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# On Generalized Topology and Minimal Structure Spaces

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**Abstract**

In this paper, we introduce the concepts of generalized topology and minimal structure spaces and their closed sets. We investigate some properties of closed sets on this space. Moreover, we give the concepts of $T_1$-GTMS spaces and $T_2$-GTMS spaces and their characterizations.

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**Keywords:** generalized topology and minimal structure space, $T_1$-GTMS spaces, $T_2$-GTMS spaces

## 1 Introduction

In 2000, V. Popa and T. Noiri [8] introduced the notion of minimal structure. Also they introduced the notion of $m_X$-open sets and $m_X$-closed sets and characterized those sets using $m_X$-closure and $m_X$-interior operators, respectively. V. Popa and T. Noiri [6] obtained the definitions and characterizations of separation axioms by using the concept of minimal structure. C. Boonpok [3] introduced the concept of biminimal structure spaces and studied $m_X^1m_X^2$-closed sets and $m_X^1m_X^2$-open sets in bimanual structure spaces.

In 2002, Á. Császár [1] introduced the notion of generalized neighborhood systems and generalized topological spaces. He also introduced the notions of continuous functions and associated interior and closure operators on generalized neighborhood systems and generalized topological spaces. Moreover, he studied the simplest separation axioms for generalized topologies in [2]. C.

Boonpok [4] introduced the concept of bigeneralized topological spaces and studied $(m, n)$-closed sets and $(m, n)$-open sets in bigeneralized topological spaces. In this paper, we study a new space which consists of a set $X$, generalized topology on $X$ and minimal structure on $X$. We will call it that generalized topology and minimal structure spaces (briefly GTMS space). And we study some properties of closed sets on the space. Finally, we introduce the concepts of the separation axioms on generalized topology and minimal structure spaces.

# 2 Preliminaries

**Definition 2.1.** [1] Let $X$ be a nonempty set and $g$ a collection of subsets of X. Then $g$ is called a *generalized topology* (briefly $GT$) on X if and only if $\emptyset \in g$ and $G_i \in g$ for $i \in I \neq \emptyset$ implies $\cup_{i\in I} G_i \in g$. We call the pair $(X, g)$ a *generalized topological space* (briefly $GTS$) on X. The elements of $g$ are called *g-open sets* and the complements are called *g-closed sets*.

The closure of a subset $A$ in a generalized topological space $(X, g)$, denoted by $g\text{-}Cl(A)$, is the intersection of generalized closed sets including $A$. And the interior of $A$, denoted by $g\text{-}Int(A)$, is the union of generalized open sets contained in $A$.

**Theorem 2.2.** [1] *Let $(X, g)$ be a generalized topological space. Then*

*(1) $g\text{-}Cl(A) = X - g\text{-}Int(X - A)$;*

*(2) $g\text{-}Int(A) = X - g\text{-}Cl(X - A)$.*

**Proposition 2.3.** [9] *Let $(X, g)$ be a generalized topological space and $A \subseteq X$. Then*

*(1) $x \in g\text{-}Int(A)$ if and only if there exists $V \in g$ such that $x \in V \subseteq A$;*

*(2) $x \in g\text{-}Cl(A)$ if and only if $V \cap A \neq \emptyset$ for every g-open set $V$ containing $x$.*

**Proposition 2.4.** [9] *Let $(X, g)$ be a generalized topological space. For subsets $A$ and $B$ of $X$, the following properties hold:*

*(1) $g\text{-}Cl(X - A) = X - g\text{-}Int(A)$ and $g\text{-}Int(X - A) = X - g\text{-}Cl(A)$;*

*(2) if $X - A \in g$, then $g\text{-}Cl(A) = A$ and if $A \in g$, then $g\text{-}Int(A) = A$;*

*(3) if $A \subseteq B$, then $g\text{-}Cl(A) \subseteq g\text{-}Cl(B)$ and $g\text{-}Int(A) \subseteq g\text{-}Int(B)$;*

*(4) $A \subseteq g\text{-}Cl(A)$ and $g\text{-}Int(A) \subseteq A$;*

*(5)* $g\text{-}Cl(g\text{-}Cl(A)) = g\text{-}Cl(A)$ *and* $g\text{-}Int(g\text{-}Int(A)) = g\text{-}Int(A)$.

