n), \rho(y_{n+1}, y_{n+2}), d(x_{n-1}, x_{n+1})) =$$
$$= cf_2(\rho_n, \rho_{n+1}, d(x_{n-1}, x_{n+1})) \le c \max\{c^n d(x_0, x_1), d(x_{n-1}, x_{n+1})\}$$

and so

$$\rho(y_n, y_{n+2}) \le \max\{c^n d(x_0, x_1), cd(x_{n-1}, x_{n+1})\}$$

Similarly, using (1), we obtain

$$d(x_n, x_{n+2}) \le \max\{c^n d(x_0, x_1), cd(x_{n-1}, x_{n+1})\}$$

Using the mathematical induction, we get:

$$d(x_n, x_{n+2}) \le \max\{c^n d(x_0, x_1), cd(x_{n-1}, x_{n+1})\} \le \max\{c^n d(x_0, x_1), c^2 d(x_{n-2}, x_n)\} \le$$
$$\le \ldots \le \max\{c^n d(x_0, x_1), c^n d(x_0, x_2)\} = c^n \max\{d(x_0, x_1), d(x_0, x_2)\} = c^n l$$

so

$$d(x_n, x_{n+2}) \le c^n l$$

And similarly

$$\rho(y_n, y_{n+2}) \le c^n l$$

where $l = \max\{d(x_0, x_1), d(x_0, x_2)\}$

We divide the proof into two cases.

Case I: Suppose $x_p = x_q$ *for some* $p, q \in N, p \ne q$. Let $p > q$. Then $(ST)^p x_0 = (ST)^{p-q}(ST)^q x_0 = (ST)^q x_0$ i.e. $(ST)^n \alpha = \alpha$ where $n = p - q$ and $(ST)^q x_0 = \alpha$. Now if $n > 1$, by (5), we have

$$d(\alpha, ST\alpha) = d[(ST)^n \alpha, (ST)^{n+1}\alpha] \le c^n d(\alpha, ST\alpha)$$

Since $0 < c < 1$, $d(\alpha, ST\alpha) = 0$. So $ST\alpha = \alpha$ and hence α is a fixed point of *ST*.

By the equality $x_p = x_q$ it follows that $y_{p+1} = y_{q+1}$. We take $\beta = (TS)^q Tx_0$ and, in similar way we prove that β is a fixed point of *TS*.

Case II: Assume that $x_n \ne x_m$ for all $n \ne m$. Then $(x_n) = ((ST)^n x_0)$ is a sequence of distinct point and for $m > n+1$, we have

$(*)$. If $m > 2$ is odd then writing $m = 2k+1, k \ge 1$ (by Rectangular property) we can easily show that

$$d(x_n, x_{n+m}) \le [d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + \ldots + d(x_{n+2k}, x_{n+2k+1})] \le$$
$$\le c^n l + c^{n+1} l + c^{n+2} l + \ldots + c^{n+2k} l = c^n l \frac{1 - c^{2k+1}}{1 - c} < c^n \frac{l}{1 - c}$$

(**). If $m>2$ is even then writing $m=2k, k\geq 2$ and using the same arguments as before we can get

$$d(x_n, x_{n+m}) \leq [d(x_n, x_{n+2}) + d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + ... + d(x_{n+2k-1}, x_{n+2k})] \leq$$

$$\leq c^n l + c^{n+1} l + c^{n+2} l + ... + c^{n+2k} l = c^n l \frac{1-c^{2k+1}}{1-c} < c^n \frac{l}{1-c}$$

Thus combining all the cases we have $d(x_n, x_{n+m}) < c^n \dfrac{l}{1-c}$ for all $n, m \in N$.

Therefore, $\lim_{n\to\infty} d(x_n, x_{n+m}) = 0$. It implies that (x_n) is a Cauchy sequence in X. Since $O(x_0)$ *is ST-orbitally complete*, there exists a $\alpha \in X$ such that $\lim_{n\to\infty} x_n = \alpha$. In the same way, we show that the sequence (y_n) is a Cauchy sequence and exists a $\beta \in Y$ such that $\lim_{n\to\infty} y_n = \beta$. The limits α and β are unique. Assume that $\alpha' \neq \alpha$ is also $\lim_{n\to\infty} x_n$. Since $x_n \neq x_m$ for all $n \neq m$, exists a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \neq \alpha$ and $x_{n_k} \neq \alpha'$ for all $k \in N$. Without loss of generality, assume that (x_n) is this subsequence. Then by Tetrahedral property of Definition 1.1 we obtain

$$d(\alpha, \alpha') \leq d(\alpha, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, \alpha')$$

Letting n tend to infinity we get $d(\alpha, \alpha') = 0$ and so $\alpha = \alpha'$. In the same way for β.

