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A Nonlocal Problem for a Multi-Term Fractional-Order Differential Equation

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Abstract

In this paper we study the existence of solution for a nonlocal problems of a multi-term arbitrary (fractional) orders differential equation. The corresponding integral condition problem will be considered.

Keywords: Fractional calculus, nonlocal condition, integral condition, multi-term differential equation, multi-term fractional-orders functional integral equation

1 Introduction

Problems with non-local conditions have been extensively studied by several authors in the last two decades. The reader is referred to ([1]- [6] and [9]- [12]), and references therein. Recently, the authors (see [10]) studied the the existence of solution for the nonlocal problem

$$\frac{dx}{dt} \ = \ f(t, D^{\alpha} x(t)), \quad t \in (0,1] \ \text{ and } \ \alpha \in (0,1] \tag{1}$$

$$x(0) + \sum_{k\,=\,1}^{m} a_k \, x(t_k) \ = \ x_o, \quad t_k \in (0,1] \tag{2}$$

when f is L^1-Caratheodory.
In this work we study the existence of at least one solution for the nonlocal

problem of the arbitrary (fractional) order differential equation

$$x'(t) \ = \ f(t, \ D^{\alpha_1}x(t), \ D^{\alpha_2}x(t), \ \cdots, \ D^{\alpha_n}x(t)) \ , \alpha_i \in (0, \ 1), \ \text{a.e. } t \in (0, \ 1) \tag{3}$$

with the nonlocal condition

$$\sum_{k=1}^{m} a_k \ x(\tau_k) \ = \ x_o, \qquad , \ \tau_k \in (a, \ b) \subset \ (0, \ 1). \tag{4}$$

As an application, we deduce the existence of solution for the nonlocal problem of the differential equation (3) with the nonlocal integral condition

$$\int_a^b x(s) \ ds \ = \ x_o, \quad (a, \ b) \subset \ (0, \ 1). \tag{5}$$

2 preliminaries

Let $L^1(I)$ denotes the class of Lebesgue integrable functions on the interval $I \ = \ [0,1]$,where $0 \leq \ a < b < \infty$ and let $\Gamma(.)$ denotes the gamma function.
Definition 2.1 The fractional-order integral of the function $f \in L_1[a,b]$ of order $\beta > 0$ is defined by (see [14])

$$I_a^\beta \ f(t) \ = \ \int_a^t \frac{(t \ - \ s)^{\beta \ - \ 1}}{\Gamma(\beta)} \ f(s) \ ds,$$

Definition 2.2 The Caputo fractional-order derivative of $f(t)$ of order $\alpha \in (0,1]$ is defined as (see [13] and [14])

$$D_a^\alpha \ f(t) \ = \ I_a^{1-\alpha} \ \frac{d}{dt} \ f(t) \ = \ \int_a^t \ \frac{(t \ - \ s)^{- \ \alpha}}{\Gamma(1 \ - \ \alpha)} \ \frac{d}{ds} f(s) \ ds.$$

The following theorems will be needed
Theorem 2.1 (Schauder fixed point theorem [7])
Let E be a Banach space and Q be a convex subset of E, and $T : Q \longrightarrow \ Q$ is compact, continuous map, Then T has at least one fixed point in Q.
Theorem 2.2 (Kolmogorov compactness criterion [8])
Let $\Omega \ \subseteq \ L^p \ (0,1), \ 1 \ \leq \ p \ < \ \infty$. If

(i) Ω is bounded in $L^p \ (0,1)$, and

(ii) $u_h \ \rightarrow \ u$ as $h \ \rightarrow \ 0$ uniformly with respect to $u \ \in \ \Omega$, then Ω is relatively compact in $L^p \ (0,1)$, where

$$u_h(t) \ = \ \frac{1}{h} \int_t^{t+h} \ u(s) \ ds.$$

3 Main results

Consider firstly the fractional-order functional integral equation

$$y(t) = f(t,\ I^{1-\alpha_1}y(t), \cdots,\ I^{1-\alpha_n}y(t)), \tag{6}$$

Definition 3.1 The function y is called a solution of the fractional-order functional integral equation (6), if $y \in L_1[0,1]$ and satisfies (6).
Consider the following assumption