**Definition 2.5.** [7] Let $X$ be a nonempty set and $P(X)$ the power set of $X$. A subfamily $m_X$ of $P(X)$ is called a *minimal structure* (briefly *m-structure*) on $X$ if $\emptyset \in m_X$ and $X \in m_X$.

By $(X, m_X)$, we denote a nonempty set $X$ with an $m$-structure $m_X$ on $X$ and it is called an *m-space*. Each member of $m_X$ is said to be *$m_X$-open* and the complement of an $m_X$-open set is said to be *$m_X$-closed*.

**Definition 2.6.** [7] Let $X$ be a nonempty set and $m_X$ an $m$-structure on $X$. For a subset $A$ of $X$, the $m_X$-closure of $A$ denoted by $m_X\text{-}Cl(A)$ and the $m_X$-interior of $A$ denoted by $m_X\text{-}Int(A)$, are defined as follows:

(1) $m_X\text{-}Cl(A) = \cap\{F : A \subseteq F, X - F \in m_X\}$,

(2) $m_X\text{-}Int(A) = \cap\{U : U \subseteq A, U \in m_X\}$.

**Lemma 2.7.** [5] *Let $X \neq \emptyset$ and $m_X$ a m-structure on $X$. For $A, B \subseteq X$ the following properties hold:*

*(1)* $m_X\text{-}Cl(X-A) = X-m_X\text{-}Int(A)$ *and* $m_X\text{-}Int(X-A) = X-m_X\text{-}Cl(A)$,

*(2) if* $X - A \in m_X$, *then* $m_X\text{-}Cl(A) = A$ *and if* $A \in m_X$, *then* $m_X\text{-}Int(A) = A$,

*(3)* $m_X\text{-}Cl(\emptyset) = \emptyset$, $m_X\text{-}Cl(X) = X$, $m_X\text{-}Int(\emptyset) = \emptyset$ *and* $m_X\text{-}Int(X) = X$,

*(4) if* $A \subseteq B$, *then* $m_X\text{-}Cl(A) \subseteq m_X\text{-}Cl(B)$ *and* $m_X\text{-}Int(A) \subseteq m_X\text{-}Int(B)$,

*(5)* $A \subseteq m_X\text{-}Cl(A)$ *and* $m_X\text{-}Int(A) \subseteq A$,

*(6)* $m_X\text{-}Cl(m_X\text{-}Cl(A)) = m_X\text{-}Cl(A)$ *and* $m_X\text{-}Int(m_X\text{-}Int(A)) = m_X\text{-}Int(A)$.

**Lemma 2.8.** [5] *Let $X)$ be a nonempty set with a minimal structure $m_X$ and $A$ a subset of $X$. Then $x \in m_X\text{-}Cl(A)$ if and only if $U \cap A \neq \emptyset$ for every $m_X$-open set $U$ containing $x$.*

# 3 Generalized topology and minimal structure spaces

In this section, we introduce the notions of generalized topology and minimal structure spaces and closed sets on this space. Next, we study some properties of closed sets.

**Definition 3.1.** Let $X$ be a nonempty set and let $g_X$ be a generalized topology and $m_X$ a minimal structure on $X$. A triple $(X, g_X, m_X)$ is called a *generalized topology and minimal structure space* (briefly *GTMS space*).

Let $(X, g_X, m_X)$ be a generalized topology and minimal structure space and $A$ a subset of $X$. The closure and interior of $A$ in $g_X$ are denote by $g_X$-$Cl(A)$ and $g_X$-$Int(A)$, respectively. And the closure and interior of $A$ in $m_X$ are denote by $m_X$-$Cl(A)$ and $m_X$-$Int(A)$, respectively.

**Definition 3.2.** Let $(X, g_X, m_X)$ be a generalized topology and minimal structure space. A subset $A$ of $X$ is said to be a *$g_X m_X$-closed set* if $g_X$-$Cl(m_X$-$Cl(A))$ $= A$. And a subset $A$ of $X$ is said to be a *$m_X g_X$-closed set* if $m_X$-$Cl(g_X$-$Cl(A))$ $= A$. The complement of a $g_X m_X$-closed (resp. $m_X g_X$-closed) set is said to be *$g_X m_X$-open* (resp. *$m_X g_X$-open*).