Let we prove now that α is a fixed point of ST. First we prove that $\beta = T\alpha$. In contrary, if $\beta \neq T\alpha$, the sequence (y_n) does not converge to $T\alpha$ and there exists a subsequence (y_{n_k}) of (y_n) such that $y_{n_k} \neq T\alpha$ for all $k \in N$. Then by Tetrahedral property of Definition 1.1 we obtain

$$\rho(\beta, T\alpha) \leq \rho(\beta, y_{n_{k-1}}) + \rho(y_{n_{k-1}}, y_{n_k}) + \rho(y_{n_k}, T\alpha)$$

Then if $k \to \infty$, we get

$$\rho(\beta, T\alpha) \leq \overline{\lim_{k\to\infty}} \rho(y_{n_k}, T\alpha) \qquad (8)$$

Using the inequality (2), for $x=\alpha$ and $y = y_{n-1}$ we obtain:

$$\rho(T\alpha, y_n) = \rho(T\alpha, TSy_{n-1}) \leq cf_2(\rho(y_{n-1}, T\alpha), \rho(y_{n-1}, TSy_{n-1}), d(\alpha, Sy_{n-1})) =$$

$$= cf_2(\rho(y_{n-1}, T\alpha), \rho(y_{n-1}, y_n), d(\alpha, x_{n-1}))$$

Letting n tend to infinity we get

$$\overline{\lim_{n\to\infty}} \rho(T\alpha, y_n) \leq cf_2(\overline{\lim_{n\to\infty}} \rho(y_{n-1}, T\alpha), 0, 0)$$

And so,

$$\overline{\lim_{n\to\infty}} \rho(T\alpha, y_n) = 0 \qquad (9)$$

Since $\overline{\lim_{k\to\infty}}\rho(y_{n_k},T\alpha)\le\overline{\lim_{n\to\infty}}\rho(T\alpha,y_n)$, By (9) and (8), we have $\rho(\beta,T\alpha)=0$ and so
$T\alpha=\beta$. (10)
It follows similarly that
$S\beta=\alpha$ (11)
By (10) and (11) we obtain:
$ST\alpha=S\beta=\alpha$ *and* $TS\beta=T\alpha=\beta$
Thus, we proved that the points α *and* β are fixed points of ST *and* TS respectively.
Let we prove now the uniqueness (for case I and II in the same time). Assume that $\alpha'\neq\alpha$ is also a fixed point of ST. By (1) for $x=\alpha'$ and $y=\beta$ we get:
$d(\alpha,\alpha')=d(S\beta,ST\alpha')\le cf_1(d(\alpha',\alpha),0,\rho(T\alpha,T\alpha'))$
And so, we have
$d(\alpha,\alpha')\le c\rho(T\alpha,T\alpha')$ (12)
If $T\alpha\neq T\alpha'$, in similar way by (2) for $x=ST\alpha$ and $y=T\alpha'$, we have:
$\rho(T\alpha,T\alpha')\le cd(\alpha,\alpha')$ (13)
By (12) and (13) we get: $d(\alpha,\alpha')=0$. Thus, we have again $\alpha=\alpha'$. The uniqueness of β follows similarly. This completes the proof of the theorem.

3. Corollaries

For different expressions of f_1 and f_2 we get different theorems. If in theorem 2.1 we take $f_1=f_2=f$, where $f(t_1,t_2,t_3)=\max\{t_1,t_2,t_3\}$ we obtain the extension of Fisher theorem [6] to generalized metric spaces:

Corollary 3.1 Let $(X,d),(Y,\rho)$ *be two generalized metric spaces. Let* T *is a mapping of* X *into* Y *and* S *is a mapping of* Y *into* X *satisfying the inequalities:*
$d(Sy,STx)\le c\max\{d(x,Sy),d(x,STx),\rho(y,Tx)\}$
$\rho(Tx,TSy)\le c\max\{\rho(y,Tx),\rho(y,TSy),d(x,Sy)\}$
for all $x\in X$ *and* $y\in Y$, where $0<c<1$, $f_1,f_2\in F_3$. If there exists $x_0\in X$ such that $O(x_0)$ is ST-orbitally complete in X and $O(Tx_0)$ is TS-orbitally complete in Y, then ST has a unique fixed point α *in* X and TS has a unique fixed point β *in* Y. Further, $T\alpha=\beta$ *and* $S\beta=\alpha$.