(i) $f : [0,1] \times R_n \to R$ be a function with the following properties:

 (a) for each $t \in [0,1]$, $f(t,.)$ is continuous,

 (b) for each $x \in R_n$, $f(.,x)$ is measurable,

(ii) there exists an integrable function a, $a \in L_1[0,1]$ and constants $b_i > 0$, $i = 1,\ 2,$ such that

$$|f(t,x)| \ \leq\ a(t)\ +\ \sum_{i=1}^{n} b_i\ |x_i|, \quad \text{for each } \ t \in [0,1], \quad x \in R_n,$$

Theorem 3.1 Let the assumptions (i) and (ii) are satisfied. If $\sum_{i=1}^{n} \frac{b_i}{\Gamma(2-\alpha_i)} < 1$, then the fractional-order functional integral equation (6) has at least one solution $y \in L_1[0,1]$.
Proof. Define the operator T associated with equation (6) by

$$Ty(t) = f(t,\ I^{1-\alpha_1}y(t), \cdots,\ I^{1-\alpha_n}y(t)).$$

Let $B_r = \{y \in L_1(I) : \|y\| < r,\ r > 0\}$ and let y be an arbitrary element in B_r. Then from the assumptions (i) and (ii), we obtain

$$\begin{aligned}
||Ty||_{L_1} &= \int_0^1 |Ty(t)|\ dt \ \leq\ \int_0^1 |f(t,\ I^{1-\alpha_1}y(t), \cdots,\ I^{1-\alpha_n}y(t))|dt \\
&\leq \int_0^1 |a(t)|dt\ +\ \sum_{i=1}^{n} b_i \int_0^1 \int_0^t \frac{(t-s)^{-\alpha_i}}{\Gamma(1-\alpha_i)}|y(s)|\ ds\ dt \\
&\leq \|a\|\ +\ \sum_{i=1}^{n} b_i \int_0^1 \int_s^1 \frac{(t-s)^{-\alpha_i}}{\Gamma(1-\alpha_i)} dt|y(s)|\ ds \\
&\leq \|a\|\ +\ \sum_{i=1}^{n} b_i \int_0^1 -\frac{(t-s)^{1-\alpha_i}}{(1-\alpha_i)\Gamma(1-\alpha_i)}|_s^1|y(s)|\ ds \\
&\leq \|a\|\ +\ \sum_{i=1}^{n} b_i \int_0^1 \frac{1}{\Gamma(2-\alpha_i)}|y(s)|\ ds \\
&\leq \|a\| + \sum_{i=1}^{n} \frac{b_i}{\Gamma(2-\alpha_i)}\|y\|_{L_1} \leq\ r,
\end{aligned}$$

which implies that the operator T maps B_r into it self. Assumption (i) implies that T is continuous. Now, we will show that T is compact, to apply Theorem 2.2. So, let Ω be a bounded subset of B_r. Then $T(\Omega)$ is bounded in $L_1[0,1]$, i.e. condition (i) of Theorem 2.2 is satisfied. It remains to show that $(Ty)_h \to Ty$ in $L_1[0,1]$ as $h \to 0$, uniformly with respect to $Ty \in T\ \Omega$. Now

$$\begin{aligned}
||(Ty)_h \ - \ Ty|| &= \int_0^1 |(Ty)_h(t) - (Ty)(t)|\ dt \\
&= \int_0^1 \ |\frac{1}{h} \int_t^{t+h} \ (Ty)(s)\ ds \ - \ (Ty)(t)|\ dt \\
&\le \int_0^1 \ \left(\frac{1}{h} \int_t^{t+h} \ |(Ty)(s) \ - (Ty)(t)|\ ds\right)\ dt \\
&\le \int_0^1 \ \frac{1}{h} \int_t^{t+h} |f(s,\ I^{1-\alpha_1}y(t), \cdots,\ I^{1-\alpha_n}y(t)) - f(t,\ I^{1-\alpha_1}y(t), \cdots,\ I^{1-\alpha_n}y(t))|\ ds\ dt.
\end{aligned}$$