**Lemma 3.3.** *Let $(X, g_X, m_X)$ be a generalized topology and minimal structure space and $A \subseteq X$. Then $A$ is $g_X m_X$-closed if and only if $m_X$-$Cl(A) = A$ and $g_X$-$Cl(A) = A$.*

*Proof.* Assume that $A$ is a $g_X m_X$-closed set. Then $g_X$-$Cl(m_X$-$Cl(A)) = A$. Since $m_X$-$Cl(A) \subseteq g_X$-$Cl(m_X$-$Cl(A)) = A$ we have $m_X$-$Cl(A) = A$. From $A \subseteq m_X$-$Cl(A)$, we obtain that $g_X$-$Cl(A) \subseteq g_X$-$Cl(m_X$-$Cl(A)) = A$. Hence, $g_X$-$Cl(A) = A$.

Conversely, let $A$ be a subset of $X$ such that $m_X$-$Cl(A) = A$ and $g_X$-$Cl(A)$ $= A$. Then $A = g_X$-$Cl(A) = g_X$-$Cl(m_X$-$Cl(A))$. Hence, $A$ is $g_X m_X$-closed. □

**Lemma 3.4.** *Let $(X, g_X, m_X)$ be a generalized topology and minimal structure space and $A \subseteq X$. Then $A$ is $m_X g_X$-closed if and only if $m_X$-$Cl(A) = A$ and $g_X$-$Cl(A) = A$.*

*Proof.* The proof of this Lemma is similar to the proof of Lemma 3.3. □

**Proposition 3.5.** *Let $(X, g_X, m_X)$ be a generalized topology and minimal structure space and $A \subseteq X$. Then $A$ is $g_X m_X$-closed if and only if $A$ is $m_X g_X$-closed.*

*Proof.* It follows from Lemma 3.3 and Lemma 3.4. □

**Definition 3.6.** Let $(X, g_X, m_X)$ be a generalized topology and minimal structure space and $A$ a subset of $X$. Then $A$ is said to be *closed* if $A$ is $g_X m_X$-closed. The complement of a closed set is said to be an *open set*.

**Remark 3.7.** *Let $(X, g_X, m_X)$ be a generalized topology and minimal structure space and $A$ a subset of $X$. Then $A$ is closed if and only if $A$ is $m_X g_X$-closed.*

**Proposition 3.8.** *Let $(X, g_X, m_X)$ be a generalized topology and minimal structure space. If $A$ and $B$ are closed, then $A \cap B$ is closed.*

*Proof.* Assume that $A$ and $B$ are closed. Then $g_X\text{-}Cl(m_X\text{-}Cl(A)) = A$ and $g_X\text{-}Cl(m_X\text{-}Cl(B)) = B$. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, $m_X\text{-}Cl(A \cap B) \subseteq m_X\text{-}Cl(A)$ and $m_X\text{-}Cl(A \cap B) \subseteq m_X\text{-}Cl(B)$. Thus $g_X\text{-}Cl(m_X\text{-}Cl(A \cap B)) \subseteq g_X\text{-}Cl(m_X\text{-}Cl(A)) = A$ and $g_X\text{-}Cl(m_X\text{-}Cl(A \cap B)) \subseteq g_X\text{-}Cl(m_X\text{-}Cl(B)) = B$. Hence, $g_X\text{-}Cl(m_X\text{-}Cl(A \cap B)) \subseteq A \cap B$. Since $A \cap B \subseteq m_X\text{-}Cl(A \cap B) \subseteq g_X\text{-}Cl(m_X\text{-}Cl(A \cap B))$, we have $g_X\text{-}Cl(m_X\text{-}Cl(A \cap B)) = A \cap B$. Therefore, $A \cap B$ is $g_X m_X$-closed, and so $A \cap B$ is closed. □

**Remark 3.9.** *The union of two closed sets is not a closed set in general as can be seen from the following example.*

**Example 3.10.** *Let $X = \{a, b, c, d\}$. We define generalized topology $g_X$ and minimal structure space $m_X$ on $X$ as follow: $g_X = \{\emptyset, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$ and $m_X = \{\emptyset, \{a, c, d\}, \{b, c, d\}, X\}$. Then $\{a\}$ and $\{b\}$ are closed but $\{a\} \cup \{b\} = \{a, b\}$ is not closed.*

**Proposition 3.11.** *Let $(X, g_X, m_X)$ be a generalized topology and minimal structure space. Then $A$ is open if and only if $A = g_X\text{-}Int(m_X\text{-}Int(A))$.*

*Proof.* Assume that $A$ is open. Then $X - A$ is closed, and so $g_X\text{-}Cl(m_X\text{-}Cl(X - A)) = X - A$. By Lemma 2.7, $g_X\text{-}Cl(X - m_X\text{-}Int(A)) = X - A$ Thus $X - g_X\text{-}Int(m_X\text{-}Int(A)) = X - A$. Hence, $g_X\text{-}Int(m_X\text{-}Int(A)) = A$.