If in theorem 2.1 we take $f_1 = f_2 = f$, where $f(t_1,t_2,t_3) = [\max\{t_1t_2, t_2t_3, t_3t_1\}]^{1/2}$, we obtain the extension of Popa result (Theorem 2 [11]) for metric spaces to generalized metric spaces:

Corollary 3.2 Let $(X,d),(Y,\rho)$ *be two generalized metric spaces. Let* T *is a mapping of* X *into* Y *and* S *is a mapping of* Y *into* X *satisfying the inequalities:*

$$d^2(Sy,STx) \le c_2 \max\{\rho(y,Tx)d(x,Sy), \rho(y,Tx)d(x,STx), d(x,Sy)d(x,STx)\}$$
$$\rho^2(Tx,TSy) \le c_1 \max\{d(x,Sy)\rho(y,Tx), d(x,Sy)\rho(y,TSy), \rho(y,Tx)\rho(y,TSy)\}$$

for all $x \in X$ *and* $y \in Y$, *where* $0 < c_1, c_2 < 1$. *If there exists* $x_0 \in X$ *such that* $O(x_0)$ *is ST-orbitally complete in X and* $O(Tx_0)$ *is TS-orbitally complete in Y, then ST has a unique fixed point* α *in* X *and TS has a unique fixed point* β *in* Y. *Further,* $T\alpha = \beta$ *and* $S\beta = \alpha$.

If in theorem 2.1 we take $f_1 = f_2 = f$ where $f(t_1,t_2,t_3) = [\max\{t_1^p, t_2^p, t_3^p\}]^{1/p}$, $p > 0$, we obtain a generalization of Corollary 3.1 which is taken for $p = 1$.

Corollary 3.3 Let $(X,d),(Y,\rho)$ *be two generalized metric spaces. Let* T *is a mapping of* X *into* Y *and* S *is a mapping of* Y *into* X *satisfying the inequalities:*

$$d^p(Sy,STx) \le c \max\{d^p(x,Sy), d^p(x,STx), \rho^p(y,Tx)\}$$
$$\rho^p(Tx,TSy) \le c \max\{\rho^p(y,Tx), \rho^p(y,TSy), d^p(x,Sy)\}$$

for all $x \in X$ *and* $y \in Y$, *where* $0 < c < 1$, $f_1, f_2 \in \mathrm{F}_3$ *If there exists* $x_0 \in X$ *such that* $O(x_0)$ *is ST-orbitally complete in X and* $O(Tx_0)$ *is TS-orbitally complete in Y, then ST has a unique fixed point* α *in* X *and TS has a unique fixed point* β *in* Y. *Further,* $T\alpha = \beta$ *and* $S\beta = \alpha$.

If in theorem 2.1 we take $f_1(t_1,t_2,t_3) = (a_1t_1t_2 + b_1t_2t_3 + c_1t_3t_1)^{1/2}$ and $f_2(t_1,t_2,t_3) = (a_2t_1t_2 + b_2t_2t_3 + c_2t_3t_1)^{1/2}$ we obtain the extension of Popa result (Corollary 2 in [7]) for metric spaces to generalized metric spaces:

Corollary 3.4 Let $(X,d),(Y,\rho)$ *be two generalized metric spaces. Let* T *is a mapping of* X *into* Y *and* S *is a mapping of* Y *into* X *satisfying the inequalities:*

$$d^2(Sy,STx) \le a_1\rho(y,Tx)d(x,Sy) + b_1\rho(y,Tx)d(x,STx) + c_1d(x,Sy)d(x,STx)$$
$$\rho^2(Tx,TSy) \le a_2d(x,Sy)\rho(y,Tx) + b_2d(x,Sy)\rho(y,TSy) + c_2\rho(y,Tx)\rho(y,TSy)$$

for all $x \in X$ *and* $y \in Y$, *where* $a_1, a_2, b_1, b_2, c_1, c_2 \geq 0$ *and* $a_1 + b_1 + c_1, a_2 + b_2 + c_2 < 1$. *If there exists* $x_0 \in X$ *such that* $O(x_0)$ *is ST-orbitally complete in X and* $O(Tx_0)$ *is TS-orbitally complete in Y, then ST has a unique fixed point* α *in* X *and TS has a unique fixed point* β *in* Y. *Further,* $T\alpha = \beta$ *and* $S\beta = \alpha$.

Remark 3.5 We can obtain many other similar results for different f.

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