And $y \in \ \Omega$ implies (by assumption (ii)) that $f \in L_1(0,1)$, then

$$\frac{1}{h} - \int_t^{t+h} -|f(s,\ I^{1-\alpha_1}y(t), \cdots,\ I^{1-\alpha_n}y(t)) - f(t,\ I^{1-\alpha_1}y(t), \cdots,\ I^{1-\alpha_n}y(t))|\ ds \ \to \ 0$$

Therefore, by Theorem 2.2, we have that $T(\Omega)$ is relatively compact, that is, T is a compact operator, then the operator T has a fixed point in B_r, which proves the existence of solution $y \in \ L_1[0,1]$ equation (6).■

For the existence of solution of the nonlocal problem (3)- (4) we have the following theorem

Theorem 3.2 Let the assumptions of Theorem 3.1 are satisfied. Then nonlocal problem (3)- (4) has at least one solution $x \in AC[0,1]$.

Proof. Consider the nonlocal fractional differential equation

$$x'(t) \ = \ f(t,\ D^{\alpha_1}x(t),\ D^{\alpha_2}x(t),\ \cdots,\ D^{\alpha_n}x(t))\ , \alpha_i \in (0,\ 1),\ \text{a.e. } t \in (0,\ 1),$$

Let $y(t) \ = \ x'(t)$, then

$$x(t) \ = \ x(0) \ + \ Iy(t) \tag{7}$$

and y is the solution of the fractional-order functional integral equation (6). Let $t \ = \tau_k \ \in (a,b)$ in equation (7), we get

$$x(\tau_k) = \int_0^{\tau_k} \ y(s)\ ds + x(0)$$

and

$$\sum_{k=1}^{m} \ a_k x(\tau_k) = \sum_{k=1}^{m} \ a_k \int_0^{\tau_k} \ y(s)\ ds + x(0) \sum_{k=1}^{m} \ a_k.$$

From equation (4), we get

$$x_o = \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\ ds + x(0) \sum_{k=1}^{m} a_k.$$

Then we obtain

$$x(0) \ = A \left(x_o \ - \ \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\ ds \right), \quad \text{where} \ \ A = (\sum_{k=1}^{m} a_k)^{-1}.$$

and

$$x(t) \ = A\ x_o \ - \ A \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\ ds + \ \int_0^{t} y(s)\ ds \tag{8}$$

which, by Theorem 3.1, has at least one solution $x \in AC(0,1)$.
Now, from equation (8), we have

$$x(0) = \lim_{t\to\ 0^+} x(t) = A\ x_o \ - \ A \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\ ds$$

$$x(1) = \lim_{t\to\ 1^-} x(t) = A\ x_o \ - \ A \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\ ds + \ \int_0^{1} y(s)\ ds$$

from which we deduce that equation (8) has at least one solution $x \in AC[0,1]$.
To complete the proof we prove that equation (8) satisfies the nonlocal problem (3)- (4). Differentiating (8), we obtain

$$D^{\alpha_i} x(t) = I^{1-\alpha_i} \ \frac{d}{dt} x(t) = I^{1-\alpha_i} y(t), \quad i \ = 1, 2 \cdots$$

and

$$x'(t) \ = \ \frac{dx}{dt} \ = \ y(t) \ = \ f(t,\ D^{\alpha_1}x(t),\ D^{\alpha_2}x(t),\ \cdots,\ D^{\alpha_n}x(t)).$$

Also from (8), we have

$$\sum_{k=1}^{m} a_k\, x(\tau_k) \ = \ x_o - \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\ ds + \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\ ds \ = \ x_o, \ \ \tau_k \in (a,b)$$