Conversely, let $A$ be a subset of $X$ such that $A = g_X\text{-}Int(m_X\text{-}Int(A))$. Then $X - A = X - g_X\text{-}Int(m_X\text{-}Int(A))$. By Lemma 2.4, $X - A = g_X\text{-}Cl(X - m_X\text{-}Int(A))$. Thus $X - A = g_X\text{-}Cl(m_X\text{-}Cl(X - A))$. Hence, $X - A$ is closed, and so $A$ is open. □

**Proposition 3.12.** *Let $(X, g_X, m_X)$ be a generalized topology and minimal structure space. If $A$ and $B$ are open, then $A \cup B$ is open.*

*Proof.* Assume that $A$ and $B$ are open. Then $g_X\text{-}Int(m_X\text{-}Int(A)) = A$ and $g_X\text{-}Int(m_X\text{-}Int(B)) = B$. Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, we have $m_X\text{-}Int(A) \subseteq m_X\text{-}Int(A \cup B)$ and $m_X\text{-}Int(B) \subseteq m_X\text{-}Int(A \cup B)$. Thus $A = g_X\text{-}Int(m_X\text{-}Int(A)) \subseteq g_X\text{-}Int(m_X\text{-}Int(A \cup B))$ and $B = g_X\text{-}Int(m_X\text{-}Int(B)) \subseteq g_X\text{-}Int(m_X\text{-}Int(A \cup B))$. Hence, $A \cup B \subseteq g_X\text{-}Int(m_X\text{-}Int(A \cup B))$. But $g_X\text{-}Int(m_X\text{-}Int(A \cup B)) \subseteq m_X\text{-}Int(A \cup B) \subseteq A \cup B$, we obtain that $A \cup B = g_X\text{-}Int(m_X\text{-}Int(A \cup B))$. Therefore, $A \cup B$ is open. □

**Remark 3.13.** *The intersection of two open sets is not a open set in general as can be seen from the following example.*

**Example 3.14.** *Let $X = \{a, b, c\}$. We define generalized topology $g_X$ and minimal structure space $m_X$ on $X$ as follow: $g_X = \{\emptyset, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$ and $m_X = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, X\}$. Then $\{a, b\}$ and $\{a, c\}$ are open but $\{a, b\} \cap \{a, c\} = \{a\}$ is not open.*

**Definition 3.15.** Let $(X, g_X, m_X)$ be a generalized topology and minimal structure space and $A$ a subset of $X$. Then $A$ is said to be *s-closed* if $g_X\text{-}Cl(A) = m_X\text{-}Cl(A)$. And $A$ is said to be *c-closed* if $g_X\text{-}Cl(m_X\text{-}Cl(A)) = m_X\text{-}Cl(g_X\text{-}Cl(A))$. The complement of a s-closed (resp. c-closed) set is called a *s-open* (resp. *c-open*) set.

**Proposition 3.16.** *Let $(X, g_X, m_X)$ be a generalized topology and minimal structure space and $A \subseteq X$. If $A$ is closed, then $A$ is s-closed.*

*Proof.* Assume that $A$ is closed. Then $A$ is $g_X m_X$-closed. By Lemma 3.3, we have $m_X\text{-}Cl(A) = A$ and $g_X\text{-}Cl(A) = A$. Thus $g_X\text{-}Cl(A) = m_X\text{-}Cl(A)$. Hence, $A$ is s-closed. □

**Remark 3.17.** *The converse of Proposition 3.16 is not true. We can be seen from the following example.*

**Example 3.18.** *Let $X = \{a, b, c, d\}$. We define generalized topology $g_X$ and minimal structure space $m_X$ on $X$ as follow: $g_X = \{\emptyset, \{a\}, \{a, c\}\}$ and $m_X = \{\emptyset, \{a\}, \{b, c\}, X\}$. Then $g_X\text{-}Cl(\{c\}) = \{b, c\} = m_X\text{-}Cl(\{c\})$. But $g_X\text{-}Cl(m_X\text{-}Cl(\{c\})) = \{b, c\} \neq \{c\}$.*

**Proposition 3.19.** *Let $(X, g_X, m_X)$ be a generalized topology and minimal structure space and $A \subseteq X$. If $A$ is s-closed, then $A$ is c-closed.*

*Proof.* Assume that $A$ is s-closed. Then $g_X\text{-}Cl(A) = m_X\text{-}Cl(A)$. It follows that $g_X\text{-}Cl(m_X\text{-}Cl(A)) = g_X\text{-}Cl(g_X\text{-}Cl(A)) = g_X\text{-}Cl(A) = m_X\text{-}Cl(A) = m_X\text{-}Cl(m_X\text{-}Cl(A)) = m_X\text{-}Cl(g_X\text{-}Cl(A))$. Hence, $A$ is c-closed. □

**Remark 3.20.** *The converse of Proposition 3.19 is not true. We can be seen from the following example.*

**Example 3.21.** *Let $X = \{a, b, c\}$. We define generalized topology $g_X$ and minimal structure space $m_X$ on $X$ as follow: $g_X = \{\emptyset, \{a\}, \{a, c\}\}$ and $m_X = \{\emptyset, \{b\}, \{b, c\}, X\}$. Then $g_X\text{-}Cl(m_X\text{-}Cl(\{c\})) = X = m_X\text{-}Cl(g_X\text{-}Cl(\{c\}))$. But $g_X\text{-}Cl(\{c\}) = \{b, c\} \neq \{a, c\} = m_X\text{-}Cl(\{c\})$.*

**Proposition 3.22.** *Let $(X, g_X, m_X)$ be a generalized topology and minimal structure space and $A \subseteq X$. Then*

*(1) $A$ is s-open if and only if $g_X\text{-}Int(A) = m_X\text{-}Int(A)$,*

*(2) A is c-open if and only if*

$g_X\text{-}Int(m_X\text{-}Int(A)) = m_X\text{-}Int(g_X\text{-}Int(A))$.

**Proposition 3.23.** *Let* $(X, g_X, m_X)$ *be a generalized topology and minimal structure space and* $A \subseteq X$. *If* $A$ *is open, then* $A$ *is s-open.*

**Proposition 3.24.** *Let* $(X, g_X, m_X)$ *be a generalized topology and minimal structure space and* $A \subseteq X$. *If* $A$ *is s-open, then* $A$ *is c-open.*

**Lemma 3.25.** *Let* $(X, g_X, m_X)$ *be a generalized topology and minimal structure space and* $A \subseteq X$. *If* $A$ *is s-closed, then* $g_X\text{-}Cl(A)$ *is closed.*

*Proof.* Assume that $A$ is s-closed. Then $g_X\text{-}Cl(A) = m_X\text{-}Cl(A)$, which implies that $m_X\text{-}Cl(g_X\text{-}Cl(A)) = m_X\text{-}Cl(m_X\text{-}Cl(A)) = m_X\text{-}Cl(A) = g_X\text{-}Cl(A)$. Thus $g_X\text{-}Cl(m_X\text{-}Cl(g_X\text{-}Cl(A))) = g_X\text{-}Cl(g_X\text{-}Cl(A)) = g_X\text{-}Cl(A)$. Hence, $g_X\text{-}Cl(A)$ is closed. □

**Remark 3.26.** *Let* $(X, g_X, m_X)$ *be a generalized topology and minimal structure space and* $A \subseteq X$. *If* $A$ *is s-closed, then* $m_X\text{-}Cl(A)$ *is closed.*

**Theorem 3.27.** *Let* $(X, g_X, m_X)$ *be a generalized topology and minimal structure space and* $A \subseteq X$. *Then* $A$ *is closed if and only if there exists a s-closed set* $B$ *such that* $g_X\text{-}Cl(B) = A$.

*Proof.* Assume that $A$ is closed. Thus $A$ is a s-closed and $A = g_X\text{-}Cl(A) = m_X\text{-}Cl(A)$. Set $B = A$. Then $B$ is s-closed and $A = g_X\text{-}Cl(B)$.