4 Nonlocal integral condition

Let $x \in AC[0,1]$ be the solution of the nonlocal problem (1)-(2).
Let $a_k \ = \ t_k - t_{k-1}, \ \ \tau_k \in (t_{k-1}, t_k), \ a \ = \ t_0 < t_1 < t_2, \cdots < t_n \ = \ b,$ then the nonlocal condition (2) will be

$$\sum_{k=1}^{m} (t_k - t_{k-1})\ x(\tau_k) \ = \ x_o.$$

From the continuity of the solution x of the nonlocal problem (1)-(2) we can obtain

$$\lim_{m\to\infty} \sum_{k=1}^{m} (t_k - t_{k-1})\; x(\tau_k) \;=\; \int_a^b x(s)\; ds.$$

and the nonlocal condition (2) transformed to the integral one

$$\int_a^b x(s)\; ds \;=\; x_o.$$

Also from the continuity of the function $Iy(t)$, where y is the solution of the functional integral equation (6), we deduce that the solution (8) will be

$$x(t) \;=\; (b-a)^{-1}\; (x_o \;-\; \int_a^b \int_0^s y(\theta)\; d\theta\; ds) \;+\; \int_0^t y(s)\; ds.$$

Now, we have the following Theorem

Theorem 4.1 Let the assumptions of Theorem 3.2 are satisfied. Then there exist at least one solution $x \in AC[0,1]$ of the nonlocal problem with integral condition,

$$x'(t) \;=\; f(t,\; D^{\alpha_1}x(t),\; D^{\alpha_2}x(t),\; \cdots,\; D^{\alpha_n}x(t))\;, \alpha_i \in (0,\; 1),\; \text{a.e. } t \in (0,\; 1),$$

$$\int_a^b x(s)\; ds \;=\; x_o, \quad (a,b) \in (0,1).$$

References

[1] Boucherif, A. First-order differential inclusions with nonlocal initial conditions, Applied Mathematics Letters Vol.15 (2002) 409-414.

[2] A. Boucherif, Nonlocal Cauchy problems for first-order multivalued differential equations, Electronic Journal of Differential Equations Vol. 2002 (2002), No. 47, pp. 1-9.

[3] Boucherif, A and Precup,R. On The nonlocal initial value problem for first order differential equations, Fixed Point Theory Vol. 4, No 2, (2003) 205-212.

[4] Boucherif, A. Semilinear evolution inclusions with nonlocal conditions, Applied Mathematics Letters Vol.22 (2009) 1145-1149.

[5] M. Benchohra, E.P. Gatsori and S.K. Ntouyas, Existence results for seme-linear integrodifferential inclusions with nonlocal conditions. Rocky Mountain J. Mat. Vol. 34, No. 3, Fall 2004

[6] M. Benchohra, S. Hamani, S. Ntouyas, Boundary value problems for differential equations with fractional order and nonlocal conditions, Nonlinear Analysis Vol.71 (2009) 23912396

[7] Deimling, K. Nonlinear Functional Analysis, Springer-Verlag (1985).

[8] Dugundji, J. and Granas, A. Fixed Point Theory, Monografie Mathematyczne, PWN, Warsaw (1982).

[9] El-Sayed, A. M. A. and Abd El-Salam, Sh. A. On the stability of a fractional order differential equation with nonlocal initial condtion, EJQTDE Vol. 2009 No. 29 (2008) 1-8.

[10] El-Sayed, A.M.A and Bin-Taher E.O A nonlocal problem of an arbitrary (fractional) orders differential equation, Alexandria j. of Math. Vol. 1 No. 2 (2010) In press.

[11] Gatsori. E, Ntouyas. S. K, and Sficas. Y.G. On a nonlocal cauchy problem for differential inclusions, Abstract and Applied Analysis (2004) 425-434.

[12] Guerekata, G. M. A Cauchy problem for some fractional abstract differential equation with non local conditions, Nonlinear Analysis 70 (2009) 1873-1876.

[13] Podlubny, I. and EL-Sayed, A. M. A. On two definitions of fractional calculus, Preprint UEF 03-96 (ISBN 80-7099-252-2), Slovak Academy of Science-Institute of Experimental phys. (1996).

[14] Podlubny, I. Fractional Differential Equations, Acad. press, San Diego-New York-London (1999).

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