Conversely, assume that there exists a s-closed set $B$ such that $g_X\text{-}Cl(B) = A$. By Lemma 3.25, we obtain that $A = g_X\text{-}Cl(B)$ is closed. □

**Lemma 3.28.** *Let* $(X, g_X, m_X)$ *be a generalized topology and minimal structure space and* $A \subseteq X$. *If* $A$ *is c-closed, then* $g_X\text{-}Cl(m_X\text{-}Cl(A))$ *is closed.*

*Proof.* Assume that $A$ is c-closed and set $B = g_X\text{-}Cl(m_X\text{-}Cl(A))$. Since $m_X\text{-}Cl(B) = m_X\text{-}Cl(m_X\text{-}Cl(g_X\text{-}Cl(A))) = m_X\text{-}Cl(g_X\text{-}Cl(A))$, we obtain that $g_X\text{-}Cl(m_X\text{-}Cl(B)) = g_X\text{-}Cl(m_X\text{-}Cl(g_X\text{-}Cl(A))) = g_X\text{-}Cl(g_X\text{-}Cl(m_X\text{-}Cl(A))) = g_X\text{-}Cl(m_X\text{-}Cl(A)) = B$. Hence, $g_X\text{-}Cl(m_X\text{-}Cl(A)) = B$ is closed. □

**Remark 3.29.** *Let* $(X, g_X, m_X)$ *be a generalized topology and minimal structure space and* $A \subseteq X$. *If* $A$ *is c-closed, then* $m_X\text{-}Cl(g_X\text{-}Cl(A))$ *is closed.*

**Theorem 3.30.** *Let* $(X, g_X, m_X)$ *be a generalized topology and minimal structure space and* $A \subseteq X$. *Then* $A$ *is closed if and only if there exists a c-closed set* $B$ *such that* $A = g_X\text{-}Cl(m_X\text{-}Cl(B))$.

*Proof.* Assume that $A$ is closed. Thus $A = g_X\text{-}Cl(m_X\text{-}Cl(A))$ and $A$ is a c-closed. Set $B = A$. Then $B$ is c-closed and $A = g_X\text{-}Cl(m_X\text{-}Cl(B))$.

Conversely, assume that there exists a c-closed set $B$ such that $A = g_X\text{-}Cl(m_X\text{-}Cl(B))$. By Lemma 3.28, we obtain that $A = g_X\text{-}Cl(m_X\text{-}Cl(B))$ is closed. □

# 4 $T_1$-GTMS space and $T_2$-GTMS space

In this section, we introduce the concepts of $T_1$-GTMS spaces and $T_2$-GTMS spaces and we give their characterizations.

**Definition 4.1.** A GTMS space $(X, g_X, m_X)$ is called a *$T_1$-GTMS space* if for any pair of distinct points $x$ and $y$ in $X$, there exist a $g_X$-open set $U$ and a $m_X$-open set $V$ such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.

**Lemma 4.2.** *Let $(X, g_X, m_X)$ be a GTMS space and let $x, y \in X$ be such that $x \neq y$. If $(X, g_X, m_X)$ is a $T_1$-GTMS space, then there exist a $g_X$-open set $U$ containing $x$ but not $y$ and a $m_X$-open set $V$ containing $x$ but not $y$.*

**Theorem 4.3.** *Let $(X, g_X, m_X)$ be a GTMS space. Then $X$ is a $T_1$-GTMS space if and only if every singleton subset of $X$ is closed.*

*Proof.* Assume that $X$ is a $T_1$-GTMS space and let $x \in X$. We will show that $\{x\} = g_X\text{-}Cl(\{x\})$ and $\{x\} = m_X\text{-}Cl(\{x\})$. Let $y \in X$ be such that $y \neq x$. By Lemma 4.2, there exist a $g_X$-open set $U$ and a $m_X$-open set $V$ such that $y \in U$, $x \notin U$ and $y \in V$, $x \notin V$. Thus $U \cap \{x\} = \emptyset$ and $V \cap \{x\} = \emptyset$ implies that $y \notin g_X\text{-}Cl(\{x\})$ and $y \notin m_X\text{-}Cl(\{x\})$. Hence, $\{x\} = g_X\text{-}Cl(\{x\})$ and $\{x\} = m_X\text{-}Cl(\{x\})$, and so $\{x\}$ is closed.

Conversely, assume that every singleton subset of $X$ is closed. Let $x, y \in X$ be such that $x \neq y$. By assumption, we have $\{x\} = m_X\text{-}Cl(\{x\})$ and $\{y\} = g_X\text{-}Cl(\{y\})$. Since $x \notin g_X\text{-}Cl(\{y\})$ and $y \notin m_X\text{-}Cl(\{x\})$, there exist a $g_X$-open set $U$ and a $m_X$-open set $V$ such that $x \in U$, $U \cap \{y\} = \emptyset$ and $y \in V$, $V \cap \{x\} = \emptyset$. Then $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$. Hence, $X$ is a $T_1$-GTMS space. □

**Definition 4.4.** A GTMS space $(X, g_X, m_X)$ is called a *Hausdorff GTMS space* or *$T_2$-GTMS space* if for any pair of distinct points $x$ and $y$ in $X$, there exist a $g_X$-open set $U$ and a $m_X$-open set $V$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

**Proposition 4.5.** *Let $(X, g_X, m_X)$ be a GTMS space. If $X$ is a $T_2$-GTMS space, then $X$ is a $T_1$-GTMS space.*

**Remark 4.6.** *The converse of Proposition 4.5 is not true. We can be seen from the following example.*

**Example 4.7.** *Let $X = \{a, b, c\}$. We define generalized topology $g_X$ and minimal structure space $m_X$ on $X$ as follow: $g_X = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$ and $m_X = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. Then $X$ is a $T_1$-GTMS space. But $X$ is not a $T_2$-GTMS space.*

**Theorem 4.8.** *Let $(X, g_X, m_X)$ be a GTMS space. Then the following are equivalent:*

*(1) $X$ is a $T_2$-GTMS space.*

*(2) If $x \in X$, then for each $y \neq x$, there exists a $g_X$-open set $U$ containing $x$ such that $y \notin m_X$-$Cl(U)$.*

*(3) For each $x \in X$, $\{x\} = \cap\{m_X\text{-}Cl(U) : U \in g_X$ and $x \in U\}$.*

*Proof.* (1) ⇒ (2) Assume that $X$ is a $T_2$-GTMS space and $x \in X$. Let $y \in X$ be such that $y \neq x$. Then there exist a $g_X$-open set $U$ and a $m_X$-open set $V$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Thus $y \notin m_X$-$Cl(U)$.

(2) ⇒ (3) Let $x \in X$. We will prove that $\{x\} = \cap\{m_X\text{-}Cl(U) : U \in g_X \text{ and } x \in U\}$. It is clear that $\{x\} \subseteq \cap\{m_X\text{-}Cl(U) : U \in g_X \text{ and } x \in U\}$. Let $y \in X$ be such that $y \neq x$. By assumption, there exists a $g_X$-open set $U_1$ containing $x$ such that $y \notin m_X$-$Cl(U_1)$. Then $y \notin \cap\{m_X\text{-}Cl(U) : U \in g_X \text{ and } x \in U\}$. Thus $\cap\{m_X\text{-}Cl(U) : U \in g_X \text{ and } x \in U\} \subseteq \{x\}$. Therefore, $\{x\} = \cap\{m_X\text{-}Cl(U) : U \in g_X \text{ and } x \in U\}$.

(2) ⇒ (3) Assume that $\{x\} = \cap\{m_X\text{-}Cl(U) : U \in g_X \text{ and } x \in U\}$ for each $x \in X$. Let $x, y \in X$ be such that $x \neq y$. Since $y \notin \{x\} = \cap\{m_X\text{-}Cl(U) : U \in g_X \text{ and } x \in U\}$, there exists $U_1 \in g_X$ such that $x \in U_1$ and $y \notin m_X$-$Cl(U_1)$. Since $y \notin m_X$-$Cl(U_1)$, there exists $V_1 \in m_X$ such that $y \in V_1$ and $U_1 \cap V_1 = \emptyset$. Then $x \in U$, $y \in V_1$ and $U_1 \cap V_1 = \emptyset$. Hence, $X$ is a $T_2$-GTMS space. □

**Theorem 4.9.** *Let $(X, g_X, m_X)$ be a GTMS space. Then the following are equivalent:*

*(1) $X$ is a $T_2$-GTMS space.*

*(2) If $x \in X$, then for each $y \neq x$, there exists a $m_X$-open set $V$ containing $x$ such that $y \notin g_X$-$Cl(V)$.*

*(3) For each $x \in X$, $\{x\} = \cap\{g_X\text{-}Cl(V) : V \in m_X$ and $x \in V\}$.*

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